Technische Universität Berlin SS 2017
Institut für Mathematik
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1. Exercise sheet

FV/FD-Methods for the solution of pde's

Discussion: 5.6.17-9.6.17

1) Exercise
a) Show that the Laplace operator Δ in the polar coordinate system (r, φ) is of the form

$$\Delta_{(r,\varphi)} u = \frac{1}{r} u_r + u_{rr} + \frac{1}{r^2} u_{\varphi\varphi} \ .$$

b) Find a solution of the problem

$$-\Delta u = f \ , \quad \in \Omega, \tag{1}$$
$$u = g \ , \quad \text{on } \Gamma = \partial\Omega \ , \tag{2}$$

with

$$\Omega = \{(x, y) \mid x = r\cos\varphi,\ y = r\sin\varphi,\ 0 < r < \rho,\ 0 < \varphi < \pi/\alpha\ ,\ \alpha \geq 1\},$$

$f = 0$ and

$$g(r, \varphi) = \begin{cases} 0 & \text{for } 0 \leq r < \rho \text{ and } \varphi = 0,\ \varphi = \pi/\alpha, \\ \varphi(\pi/\alpha - \varphi) & \text{for } 0 < \varphi < \pi/\alpha \text{ and } r = \rho \end{cases} \quad ,$$

by a separation Ansatz with polar coordinates.

2) Exercise
We consider the Poisson equation

$$-\Delta u = f \ , \quad \in \Omega =]0, 1[\times]0, 1[, \tag{3}$$
$$u = 0 \ , \quad \text{on } \Gamma = \partial\Omega \ , \tag{4}$$

with

$$f(x, y) = -\sin(2\pi x)\sin(2\pi y) \ .$$

(a) Verify

$$u(x, y) = -\frac{1}{8\pi^2}\sin(2\pi x)\sin(2\pi y)$$

as the exact solution of (3).

(b) Construct a FD-method to solve the problem (3) numerically. Use an equidistant grid with the uniform grid space h. Realize the method by a Matlab/Octave-program and demonstrate the convergence behavior for $h = 1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32$ with a plot of the error e_h vs. grid space (maybe with double logarithmic scale of the h- and the e_h-axis).
Solve the linear equation system $A_h u_h = f_h$ with the Matlab/Octave backslash-command.

(c) Plot the exact solution and the numerical solution for $h = 1/4,\ 1/16,\ 1/32$.

3) Exercise

Solve the problem (1) with

$$\Omega = \{(x, y) \mid x = r\cos\varphi,\ y = r\sin\varphi,\ 1 < r < 2,\ 0 < \varphi \leq 2\pi\}\ ,$$

and

$$f(r, \varphi) = (2 - r)(1 - r) \text{ on } \Omega\ ,$$

and homogeneous Dirichlet boundary conditions.