Small deviations of general Lévy processes

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joint work with Steffen Dereich (Berlin)

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Outline

1. Statement of the problem
2. Main results
3. Examples
4. Ideas of the proofs; Multi-dimensional extension

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The small deviation problem

Let $X(t), t \geq 0$ be a stochastic process. We search for the rate of

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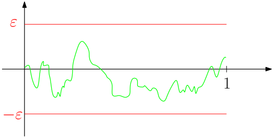

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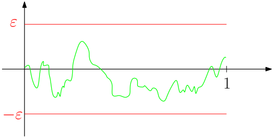

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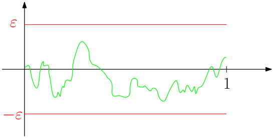

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- relation to approximation quantities for the stochastic process

Previous results for Lévy processes

The rate of

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right)$$

was known for

- Brownian motion (classical)
- strictly α-stable Lévy processes (by scaling, see Bertoin's book)
- subordinators (trivial)
- LP obtained by subordinating to Brownian motion (Linde/Shi04)
- When is $\mathbb{P}(\ldots \le \varepsilon) > 0$? (Simon01)

...now: general framework that can be used for all LPs

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Notation

Let X be a Lévy process with triplet (ν, σ^2, b), i.e. Lévy-Khinchine repr.

$$\log \mathbb{E} e^{izX(t)} = t \left(-\frac{\sigma^2}{2} z^2 + ibz + \int_{\mathbb{R}} (e^{izx} - 1 - izx 1_{|x| \leq 1}) \, d\nu(x) \right)$$

Goal: Express the rate in

$$-\log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X(t)| \leq \varepsilon \right) \approx \ ?$$

in terms of ν, σ^2, and b!

Here and everywhere $f \approx g$ means $0 < \liminf \frac{f}{g} \leq \limsup \frac{f}{g} < \infty$.

Three effects

The small deviation order

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right) \approx \ ?$$

is governed by three effects:

(1) the cost of having no large jumps

(2) the cost of eliminating the drift

(3) the cost of the oscillations of the process

Effect 1: no large jumps

- obviously, if

$$\sup_{0\leq t\leq 1} |X(t)| \leq \frac{\varepsilon}{2}$$

  then the process has no jump larger than ε

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- this implies

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right) = e^{-\nu([-\varepsilon,\varepsilon]^c)}\, \mathbb{P}'\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right)$$

- where $\mathbb{P}'$ is the law of the Lévy process with triplet

$$\left(\nu|_{[-\varepsilon,\varepsilon]}, \sigma^2, b - \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} x\, d\nu(x)\right)$$

- so, in the next step we can assume that ν is supported on $[-\varepsilon,\varepsilon]$

Effect 2: drift of the process

eliminate the drift of the process by exponential change of measure

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Proposition

Let X be a LP with ν supported by $\{|x| \leq 1\}$. Assume that $u^* \in \mathbb{R}$ is a solution of $\Lambda'(u^*) = 0$ where

$$\Lambda(u) := \frac{\sigma^2}{2} u^2 + bu + \int [e^{ux} - 1 - ux] \, d\nu(x)$$

denotes the logarithmic MGF of X_1. Then the Esscher transf. given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{u^* X_1 - \Lambda(u^*)}$$

is a probability measure such that, for all $\varepsilon > 0$,

$$e^{\Lambda(u^*) - \varepsilon |u^*|} \mathbb{Q}(||X|| \leq \varepsilon) \leq \mathbb{P}(||X|| \leq \varepsilon) \leq e^{\Lambda(u^*) + \varepsilon |u^*|} \mathbb{Q}(||X|| \leq \varepsilon)$$

and X is a $(e^{u^* x} \cdot \nu(dx), \sigma^2, 0)$-Lévy martingale under $\mathbb{Q}$.

Effect 2: drift of the process

eliminate the drift of the process by exponential change of measure

- When does $\Lambda'(u^*) = 0$ have a solution for all $\varepsilon > 0$?
- if and only if

$$\mathbb{P}(\sup_{0\leq t\leq 1} |X(t)| \leq \varepsilon) > 0 \qquad \forall \varepsilon > 0$$

and X is not a subordinator (or $-X$ subord.)

- i.e. in all cases of interest!

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Theorem

Let $\varepsilon > 0$ *and denote by* X *a* $(\nu, \sigma^2, 0)$*-Lévy martingale with* ν *supported on* $[-\varepsilon, \varepsilon]$*. Then*

$$e^{-10\, F(\varepsilon/3)-3} \leq \mathbb{P}(\sup_{0\leq t\leq 1} |X(t)| \leq \varepsilon) \leq e^{-\frac{1}{12} F(2\varepsilon)+1},$$

where

$$F(\varepsilon) := \frac{1}{\varepsilon^2} \left[\sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \, d\nu(x) \right].$$

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- the function F can be computed explicitly from the triplet
- this is the same small deviation behaviour as if we replaced X by a Brownian motion with the same variance

Overview

Essentially

$$-\log \mathbb{P}\Big(\sup_{0\le t\le 1} |X(t)| \le \varepsilon \Big) \approx \nu([-\varepsilon,\varepsilon]^c) - \Lambda_\varepsilon(u_\varepsilon^*) \pm \varepsilon |u_\varepsilon^*| + \bar{F}(\varepsilon),$$

where

$$\Lambda_\varepsilon(u) = \frac{\sigma^2}{2} u^2 + b_\varepsilon u + \int_{-\varepsilon}^{\varepsilon} [e^{ux} - 1 - ux] \, d\nu(x)$$

the term u_ε^* solves $\Lambda'(u_\varepsilon^*) = 0$,

$$\bar{F}(\varepsilon) = \frac{1}{\varepsilon^2} \left[\sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 e^{u_\varepsilon^* x} \, d\nu(x) \right] = \varepsilon^{-2} \Lambda_\varepsilon''(u_\varepsilon^*),$$

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$$b_\varepsilon = b - \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} x \, d\nu(x)$$

Everything is explicitly computable from (ν, σ^2, b)!

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Lévy processes with Gaussian component

The results imply

Corollary

Let X be a (ν, σ^2, b)-Lévy process with $\sigma \neq 0$. Then

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right) \sim \frac{\pi^2}{8}\,\sigma^2 \varepsilon^{-2}.$$

Here, $f \sim g$ is $\lim \frac{f}{g} = 1$.

Symmetric Lévy processes

If X is symmetric (i.e. $\mathcal{L}(X_1) = \mathcal{L}(-X_1)$), Effect 2 does not appear (no change of measure needed)

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Corollary

Let X be a symmetric Lévy process. Then

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right) \approx \nu([-\varepsilon,\varepsilon]^c) + \frac{1}{\varepsilon^2}\left[\sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \, d\nu(x)\right]$$

Subordinators with negative drift

Let X be an α-stable subordinator, $0 < \alpha < 1$ (strictly increasing):

$$\mathbb{E}e^{izX(t)} = \exp\left(t\int_0^\infty (e^{izx} - 1)\,\frac{dx}{x^{1+\alpha}}\right)$$

Then, as $\varepsilon \to 0$,

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t) + \mu t| \le \varepsilon\right) \approx \begin{cases} = \infty & \mu > 0 \\ \varepsilon^{-\alpha/(1-\alpha)} & \mu = 0 \\ \varepsilon^{-1}|\log \varepsilon| & \mu < 0. \end{cases}$$

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Let $N_\varepsilon :=$ no. of jumps $< \varepsilon$. Then $X(1) \le -\mu + \varepsilon N_\varepsilon +$ “large jumps”. Thus, discounting the large jumps,

$$\begin{aligned} \mathbb{P}(\sup \le \varepsilon) &\le \mathbb{P}(X(1) > -\varepsilon) \le \mathbb{P}(-\mu + \varepsilon N_\varepsilon > -\varepsilon) \\ &\le \mathbb{P}\left(N_\varepsilon > \frac{\mu}{\varepsilon} - 1 \right) \le \exp(-\mu\varepsilon^{-1} |\log \varepsilon| - \text{“other terms”}) \end{aligned}$$

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LP with ν supp. on $[-\varepsilon,\varepsilon]$ and that is a martingale. Want to show:

$$e^{-10\,F(\varepsilon/3)-3} \le \mathbb{P}(\sup_{0\le t\le 1} |X(t)| \le \varepsilon) \le e^{-\frac{1}{12}F(2\varepsilon)+1},$$

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Let

$$\tau := \text{first exit time from } [-\varepsilon, \varepsilon]$$

Then one can see easily that $\mathbb{E}\tau \approx 1/F(\varepsilon)$. Therefore,

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \frac{\varepsilon}{2}\right) \le \mathbb{P}\left(\sup_{\frac{i}{n}\le t\le \frac{i+1}{n}, i=0,\dots,n-1} |X(t) - X(i/n)| \le \varepsilon\right)$$

$$\le \mathbb{P}\left(\tau \ge 1/n\right)^n \le \left(\frac{\mathbb{E}\tau}{1/n}\right)^n = e^{-n},$$

if $n \approx F(\varepsilon)$.

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Let τ be as above, recall $\mathbb{E}\tau \approx 1/F(\varepsilon)$.

$$\mathbb{P}(||X|| \leq 3\varepsilon) \geq \mathbb{P}\left(\text{“scenario”}\right) \approx \left(\frac{1}{3}\right)^{\text{typical no. of exits in } [0,1]} \approx \left(\frac{1}{3}\right)^{F(\varepsilon)}.$$

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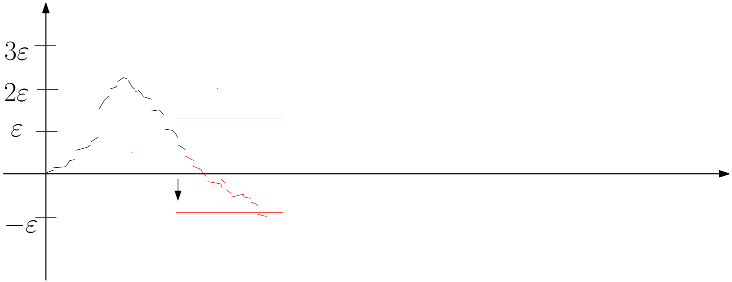

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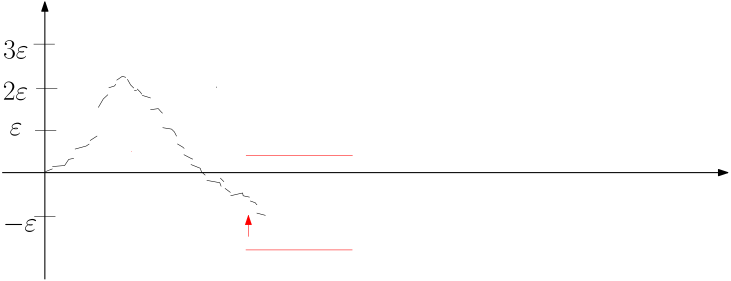

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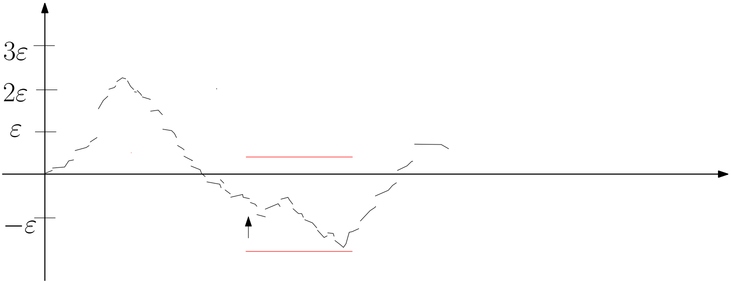

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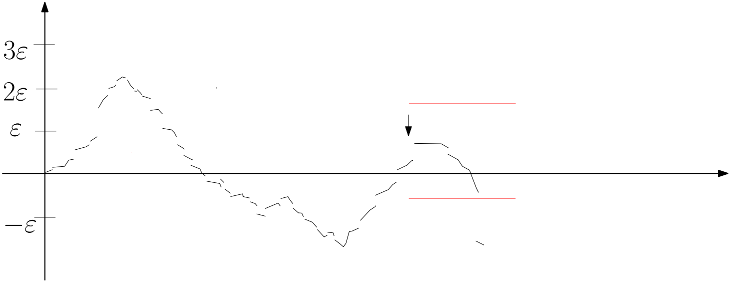

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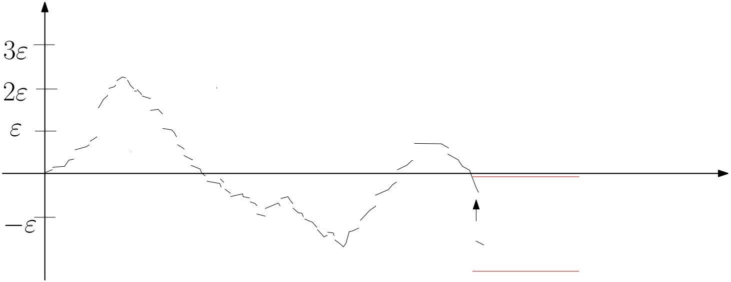

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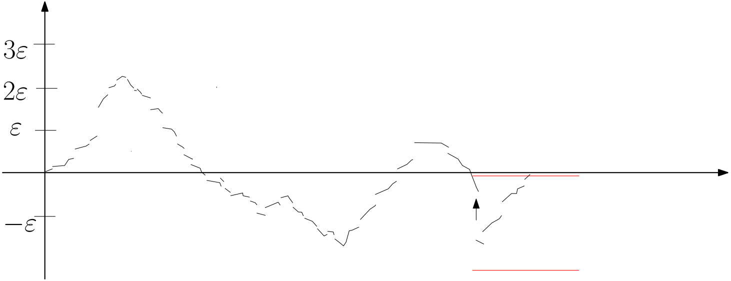

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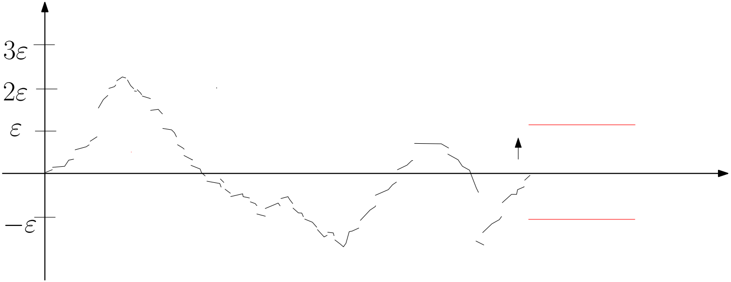

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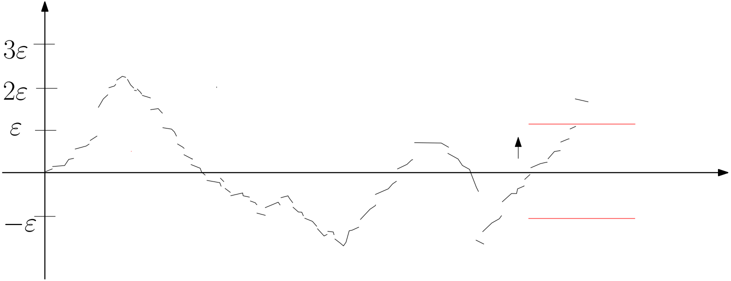

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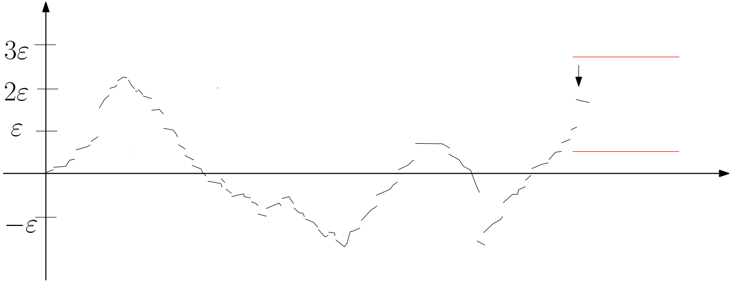

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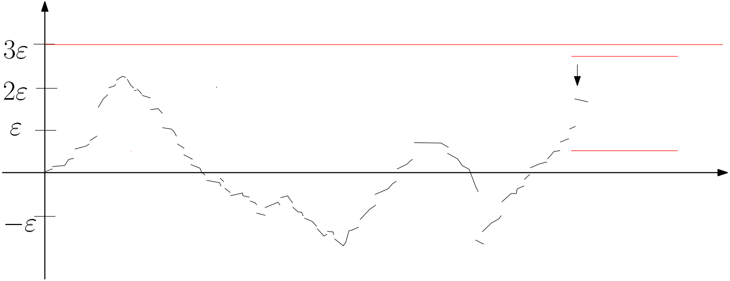
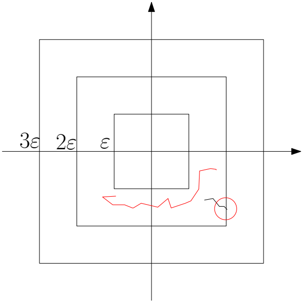


Problem for *d*-dimensional LP

$$\mathbb{P}(||X|| \leq 3\varepsilon) \geq \mathbb{P}\left(\text{“scenario”}\right) \approx ? \approx \left(\frac{1}{3}\right)^{\text{typical no. of exits in } [0,1]} \approx \left(\frac{1}{3}\right)^{F(\varepsilon)}.$$

Scenario?



Thank you for your attention!

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