

Small Deviation Probability via Chaining

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Abstract

We obtain several extensions of Talagrand's lower bound for the small deviation probability using metric entropy. For Gaussian processes, our investigations are focused on processes with sub-polynomial and, respectively, exponential behaviour of covering numbers. The corresponding results are also proved for non-Gaussian symmetric stable processes, both for the cases of critically small and critically large entropy. The results extensively use the classical chaining technique; at the same time they are meant to explore the limits of this method.

Key words: Small deviation, lower tail probability, chaining, metric entropy, Gaussian processes, stable processes.

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1 Introduction and main results

1.1 Motivation

General small deviation problems attracted much attention recently due to their deep relations to various mathematical topics like operator theory, quantization, strong limit laws in statistics, etc., cf. the surveys [5,7].

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The first goal of this article is to extend the well-known Talagrand lower bound for the small deviation probability to the case of Gaussian random functions with not necessarily regularly varying behaviour of their metric entropy.

Before recalling the known results and stating the new ones, let us introduce the necessary notation. Consider a centered Gaussian random function $X(t), t \in T, T \neq \emptyset$, and assume there exists a separable version of X that we consider in the sequel. Assume furthermore that the parameter set T equipped with quasi-metric $\rho(s, t)^2 = \mathbb{E}(X(t) - X(s))^2$, usually referred to as Dudley metric, is a relatively compact metric space. Let

$$N(\varepsilon) := \min\{n \in \mathbb{N} \mid \exists t_1, \dots, t_n \in T \forall t \in T \exists i : \rho(t, t_i) \leq \varepsilon\}$$

denote the covering numbers of (T, ρ) and $\sigma := \text{diam}(T)$. Obviously, $N(\varepsilon) = 1$ whenever $\varepsilon \geq \sigma$. Covering numbers present a common quantitative measure for the entropy of the space (T, ρ) .

At some places we use the following notation for strong and weak asymptotics. For two functions f and g , $f(x) \sim g(x)$, as $x \rightarrow 0$, means that $f(x)/g(x) \rightarrow 1$, as $x \rightarrow 0$. On the other hand, we use the notation $f(x) \preceq g(x)$, as $x \rightarrow 0$, if $\limsup_{x \rightarrow 0} f(x)/g(x) < \infty$. We also write $g(x) \succeq f(x)$ in this case. Furthermore, we write $f(x) \approx g(x)$, as $x \rightarrow 0$, if $f(x) \preceq g(x)$ and $g(x) \succeq f(x)$. The notation is defined analogously for sequences.

Talagrand's lower bound from [14], which became by now classical in the form given by M. Ledoux [4, p. 257], reads as follows.

Theorem 1 *Assume that $N(\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and let the bound Ψ satisfy the regularity assumptions*

$$C_1 \Psi(\varepsilon) \leq \Psi\left(\frac{\varepsilon}{2}\right), \quad \sigma > \varepsilon > 0, \quad (1)$$

$$\Psi\left(\frac{\varepsilon}{2}\right) \leq C_2 \Psi(\varepsilon), \quad \varepsilon > 0, \quad (2)$$

with some $C_2 > C_1 > 1$. Then

$$\log \mathbb{P} \left\{ \sup_{s, t \in T} |X(s) - X(t)| \leq \varepsilon \right\} \geq -K\Psi(\varepsilon), \quad \varepsilon > 0, \quad (3)$$

with $K > 0$ depending only on C_1, C_2 .

This result works perfectly well and provides sharp estimates for many cases where Ψ is a polynomial-type function. Unfortunately, on the one hand, it does not apply to slowly varying bounds, e.g. $\Psi(\varepsilon) = |\log \varepsilon|^\beta$, since $C_1 > 1$ in (1) is impossible for such functions. Neither is this theorem applicable to

exponential bounds, e.g. $\log \Psi(\varepsilon) = \varepsilon^{-\gamma} |\log \varepsilon|^\beta$, since it is not possible to find $C_2 < \infty$ in this case.

Moreover, it is easy to see (cf. e.g. Example 1 below) that in such cases the estimate (3) fails in its present form. However, recently, a number of works appeared where small deviations are studied for cases with rather arbitrary behaviour of entropy, see e.g. [9,10]. In particular, a slow increase of $N(\varepsilon)$ when ε tends to zero is not excluded at all. It is therefore desirable to have a version of Theorem 1 with a wider application range.

The objectives of this article are as follows. Firstly, we show that a more careful estimation in the original proof of Talagrand leads to a generally applicable lower bound (Theorem 2), which, in particular, in the case of slow entropy behaviour returns a correct bound.

In the case of large entropy behaviour, we complete the standard approach by combining the classical chaining arguments with the use of Laplace transform techniques. To the knowledge of the authors, this has not been applied before; and it is their belief that the idea could be used successfully in other contexts. For this reason, Section 2 is devoted to the chaining technique.

Furthermore, the considerations will show that the classical chaining idea leads to “sum of maxima” type expressions. Namely, classical chaining essentially yields estimates of the form

$$\sup_{s,t \in T} |X(s) - X(t)| \leq 2 \sum_{k=0}^{\infty} \varepsilon_k \max_{i=1, \dots, N_k} |\xi'_{k,i}|,$$

where $\xi'_{k,i}$ are – not necessarily independent – standard Gaussian random variables, $N_k = N(\varepsilon_{k+1})$, and (ε_k) is some arbitrary decreasing sequence. The above estimate could be called “uniform” chaining, as opposed to majorizing measure/generic chaining bounds, cf. [15] for a recent description of the theory.

Using the Khatri-Šidák Inequality allows to replace the $\xi'_{k,i}$ by *independent* standard Gaussian random variables $\xi_{k,i}$ giving

$$\sup_{s,t \in T} |X(s) - X(t)| \leq 2 \sum_{k=0}^{\infty} \varepsilon_k \max_{i=1, \dots, N_k} |\xi_{k,i}|, \quad (4)$$

where \leq is to be understood in law. The expression on the right-hand side is what we will call “sum of maxima” type. For the time being, this observation has nothing to do with small deviations; note e.g. that taking expectations of (4) immediately yields Dudley’s Theorem. However, as we demonstrate in this article, a careful estimation of “sum of maxima” type terms leads to reasonable small deviation results.

Finally, we apply the above-mentioned techniques also to non-Gaussian sym-

metric stable processes, where everything works analogously – with the natural limitations due to the heavy tails. In fact, the most delicate point to be adapted to the non-Gaussian case is the Khatri-Šidák Inequality used in the chaining argument. Fortunately, a version of this inequality for *symmetric* stable variables is available, see Lemma 2.1 in [12].

The paper is structured as follows. In Sections 1.2 and 1.3 we state the main results of the article, for the cases of Gaussian and symmetric α -stable random functions, respectively. In order to give a taste of the applicability of the results and to present the crucial “sum of maxima” examples, we consider some important special cases in Section 1.4.

In Section 2, we recall the classical “uniform” chaining argument and present the corresponding result for the Laplace transform. Section 3 contains the proofs of the general estimate, which works for slow and polynomial entropy behaviour. The proof is essentially the same for Gaussian and symmetric α -stable processes. Contrary to this, for the large entropy cases, we have to distinguish Gaussian and non-Gaussian stable processes, due to their distinct tail behaviour. The proofs in those cases are presented in Sections 4 and 5, respectively. The article is concluded by some remarks on further extensions and related questions in Section 6.

1.2 The Gaussian case

A version of Talagrand’s result that, in particular, includes the case of slow increase of entropy is as follows. Let

$$\tilde{\Psi}(\varepsilon) = \int_{\varepsilon}^{\sigma} \frac{\Psi(u)}{u} du, \quad 0 < \varepsilon \leq \sigma/2, \quad (5)$$

and $\tilde{\Psi}(\varepsilon) = \Psi(\varepsilon)$ for $\varepsilon \geq \sigma/2$. We prove the following.

Theorem 2 *Assume that $N(\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and let the bound Ψ be a non-increasing continuous function satisfying the regularity assumption*

$$\Psi\left(\frac{\varepsilon}{2}\right) \leq C_2 \Psi(\varepsilon), \quad \varepsilon > 0, \quad (6)$$

with some $C_2 > 1$. Then

$$\log \mathbb{P} \left\{ \sup_{s,t \in T} |X(s) - X(t)| \leq K_0 \varepsilon \right\} \geq -K \tilde{\Psi}(\varepsilon), \quad \varepsilon > 0, \quad (7)$$

with numerical constants K_0 and $K > 0$, where K depends on C_2 and K_0 is a universal constant.

Comments.

1. We first notice that Theorem 2 contains Theorem 1. Indeed, assumption (1) yields

$$\frac{\Psi(u)}{\Psi(\varepsilon)} \leq C_1 \left(\frac{u}{\varepsilon}\right)^{-h}, \quad \forall \varepsilon \leq u,$$

with $h = \log C_1 / \log 2 > 0$. We easily obtain from the latter inequality that

$$\tilde{\Psi}(\varepsilon) = \int_{\varepsilon}^{\sigma} \frac{\Psi(u)}{u} du \leq C_1 \Psi(\varepsilon) \varepsilon^h \int_{\varepsilon}^{\infty} u^{-1-h} du = C_1 h^{-1} \Psi(\varepsilon). \quad (8)$$

It is now clear that (7) implies (3).

2. Apart from polynomial-type Ψ already covered by Theorem 1, the most instructive applications of Theorem 2 are the following.

a) If $\Psi(\varepsilon) = C |\log \varepsilon|^\beta$ with some $\beta > 0$, then

$$\tilde{\Psi}(\varepsilon) \sim \frac{C}{\beta + 1} |\log \varepsilon|^{\beta+1}, \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, $N(\varepsilon) \leq C |\log \varepsilon|^\beta$ yields

$$-\log \mathbb{P} \left\{ \sup_{t,s \in T} |X(t) - X(s)| \leq \varepsilon \right\} \preceq |\log \varepsilon|^{\beta+1}, \quad \text{as } \varepsilon \rightarrow 0.$$

b) If $\Psi(\varepsilon) = C \exp \{A |\log \varepsilon|^\alpha\}$ with some $C, A > 0$ and $\alpha \in (0, 1)$, then

$$\tilde{\Psi}(\varepsilon) \sim \frac{C}{A} |\log \varepsilon|^{1-\alpha} \exp \{A |\log \varepsilon|^\alpha\}, \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, $N(\varepsilon) \leq C \exp \{A |\log \varepsilon|^\alpha\}$ yields, as $\varepsilon \rightarrow 0$,

$$-\log \mathbb{P} \left\{ \sup_{t,s \in T} |X(t) - X(s)| \leq \varepsilon \right\} \preceq |\log \varepsilon|^{1-\alpha} \exp \{A |\log \varepsilon|^\alpha\}. \quad (9)$$

We give concrete cases with the above entropy behaviour in Example 1 in Section 1.4 below.

3. As one can observe from the above-mentioned examples, the ratio of functions $\tilde{\Psi}$ and Ψ ranges between the constant and the logarithmic function. Actually, this is always true under our assumptions, since for $\varepsilon \leq \sigma/2$

$$\tilde{\Psi}(\varepsilon) = \int_{\varepsilon}^{\sigma} \frac{\Psi(u)}{u} du \leq \int_{\varepsilon}^{\sigma} \frac{du}{u} \Psi(\varepsilon) = \log \frac{\sigma}{\varepsilon} \Psi(\varepsilon)$$

and

$$\tilde{\Psi}(\varepsilon) = \int_{\varepsilon}^{\sigma} \frac{\Psi(u)}{u} du \geq \int_{\varepsilon}^{2\varepsilon} \frac{du}{u} \Psi(2\varepsilon) = \log 2 \Psi(2\varepsilon) \geq \frac{\log 2}{C_2} \Psi(\varepsilon). \quad (10)$$

4. The reader familiar with the theory of Gaussian processes (see e.g. [7]) will surely notice that the integral characteristic $\tilde{\Psi}$ has much in common with the Dudley integral – the basic entropy tool for the evaluation of *large* deviations and moduli of continuity of Gaussian processes.

Let us now come to the case of large entropy behaviour. Note that (6) restricts the application range of Theorem 2 to essentially regularly or slowly varying entropy behaviour. However, with the techniques presented in this article we can also tackle the case of exponentially increasing entropy. One possibility is the following theorem.

Theorem 3 *Let us assume that*

$$\log N(\varepsilon) \leq C\varepsilon^{-\gamma} |\log \varepsilon|^{-\beta}, \quad (11)$$

with some $0 < \gamma < 2$ or $\gamma = 2$ and $\beta > 2$. Then

$$\log \left| \log \mathbb{P} \left\{ \sup_{t,s \in T} |X_t - X_s| \leq \varepsilon \right\} \right| \leq \varepsilon^{-\frac{2\gamma}{2-\gamma}} |\log \varepsilon|^{-\frac{2\beta}{2-\gamma}}, \quad \text{for } 0 < \gamma < 2,$$

and

$$\log \log \left| \log \mathbb{P} \left\{ \sup_{t,s \in T} |X_t - X_s| \leq \varepsilon \right\} \right| \leq \varepsilon^{-\frac{2}{\beta-2}}, \quad \text{for } \gamma = 2 \text{ and } \beta > 2.$$

Note that, due to the classical Dudley Theorem, the above theorem cannot be extended beyond $\gamma = 2$ and $\beta > 2$. Furthermore, it will become clear in Examples 3 and 4 that the above bound obtained from (11) cannot be improved by “uniform” chaining methods.

1.3 Stable case

Assume now that $X(t), t \in T$, is a symmetric α -stable process, $0 < \alpha < 2$, which means that $(X(t_1), \dots, X(t_n))$ is an n -dimensional symmetric α -stable vector for all choices $t_1, \dots, t_n \in T$, cf. [13]. We define the quasi-metric related to X by letting $\rho(s, t)$ denote the scale parameter of the stable real variable $X(t) - X(s)$; in other words,

$$\mathbb{E} \exp\{iu(X(t) - X(s))\} = \exp\{-\rho(t, s)^\alpha |u|^\alpha\}.$$

Alternatively, one could choose $(\mathbb{E}|X(t) - X(s)|^r)^{1/r}$ for any fixed positive $r < \alpha$ as a quasi-metric. We assume that, as in the Gaussian case, $\sigma := \text{diam}(T) < \infty$ and (T, ρ) is a relatively compact space. In what follows, $N(\varepsilon)$ are the covering numbers of the space (T, ρ) , as defined above.

An analogue of Talagrand's Theorem, i.e. our Theorem 1, for the stable non-Gaussian case was recently obtained by the first author in [2], where it is shown that the result remains true under the additional assumption $C_2 < 2^\alpha$. Recall (cf. e.g. [13], p. 546) that admitting $C_2 > 2^\alpha$ leads to processes which may even be not bounded with probability one. Hence there is no chance to prove Talagrand's bound for the non-Gaussian case with $C_2 > 2^\alpha$ in (2). The critical case $C_2 = 2^\alpha$ merits a special consideration. It is the case with "critically large" entropy, which will be handled below.

However, first, we show that Theorem 2 admits an extension to the stable case, too. Namely, the following is true.

Theorem 4 *Let $X(t), t \in T$, be a symmetric α -stable process, $0 < \alpha < 2$. Assume that the corresponding covering numbers satisfy $N(\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and let the bound Ψ be a non-increasing continuous function satisfying the regularity assumption (6) with some $1 < C_2 < 2^\alpha$. Then*

$$\log \mathbb{P} \left\{ \sup_{s, t \in T} |X(s) - X(t)| \leq K_0 \varepsilon \right\} \geq -K \tilde{\Psi}(\varepsilon), \quad \varepsilon > 0,$$

with a universal constant $K_0 > 0$ and a constant $K > 0$ depending only on α and C_2 , and where $\tilde{\Psi}$ is defined in (5).

The next theorem excludes again slow entropy behaviour but implicitly handles the critical case, i.e. large entropy behaviour. Let us denote

$$\hat{\Psi}(\varepsilon) = \int_0^\varepsilon \left(\frac{\Psi(u)}{u} \right)^{\frac{1}{\alpha+1}} du.$$

Theorem 5 *Let $X(t), t \in T$, be a symmetric α -stable process, $0 < \alpha < 2$. Assume that the corresponding covering numbers satisfy $N(\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and let the bound Ψ be a non-increasing continuous function satisfying the regularity assumption*

$$C_1 \Psi(\varepsilon) \leq \Psi\left(\frac{\varepsilon}{2}\right), \quad \sigma \geq \varepsilon > 0, \quad (12)$$

with some $C_1 > 1$. Then

$$\log \mathbb{P} \left\{ \sup_{s, t \in T} |X(s) - X(t)| \leq K_0 \varepsilon \right\} \geq -K \varepsilon^{-\alpha} \hat{\Psi}(\varepsilon)^{\alpha+1}, \quad \varepsilon > 0,$$

with a universal constant $K_0 > 0$ and a constant $K > 0$ depending only on α and C_1 .

This theorem also provides a new sufficient condition for the boundedness of stable processes.

Corollary 6 *Let $X(t)$, $t \in T$, be a symmetric α -stable process, $0 < \alpha < 2$. Assume that the corresponding covering numbers satisfy $N(\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and let the bound Ψ be a non-increasing continuous function satisfying the regularity assumption (12). If*

$$\int_0^\sigma \left(\frac{\Psi(u)}{u} \right)^{\frac{1}{\alpha+1}} du < \infty,$$

then the process X is a.s. bounded.

Recall that for $0 < \alpha < 1$ no sufficient condition for a.s. boundedness of stable processes in terms of metric entropy had been available so far. When $1 \leq \alpha < 2$, Theorem 12.2.1 in [13] provides a sufficient condition, which is better than our Corollary 6, because the integral test is slightly weaker and no regularity assumption is required.

We can even go beyond the last theorem in the case $N(\varepsilon) \leq \Psi(\varepsilon)$ with $\Psi(\varepsilon) := C\varepsilon^{-\alpha}|\log \varepsilon|^{-\beta}$ with $\beta > 0$. Note that Theorem 5 only works for $\beta > 1 + \alpha$.

Theorem 7 *Let $N(\varepsilon) \leq C\varepsilon^{-\alpha}|\log \varepsilon|^{-\beta}$ for $\varepsilon < \sigma$. Then*

$$\log \mathbb{P} \left\{ \sup_{s,t \in T} |X(s) - X(t)| \leq \varepsilon \right\} \geq \begin{cases} -K\varepsilon^{-1/(\beta/\alpha-1)} & \max(1, \alpha) < \beta < 1 + \alpha, \\ -K\varepsilon^{-\alpha} |\log \varepsilon|^{1+\alpha} & \beta = 1 + \alpha, \\ -K\varepsilon^{-\alpha} |\log \varepsilon|^{-\beta+1+\alpha} & \beta > 1 + \alpha. \end{cases}$$

We will show below that these estimates cannot be improved in general by the chaining method. In particular, for $\beta \leq \max(1, \alpha)$ no estimate can be obtained by uniform chaining. It would be interesting to ask what can be done for stable processes using majorizing measure/generic chaining techniques.

Remark 8 *Similarly to Corollary 6, we have that $N(\varepsilon) \leq C\varepsilon^{-\alpha}|\log \varepsilon|^{-\beta}$ with $\beta > \max(1, \alpha)$ implies the a.s. boundedness of the process. Note that this recovers Dudley's Theorem (Theorem 12.2.1 in [13]) for $\alpha \geq 1$ and provides a new Dudley-type theorem for $0 < \alpha < 1$.*

1.4 Some examples

In the below examples we use, for simplicity, the term symmetric α -stable for both, the Gaussian ($\alpha = 2$) and the non-Gaussian ($0 < \alpha < 2$) case.

We start with an example that shows that Theorem 1 does not return the correct bound for slowly varying Ψ .

Example 1 (Logarithmic behaviour of entropy). Let $t_n := 2^{-n^{1/\beta}}$ with some $\beta > 0$ and let M be an independently scattered symmetric α -stable random measure on $[0, 1]$ controlled by the Lebesgue measure. We consider the process

$$X_n := M([0, t_n]), \quad n \geq 1.$$

It is easy to calculate that $N(\varepsilon) \leq C|\log \varepsilon|^\beta$.

As an example, let us consider $\beta = 1$. Note that, if Theorem 1 were applicable, it would lead to the estimate

$$\mathbb{P} \left\{ \sup_{n, m \geq 1} |X_n - X_m| \leq \varepsilon \right\} \geq C\varepsilon^K,$$

for some $K, C > 0$, which is absurd. Instead, we get

$$\log \mathbb{P} \left\{ \sup_{n, m \geq 1} |X_n - X_m| \leq K_0\varepsilon \right\} \geq -K|\log \varepsilon|^2,$$

by Theorem 2 in the Gaussian and Theorem 4 in the symmetric stable case, which in fact happens to be the correct order.

Analogous arguments give rise to the small deviation behaviour as stated in (9). Similar examples (and counterexamples) can be also obtained by using weighted sums of independent sequences that are described in Example 2 below. \square

Now we come to the most simple form of symmetric α -stable processes, namely, sequences of independent random variables. We investigate what can be said about the small deviations of such sequences in the case of large entropy behaviour.

Example 2 (Sequence of independent variables). Let us consider the stochastic process $X = (\sigma_n \xi_n)_{n \geq 1}$, where ξ_n are i.i.d. standard symmetric α -stable random variables.

In the Gaussian case, consider $\sigma_n \sim (c \log n)^{1/\gamma} (\log \log n)^{-\beta/\gamma}$. Then (3) holds. Theorem 3 only applies for $\gamma = 2$ and $\beta > 2$, whereas the problem makes sense even for $\gamma = 2$, $\beta = 0$, and $c > 2$.

In the stable case, the critical situation is obtained when considering $\sigma_n \sim n^{-1/\alpha}(\log n)^{-\beta/\alpha}$ with $\beta > 1$. It is easy to verify that $N(\varepsilon) \approx \varepsilon^{-\alpha} |\log \varepsilon|^{-\beta}$ and

$$\log \mathbb{P} \left\{ \sup_n |\sigma_n \xi_n| \leq \varepsilon \right\} \approx -\varepsilon^{-\alpha} |\log \varepsilon|^{1-\beta}, \quad \text{for all } \beta > 1, \quad (13)$$

cf. [1], Section 4.6. Our Theorem 7 gives weaker results in all cases. In particular, it only works for $\beta > \max(1, \alpha)$. \square

Let us now come to the crucial “sum of maxima” example, that – as already mentioned in the introduction – gains its importance as a prototype arising from the chaining estimate.

Example 3 (Sum of maxima). Let $\sigma_n > 0$ and let $N_k \geq 1$ be some integers. Let $(\xi_{k,i}), k, i \geq 1$, be an array of i.i.d. standard symmetric α -stable random variables. Let $T = \{(\ell, s) \in \mathbb{N}^\infty \times \{-1, +1\}^\infty : \ell_k \leq N_k, \forall k \geq 1\}$ and set

$$X(\ell, s) = \sum_{k=1}^{\infty} \sigma_k s_k \xi_{k, \ell_k}, \quad (\ell, s) \in T.$$

Note that $X(\ell, s)$ is a symmetric α -stable random variable with scale parameter $(\sum_k \sigma_k^\alpha)^{1/\alpha}$. Then

$$S = \sup_{(\ell, s) \in T} |X(\ell, s)| = \sum_{k=1}^{\infty} \sigma_k \max_{i=1, \dots, N_k} |\xi_{k,i}|. \quad (14)$$

Even if $N_k = 1$, we have a nontrivial example of an ℓ_1 -norm,

$$\sup_{(\ell, s) \in T} |X(\ell, s)| = \sum_{k=1}^{\infty} \sigma_k |\xi_{k,1}|. \quad (15)$$

Certain important cases of the “simplified” version (15) were studied in [1]. We recall only one particular case showing that “simplified” is not obvious at all. Let ξ be Gaussian and $\sigma_k = k^{-1}(\log k)^{-b}$; then X is bounded for $b > 1$ and

$$\log \left| \log \mathbb{P} \left\{ \sup_{t \in T} |X(t)| \leq \varepsilon \right\} \right| \approx \varepsilon^{-\frac{1}{b-1}}, \quad (16)$$

while the entropy satisfies $\log N(\varepsilon) \approx \varepsilon^{-2} |\log \varepsilon|^{-2b}$ and thus approaches the famous Dudley-Sudakov border between the bounded and unbounded processes. Our Theorem 3 returns the correct lower bound for (16).

As explained in the introduction, this kind of examples provides a sharp power test for the chaining method in the small deviation problem.

In the Gaussian case, we obtain the following.

Proposition 9 *Let S be the sum defined in (14) with $N_k = e^{2^{\gamma k} k^{-\beta}}$ and $\sigma_k = 2^{-k}$ with some $0 < \gamma \leq 2$. Then the order given in Theorem 3 is attained for*

$0 < \gamma < 2$ or $\gamma = 2$ and $\beta > 2$, respectively. For $\gamma = 2$ and $\beta \leq 2$, the process is a.s. unbounded.

Although formally our theorems cannot be applied here, the considerations in the introduction show that Proposition 9 yields the optimality of our theorems in the sense that classical “uniform” chaining estimates cannot lead to better estimates.

For the non-Gaussian stable case, we can get the following analog in the respective critical situation.

Proposition 10 *Let S be the sum defined in (14) with $\sigma_k = 2^{-k/\alpha} k^{-\beta/\alpha}$ and $N_k = 2^k$. Then $S < \infty$ a.s. if and only if $\beta > \max(1, \alpha)$ and we have*

$$\log \mathbb{P} \{S \leq \varepsilon\} \approx \begin{cases} -\varepsilon^{-1/(\beta/\alpha-1)} & \max(1, \alpha) < \beta < 1 + \alpha, \\ -\varepsilon^{-\alpha} |\log \varepsilon|^{1+\alpha} & \beta = 1 + \alpha, \\ -\varepsilon^{-\alpha} |\log \varepsilon|^{-\beta+1+\alpha} & \beta > 1 + \alpha. \end{cases}$$

□

Finally, let us consider an example that seems to be closely related to Example 3 and may be important in other circumstances.

Example 4 (Binary tree). Let us take an infinite binary tree and associate a standard symmetric α -stable random variable ξ_a to every edge a of this tree, where we assume all random variables to be independent. Let $|a| \geq 1$ denote the level number of an edge a . Let T be the set of all finite branches starting from the root of the tree. Furthermore, we take a non-increasing sequence of positive numbers (σ_n) and consider

$$X(t) = \sum_{a \in t} \sigma_{|a|} \xi_a, \quad t \in T.$$

Then $X(t)$ is a symmetric α -stable random variable with scale parameter $(\sum_{n \leq |t|} \sigma_n^\alpha)^{1/\alpha}$, for all $t \in T$, where $|t|$ is the length of the branch.

It is easy to see that this case partially resembles the previous example if we set there $N_n = 2^n$, although the dependence structures of the two processes are substantially different. We have the obvious majoration

$$\sup_{t \in T} |X(t)| \leq \sum_{n=1}^{\infty} \sigma_n \max_{\{a: |a|=n\}} |\xi_a|. \quad (17)$$

In the Gaussian case, let us consider the following exemplary situation.

Proposition 11 *Let X be the binary tree constructed above with standard normal i.i.d. ξ_a .*

(a) *Let $\sigma_n = 2^{-n/\gamma} n^{-\beta/\gamma}$ with $\gamma > 0$ and $\beta \in \mathbb{R}$. Then*

$$-\log \mathbb{P} \left\{ \sup_{t \in T} |X(t)| \leq \varepsilon \right\} \approx \varepsilon^{-\gamma} |\log \varepsilon|^{-\beta}.$$

(b) *Let $\sigma_n = n^{-1/2-1/\gamma} (\log n)^{-\beta/\gamma}$ with $0 < \gamma < 2$ and $\beta \in \mathbb{R}$. Then*

$$\log \left| \log \mathbb{P} \left\{ \sup_{t \in T} |X(t)| \leq \varepsilon \right\} \right| \approx \varepsilon^{-\frac{2\gamma}{2-\gamma}} |\log \varepsilon|^{-\frac{2\beta}{2-\gamma}}.$$

Assertion (b) shows that Theorem 3 cannot be improved. Note furthermore that we do not have the bounds corresponding to the case $\gamma = 2$, $\beta > 2$. So, there is a gap in the results in this case. In fact, it is not clear for which β the binary tree process is actually bounded when $\gamma = 2$; we only know from (17) and Proposition 9 that $\beta > 2$ is sufficient.

For the non-Gaussian stable case, let, in particular, $\sigma_n \sim 2^{-n/\gamma} n^{-\beta/\gamma}$ for some $\gamma > 0$, $\beta \in \mathbb{R}$. Then $N(\varepsilon) \approx \varepsilon^{-\gamma} |\log \varepsilon|^{-\beta}$. In this case, the following holds.

Proposition 12 *Let X be the binary tree constructed above with standard symmetric α -stable i.i.d. ξ_a . Let $\sigma_n = 2^{-n/\gamma} n^{-\beta/\gamma}$ with $\gamma > \alpha$ and $\beta \in \mathbb{R}$. Then*

$$-\log \mathbb{P} \left\{ \sup_{t \in T} |X(t)| \leq \varepsilon \right\} \approx \varepsilon^{-\gamma} |\log \varepsilon|^{-\beta}.$$

However, the most challenging is the stable non-Gaussian case with $\gamma = \alpha$. In view of (17), Proposition 10 provides the lower bounds for small deviation probabilities of X whenever $\beta > \max(1, \alpha)$. On the other hand, it is easy to show, by considering the oscillations on each separate level, that X is not bounded when $\beta \leq 1$. Note that, for $\alpha < 1$, the process is bounded if and only if $\beta > 1$, by Theorem 10.4.2 in [13]. Observing that

$$\mathbb{P} \left\{ \sup_{t \in T} |X(t)| \leq \varepsilon/2 \right\} \leq \mathbb{P} \left\{ \max_n \sigma_n \sup_{\{a: |a|=n\}} |\xi_a| \leq \varepsilon \right\}$$

it is easy to show that for any β

$$\log \mathbb{P} \left\{ \sup_{s, t \in T} |X(s) - X(t)| \leq \varepsilon \right\} \leq -K \varepsilon^{-\alpha} |\log \varepsilon|^{1-\beta}.$$

There is a gap between this bound and those coming from Proposition 10. Moreover, we even do not know whether $\beta \in (1, \alpha]$ corresponds to a bounded process X .

Therefore, many interesting questions related to this example remain open. We omit the proofs of Propositions 11 and 12 here; these are available from the authors upon request or can be found at <http://arxiv.org/abs/0706.2720> (see Section 4.4 there).

2 The chaining technique

This section is devoted to the basic Dudley-Talagrand chaining argument. For the reader's convenience we shall re-prove it as a separate statement. Following this, we prove a chaining statement for the corresponding Laplace transform, which turns out to be slightly stronger. However, returning from the Laplace transform to the small deviation probability via Tauberian-type theorems is only possible for regularly varying cases.

These chaining inequalities form the main ingredient of our results. The proofs of our main theorems rely on the following lemmas, appropriate optimization of the parameters in case Lemma 13 is used and appropriate estimates of the involved Laplace transforms if we use Lemma 14.

Lemma 13 *Let $(\varepsilon_k)_{k \geq 0}$ be a decreasing sequence tending to zero such that $\varepsilon_0 \geq \sigma$. Let $(b_k)_{k \geq 0}$ be an arbitrary positive sequence. Set $b = \sum_{k=0}^{\infty} b_k$. Then*

$$\mathbb{P} \left\{ \sup_{s,t \in T} |X(s) - X(t)| \leq 2b \right\} \geq \prod_{k=0}^{\infty} \mathbb{P} \{ \varepsilon_k |\xi| \leq b_k \}^{N(\varepsilon_{k+1})} \quad (18)$$

where ξ is a standard normal random variable.

Proof. For any $k \geq 0$, let T_k be a minimal ε_k -net in T . Recall that $|T_k| = N(\varepsilon_k)$. In particular, $|T_0| = N(\varepsilon_0) = 1$, since $\varepsilon_0 \geq \sigma$.

Since T_0 consists of a single element, we have

$$\sup_{s,t \in T_0} |X(s) - X(t)| = 0,$$

which provides the induction base. Now we come to the chaining induction step. For any $k \geq 0$, let $\pi_k : T_{k+1} \rightarrow T_k$ be a mapping that satisfies

$$\max_{t \in T_{k+1}} \rho(t, \pi_k(t)) \leq \varepsilon_k.$$

Such a mapping exists by the definition of T_k . Then we have the chaining inequality: for all $s, t \in T_{k+1}$

$$\begin{aligned} |X(s) - X(t)| &\leq |X(s) - X(\pi_k(s))| + |X(\pi_k(s)) - X(\pi_k(t))| \\ &\quad + |X(\pi_k(t)) - X(t)|. \end{aligned}$$

Hence,

$$\sup_{s,t \in T_{k+1}} |X(s) - X(t)| \leq 2 \sup_{s \in T_{k+1}} |X(s) - X(\pi_k(s))| + \sup_{s,t \in T_k} |X(s) - X(t)|.$$

By induction, we obtain for any $n \geq 0$,

$$\sup_{s,t \in T_{n+1}} |X(s) - X(t)| \leq 2 \sum_{k=0}^n \sup_{s \in T_{k+1}} |X(s) - X(\pi_k(s))|. \quad (19)$$

Hence, the probability

$$P_n := \mathbb{P} \left\{ \sup_{s,t \in T_{n+1}} |X(s) - X(t)| \leq 2b \right\}$$

satisfies

$$P_n \geq \mathbb{P} \{ |X(s) - X(\pi_k(s))| \leq b_k, \quad \forall s \in T_{k+1}, 0 \leq k \leq n \}.$$

By using Khatri-Šidák Inequality (see e.g. [4, p. 260]) and the main property of the mappings π_k , we get

$$P_n \geq \prod_{k=0}^n \prod_{s \in T_{k+1}} \mathbb{P} \{ |X(s) - X(\pi_k(s))| \leq b_k \} \geq \prod_{k=0}^n \mathbb{P} \{ \varepsilon_k |\xi| \leq b_k \}^{N(\varepsilon_{k+1})}.$$

Now the assertion follows by a separability argument. \square

Now let us obtain an analog of the chaining lemma, for the corresponding Laplace transform. Recall that it is well-known and has been used at many occasions that considering small deviations of a random variable and the Laplace transform at infinity is equivalent, by the use of Tauberian-type theorems. However, it will turn out that the use of the Laplace transform is technically easier and thus more powerful in a certain sense. In particular, it can be avoided to choose the sequence (b_k) , which appears when passing from (19) to deterministic bounds, which is a somewhat unnecessary step in our context.

Lemma 14 *Let $(\varepsilon_k)_{k \geq 0}$ be a decreasing sequence tending to zero such that $\varepsilon_0 \geq \sigma$. Then*

$$\mathbb{E} \exp \left\{ -\lambda \sup_{t,s \in T} |X(t) - X(s)| \right\} \geq \prod_{k=0}^{\infty} \int_0^{\infty} e^{-y} \mathbb{P} \{ 2\lambda \varepsilon_k |\xi| \leq y \}^{N(\varepsilon_{k+1})} dy. \quad (20)$$

Proof. By the chaining arguments in the proof of Lemma 13, we obtain (19). This shows that

$$\mathbb{E} e^{-\lambda \sup_{t,s \in T_n} |X(t) - X(s)|} \geq \mathbb{E} e^{-2\lambda \sum_{k=0}^n \sup_{s \in T_{k+1}} |X(s) - X(\pi_k(s))|}.$$

By separability, the left-hand side tends to the Laplace transform we wish to evaluate. The right-hand side can be written as

$$\int_{\mathbb{R}^{n+1}} e^{-\sum_{k=0}^n y_k} \mathbb{P} \left\{ 2\lambda \sup_{s \in T_{k+1}} |X(s) - X(\pi_k(s))| \leq y_k, \forall k \right\} d(y_0, \dots, y_n).$$

By the Khatri-Šidák Inequality, this is greater or equal to

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} e^{-\sum_{k=0}^n y_k} \prod_{k=0}^n \prod_{s \in T_{k+1}} \mathbb{P} \{ 2\lambda |X(s) - X(\pi_k(s))| \leq y_k \} d(y_0, \dots, y_n) \\ = \prod_{k=0}^n \int_{\mathbb{R}} e^{-y_k} \prod_{s \in T_{k+1}} \mathbb{P} \{ 2\lambda |X(s) - X(\pi_k(s))| \leq y_k \} dy_k. \end{aligned}$$

Note that this is greater or equal to

$$\prod_{k=0}^n \int_{\mathbb{R}} e^{-y} \prod_{s \in T_{k+1}} \mathbb{P} \{ 2\lambda \varepsilon_k |\xi| \leq y \} dy = \prod_{k=0}^n \int_{\mathbb{R}} e^{-y} \mathbb{P} \{ 2\lambda \varepsilon_k |\xi| \leq y \}^{N(\varepsilon_{k+1})} dy,$$

as required in (20). \square

Remark 15 *Let us make an important remark about a slightly more general chaining construction. Our calculations still work if we have, similarly to (19),*

$$\sup_{s, t \in T} |X(s) - X(t)| \leq C \sum_{k=0}^{\infty} \varepsilon_k \max_{i=1, \dots, N_{k+1}} |\xi'_{k,i}|$$

for some, possibly dependent, standard Gaussian (or, according to the context, symmetric stable) variables $\xi'_{k,i}$. In this approach, the N_n are not necessarily covering numbers. This observation will be particularly useful when considering the tree-based examples.

3 Proofs for the cases with small entropy

We now assume that the covering numbers admit a reasonable majorant Ψ and construct, under mildest possible assumptions on Ψ , the appropriate lower bounds for the products appearing in Lemma 13.

We first show that under (6) the layers with small ε_k never bring anything really different from Talagrand's bound.

Lemma 16 *Assume that Ψ satisfies (6) for $\varepsilon \leq \varepsilon_0$. Then, for any $\varepsilon \in (0, \varepsilon_0)$*

and any $r \in (\frac{1}{2}, 1)$ it is true that

$$\prod_{k=0}^{\infty} \mathbb{P} \left\{ 2^{-k} \varepsilon |\xi| \leq r^k \varepsilon \right\}^{N(2^{-k-1} \varepsilon)} \geq \exp \{ -C_3(r) \Psi(\varepsilon) \}, \quad (21)$$

where $C_3(r)$ depends only on C_2 and r .

Proof. Since $N(2^{-k-1} \varepsilon) \leq \Psi(2^{-k-1} \varepsilon) \leq C_2^{k+1} \Psi(\varepsilon)$, we have

$$\prod_{k=0}^{\infty} \mathbb{P} \left\{ 2^{-k} \varepsilon |\xi| \leq r^k \varepsilon \right\}^{N(2^{-k-1} \varepsilon)} \geq \prod_{k=0}^{\infty} \left(1 - \mathbb{P} \left\{ |\xi| > (2r)^k \right\} \right)^{C_2^{k+1} \Psi(\varepsilon)}. \quad (22)$$

Since $r > \frac{1}{2}$ and $k \geq 0$, we have $\mathbb{P} \left\{ |\xi| > (2r)^k \right\} \leq \mathbb{P} \left\{ |\xi| > 1 \right\}$, hence, by using the standard Gaussian tail estimate, we get for some numerical constant A ,

$$1 - \mathbb{P} \left\{ |\xi| > (2r)^k \right\} \geq \exp \left\{ -A \exp \left\{ -\frac{1}{2} (2r)^{2k} \right\} \right\}. \quad (23)$$

It follows that the expression in (22) is greater or equal to

$$\exp \left\{ -A \sum_{k=0}^{\infty} \exp \left\{ -\frac{1}{2} (2r)^{2k} \right\} C_2^{k+1} \Psi(\varepsilon) \right\} =: \exp \{ -C_3(r) \Psi(\varepsilon) \},$$

where the sum converges since $r > 1/2$. \square

We pass now to the evaluation of the product over the relatively large levels (small k). Let $r \in (0, 1)$ and fix any $\varepsilon > 0$. Let (ε_k) , $0 \leq k \leq n$, be a decreasing positive sequence such that $\varepsilon_n = \varepsilon$ and

$$\Psi(\varepsilon_k) \leq r \Psi(\varepsilon_{k+1}), \quad 1 \leq k \leq n-1. \quad (24)$$

Furthermore, we set $b_k = r^{n-k} \varepsilon$, for $0 \leq k < n$.

Lemma 17 *With notation introduced above and under assumption (24) we have*

$$\prod_{k=0}^{n-1} \mathbb{P} \left\{ \varepsilon_k |\xi| \leq b_k \right\}^{N(\varepsilon_{k+1})} \geq \exp \left\{ -C_4(r) \Psi(\varepsilon) - (1-r)^{-1} G \right\}, \quad (25)$$

where $C_4(r)$ depends only on r , and $G = \sum_{\ell=1}^n (\log \varepsilon_{\ell-1} - \log \varepsilon_{\ell}) \Psi(\varepsilon_{\ell})$.

Proof. Since for any $k < n$ we have

$$\frac{b_k}{\varepsilon_k} = r^{n-k} \frac{\varepsilon}{\varepsilon_k} = r^{n-k} \frac{\varepsilon_n}{\varepsilon_k} \leq 1,$$

it is true that

$$\mathbb{P} \left\{ \varepsilon_k |\xi| \leq b_k \right\} \geq c \frac{b_k}{\varepsilon_k} = c r^{n-k} \frac{\varepsilon_n}{\varepsilon_k}, \quad (26)$$

where $c = (2\pi e)^{-1/2}$ is a numerical constant. On the other hand, it follows from (24) that

$$\Psi(\varepsilon_k) \leq r^{l-k} \Psi(\varepsilon_\ell), \quad 1 \leq k \leq \ell, \quad (27)$$

in particular,

$$\Psi(\varepsilon_k) \leq r^{n-k} \Psi(\varepsilon_n) = r^{n-k} \Psi(\varepsilon), \quad 1 \leq k \leq n. \quad (28)$$

Therefore,

$$\prod_{k=0}^{n-1} \mathbb{P} \{ \varepsilon_k | \xi | \leq b_k \}^{N(\varepsilon_{k+1})} \geq \prod_{k=0}^{n-1} \left(c r^{n-k} \frac{\varepsilon_n}{\varepsilon_k} \right)^{\Psi(\varepsilon_{k+1})} = \Pi_1 \Pi_2,$$

where

$$\Pi_1 := \prod_{k=0}^{n-1} \left(c r^{n-k} \right)^{\Psi(\varepsilon_{k+1})} \quad \text{and} \quad \Pi_2 := \prod_{k=0}^{n-1} \left(\frac{\varepsilon_n}{\varepsilon_k} \right)^{\Psi(\varepsilon_{k+1})}.$$

Using (28), we have

$$\begin{aligned} |\log \Pi_1| &\leq \sum_{k=0}^{n-1} (|\log c| + |\log r|(n-k)) r^{n-k-1} \Psi(\varepsilon) \\ &\leq \sum_{\ell=1}^{\infty} (|\log c| + |\log r|\ell) r^{\ell-1} \Psi(\varepsilon) =: C_4(r) \Psi(\varepsilon). \end{aligned}$$

Similarly, by using (27) and (28), we have

$$\begin{aligned} |\log \Pi_2| &\leq \sum_{k=0}^{n-1} \sum_{\ell=k+1}^n (\log \varepsilon_{\ell-1} / \varepsilon_\ell) \Psi(\varepsilon_{k+1}) \\ &\leq \sum_{\ell=1}^n (\log \varepsilon_{\ell-1} / \varepsilon_\ell) \sum_{k=0}^{\ell-1} r^{\ell-k-1} \Psi(\varepsilon_\ell) \leq (1-r)^{-1} \sum_{\ell=1}^n (\log \varepsilon_{\ell-1} / \varepsilon_\ell) \Psi(\varepsilon_\ell), \end{aligned}$$

as claimed above. \square

Proof of Theorem 2. Let us fix $r \in (1/2, 1)$. W.l.o.g. $\Psi(\sigma/2) > \Psi(\sigma)$. Therefore, for any $\varepsilon \leq \sigma/2$, we can choose $n = n(\varepsilon) \geq 1$ such that

$$r^{n-1} \Psi(\varepsilon) > \Psi(\sigma) \geq r^n \Psi(\varepsilon).$$

We choose now the first layer by letting $\varepsilon_0 = \sigma$, and the following n layers from equation

$$\Psi(\varepsilon_\ell) = r^{n-\ell} \Psi(\varepsilon), \quad 1 \leq \ell \leq n.$$

In particular, we can choose $\varepsilon_n = \varepsilon$. The choice of ε_ℓ is possible, since the function $\Psi(\cdot)$ is continuous and

$$\Psi(\varepsilon) \geq r^{n-\ell} \Psi(\varepsilon) \geq \Psi(\sigma).$$

Since $\Psi(\cdot)$ is non-increasing, the sequence $(\varepsilon_\ell)_{0 \leq \ell \leq n}$ is non-increasing as well.

We put $b_\ell = r^{n-\ell}\varepsilon$, $0 \leq \ell < n$, and apply Lemma 17. Note that (24) is automatically satisfied by the construction of the ε_k . Notice furthermore that for any $1 \leq \ell \leq n$ we have

$$\Psi(\varepsilon_\ell) \leq r^{-1}\Psi(\varepsilon_{\ell-1})$$

with equality for $2 \leq \ell \leq n$. It follows that

$$(\log \varepsilon_{\ell-1} - \log \varepsilon_\ell) \Psi(\varepsilon_\ell) \leq r^{-1} \int_{\varepsilon_\ell}^{\varepsilon_{\ell-1}} \frac{du}{u} \Psi(\varepsilon_{\ell-1}) \leq r^{-1} \int_{\varepsilon_\ell}^{\varepsilon_{\ell-1}} \frac{\Psi(u)}{u} du.$$

By summing over ℓ we get

$$G = \sum_{\ell=1}^n (\log \varepsilon_{\ell-1} - \log \varepsilon_\ell) \Psi(\varepsilon_\ell) \leq r^{-1} \sum_{\ell=1}^n \int_{\varepsilon_\ell}^{\varepsilon_{\ell-1}} \frac{\Psi(u)}{u} du = r^{-1} \int_\varepsilon^\sigma \frac{\Psi(u)}{u} du,$$

whenever $\varepsilon \leq \sigma/2$. We obtain from (25)

$$\prod_{k=0}^{n-1} \mathbb{P} \{ \varepsilon_k | \xi | \leq b_k \}^{N(\varepsilon_{k+1})} \geq \exp \left\{ -C_4(r) \Psi(\varepsilon) - \frac{r^{-1}}{1-r} \int_\varepsilon^\sigma \frac{\Psi(u)}{u} du \right\}. \quad (29)$$

We finish the construction by letting $\varepsilon_{n+k} = 2^{-k}\varepsilon$ and $b_{n+k} = r^k\varepsilon$ for all positive integers k . By using (21) we obtain

$$\prod_{k=0}^{\infty} \mathbb{P} \{ \varepsilon_{n+k} | \xi | \leq b_{n+k} \}^{N(\varepsilon_{n+k+1})} \geq \exp \{ -C_3(r) \Psi(\varepsilon) \}. \quad (30)$$

By plugging (29) and (30) into (18) and letting $K_0 := 4 \sum_{k=0}^{\infty} r^k = 4(1-r)^{-1}$ we obtain for

$$P := \mathbb{P} \left\{ \sup_{s,t \in T} |X(s) - X(t)| \leq K_0 \varepsilon \right\}$$

that

$$P \geq \exp \left\{ -[C_4(r) + C_3(r)] \Psi(\varepsilon) - (1-r)^{-1} r^{-1} \int_\varepsilon^\sigma \frac{\Psi(u)}{u} du \right\}. \quad (31)$$

Finally, consider three cases:

a) $0 < \varepsilon \leq \sigma/2$. Then (10) and (31) yield

$$P \geq \exp \left\{ -[(C_4(r) + C_3(r)) \frac{C_2}{\log 2} + (1-r)^{-1} r^{-1}] \int_\varepsilon^\sigma \frac{\Psi(u)}{u} du \right\} =: e^{-C_5 \tilde{\Psi}(\varepsilon)}.$$

b) $\sigma/2 < \varepsilon \leq \sigma$. Then

$$\int_\varepsilon^\sigma \frac{\Psi(u)}{u} du \leq \frac{\Psi(\varepsilon)}{\varepsilon} \cdot \frac{\sigma}{2} \leq \Psi(\varepsilon),$$

and hence

$$P \geq \exp \left\{ -[C_4(r) + C_3(r) + (1-r)^{-1}r^{-1}] \Psi(\varepsilon) \right\} =: \exp \left\{ -C_6 \tilde{\Psi}(\varepsilon) \right\}.$$

c) $\varepsilon \geq \sigma/2$. In this case estimate (21) alone yields $\log P \geq -C_3(r) \Psi(\varepsilon) = -C_3(r) \tilde{\Psi}(\varepsilon)$.

We choose $K := \max\{C_3, C_5, C_6\}$ and obtain in all cases $\log P \geq -K \tilde{\Psi}(\varepsilon)$, as required. \square

Proof of Theorem 4. We only indicate here the necessary changes in the proof with respect to the Gaussian case.

The first point is the use of the Khatri-Šidák Inequality used in the chaining argument. As mentioned in the introduction, this is possible, by Lemma 2.1 in [12]. By using this lemma, it was shown in fact in [2] (following some ideas of [8]) that the chaining inequality (18) is still true with the natural replacement of a standard normal random variable ξ by a standard symmetric α -stable random variable.

The second important modification concerns the place where the tail probabilities come into play. Namely, in Lemma 16 we must assume that $C_2^{1/\alpha}/2 < r < 1$ (recall that $C_2^{1/\alpha}/2 < 1$ by our theorem's assumption). Instead of (23) we have

$$1 - \mathbb{P} \left\{ |\xi| > (2r)^k \right\} \geq \exp \left\{ -A (2r)^{-\alpha k} \right\},$$

where we use the stable tail behaviour:

$$\mathbb{P} \left\{ |\xi| \leq r \right\} \geq \exp \left\{ -Ar^{-\alpha} \right\}, \quad r > 0, \quad (32)$$

with some finite positive A . Hence this time

$$\prod_{k=0}^{\infty} \mathbb{P} \left\{ 2^{-k} \varepsilon |\xi| \leq r^k \varepsilon \right\}^{N(2^{-k-1} \varepsilon)} \geq \exp \left\{ -C_3 \Psi(\varepsilon) \right\}$$

where

$$C_3 := A \sum_{k=0}^{\infty} (2r)^{-\alpha k} C_2^{k+1} = \frac{A C_2}{1 - (2r)^{-\alpha} C_2}$$

is finite since $r > C_2^{1/\alpha}/2$.

The third point to take care of concerns the density bound used in (26). Just note that the density of a standard non-Gaussian symmetric stable variable is positive and bounded away from zero in any neighborhood of the origin. However, the numerical constant c in (26) has to be replaced appropriately. All other arguments given earlier are valid in the non-Gaussian case, too. \square

4 Gaussian case with critically large entropy

4.1 Technical lemmas

In the following, it will turn out that we have to use a Tauberian-type theorem for the Laplace transform that does not seem to be in the literature. The proof is based on, essentially, exponential Chebyshev Inequality and a similar estimate. It is in the same spirit as the one for the so-called de Bruijn Tauberian Theorem, i.e. Theorem 4.12.9 in [3], and will therefore be omitted.

Lemma 18 *Let V be a positive random variable. For $\tau > 0$ and $\theta \in \mathbb{R}$ the following relations are equivalent*

$$\begin{aligned} \log \mathbb{E}e^{-\lambda V} &\approx -\lambda(\log \lambda)^{-\tau}(\log \log \lambda)^\theta, & \lambda \rightarrow \infty, \\ \log |\log \mathbb{P}\{V \leq \varepsilon\}| &\approx \varepsilon^{-1/\tau} |\log \varepsilon|^{\theta/\tau}, & \varepsilon \rightarrow 0. \end{aligned}$$

Furthermore, let $\theta > 0$. Then the following relations are equivalent

$$\begin{aligned} \log \mathbb{E}e^{-\lambda V} &\approx -\lambda(\log \log \lambda)^{-\theta}, & \lambda \rightarrow \infty, \\ \log \log |\log \mathbb{P}\{V \leq \varepsilon\}| &\approx \varepsilon^{-1/\theta}, & \varepsilon \rightarrow 0. \end{aligned}$$

In all statements, the upper (lower) bounds in the assumptions imply lower (upper) bounds in the respective assertions.

One of the major ingredients of the proofs for the case of critically large entropy is the evaluation of the Laplace transform of the random variable $\max_{i=1, \dots, N} |\xi_i|$, where ξ_1, ξ_2, \dots are i.i.d. standard Gaussian random variables. We start with the case that the argument of the Laplace transform, L , is of lower order than N .

Lemma 19 *Let ξ_1, ξ_2, \dots be i.i.d. standard Gaussian r.v. Then there is a constant $c_1 > 0$ such that for all $L > 0$ and all integers $N \geq 1$ with $2L \leq N$ we have*

$$-\log \mathbb{E}e^{-L \max_{i=1, \dots, N} |\xi_i|} \leq c_1 L \sqrt{\log(N/L)}.$$

Additionally, there is a constant $c_2 > 0$ such that for all $L \geq 1$ and all integers $N \geq 1$ with $2L \leq N$ we have

$$-\log \mathbb{E}e^{-L \max_{i=1, \dots, N} |\xi_i|} \geq c_2 L \sqrt{\log(N/L)}.$$

Proof. In order to get the first part, note that

$$\begin{aligned}\mathbb{E}e^{-L \max_{i=1, \dots, N} |\xi_i|} &\geq \int_{L\sqrt{2\log(N/L)}}^{\infty} e^{-y} dy \cdot \mathbb{P}\left\{|\xi| \leq \sqrt{2\log(N/L)}\right\}^N \\ &\geq e^{-L\sqrt{2\log(N/L)}} e^{-C_1 N \mathbb{P}\left\{|\xi| > \sqrt{2\log(N/L)}\right\}} \geq e^{-C_3 L \sqrt{2\log(N/L)}},\end{aligned}$$

where we used the assumption $N \geq 2L$ and the Gaussian tail.

For the reverse inequality note first that

$$\begin{aligned}\int_0^{\infty} e^{-y} \mathbb{P}\{L|\xi| \leq y\}^N dy \\ \leq \int_0^{L\sqrt{2\log(N/(2L))}} e^{-y} \mathbb{P}\{L|\xi| \leq y\}^N dy + \int_{L\sqrt{2\log(N/(2L))}}^{\infty} e^{-y} dy.\end{aligned}$$

Here, the second term already admits the required estimate. In order to treat the first term, consider the function

$$f(y) := e^{-y} \mathbb{P}\{L|\xi| \leq y\}^N, \quad y \in \left[0, L\sqrt{2\log(N/(2L))}\right].$$

One can check that f is increasing and thus

$$\begin{aligned}\int_0^{L\sqrt{2\log(2N/L)}} e^{-y} \mathbb{P}\{L|\xi| \leq y\}^N dy \\ \leq \int_0^{L\sqrt{2\log(2N/L)}} f(L\sqrt{2\log(2N/L)}) dy \leq e^{-\frac{1}{2}L\sqrt{2\log(N/(2L))}},\end{aligned}$$

as long as $L \geq 1$. This shows the second assertion. \square

The case when L is of larger order than N is as follows.

Lemma 20 *Let ξ_1, ξ_2, \dots be i.i.d. standard Gaussian r.v. Then there are constants $c_3, c_4 > 0$ such that for all integers $N \geq 1$ and all $L \geq 2N$ we have*

$$c_3 N \log(L/N) \leq -\log \mathbb{E}e^{-L \max_{i=1, \dots, N} |\xi_i|} \leq c_4 N \log(L/N).$$

Proof. Note that, for some $c > 0$,

$$\begin{aligned}\mathbb{E}e^{-L \max_{i=1, \dots, N} |\xi_i|} &\geq \int_0^L e^{-y} \mathbb{P}\{L|\xi| \leq y\}^N dy \geq \int_0^L e^{-y} \left(\frac{cy}{L}\right)^N dy \\ &\geq \left(\frac{c}{L}\right)^N \int_N^{N+1} e^{-y} y^N dy \geq \left(\frac{cN}{L}\right)^N e^{-(N+1)} \geq \left(\frac{c'N}{L}\right)^N.\end{aligned}$$

Taking logarithms gives the upper bound. The lower bound is proved in the same fashion. \square

The behaviour of the Laplace transform is yet different if L is of the same order as N . The proof is analogous to that of Lemma 20.

Lemma 21 *Let ξ_1, ξ_2, \dots be i.i.d. standard Gaussian r.v. Then there are constants $\tilde{c}_3, \tilde{c}_4 > 0$ such that for all $L > 0$ and all $N \in \mathbb{N}$ with $L/2 \leq N \leq 2L$ we have*

$$\tilde{c}_3 L \leq -\log \mathbb{E} e^{-L \max_{i=1, \dots, N} |\xi_i|} \leq \tilde{c}_4 L.$$

4.2 Proof of Theorem 3

Preliminaries: We use $\varepsilon_k = 2^{-k}$ and let $\Psi(\varepsilon) := \exp \{C\varepsilon^{-\gamma} |\log \varepsilon|^{-\beta}\}$. Then (20) and the assumption (11) imply that

$$\log \mathbb{E} e^{-\lambda \sup_{t, s \in T} |X(t) - X(s)|} \geq \sum_{k=0}^{\infty} \log \mathbb{E} \exp \left\{ -2\lambda \varepsilon_k \max_{i=1, \dots, \Psi(\varepsilon_{k+1})} |\xi_i| \right\}. \quad (33)$$

Let, for the purpose of this proof, $e^r = \lambda$ and

$$F(x) := \log \left(\Psi(2^{-(x+1)}) 2^x \right) = 2^{\gamma(x+1)} (x+1)^{-\beta} + x \log 2 + \log C.$$

We split the sum (33) into three parts: namely, we define $S_1 := \sum_{\{k: \Psi(\varepsilon_{k+1}) \leq \lambda \varepsilon_k\}}$, $S_2 := \sum_{\{k: \Psi(\varepsilon_{k+1}) \leq 4\lambda \varepsilon_k \leq 4\Psi(\varepsilon_{k+1})\}}$, and $S_3 := \sum_{\{k: \Psi(\varepsilon_{k+1}) \geq 4\lambda \varepsilon_k\}}$.

Evaluation of S_1 : By Lemma 20, S_1 can be estimated from below by

$$- \sum_{\{k: \Psi(\varepsilon_{k+1}) \leq \lambda \varepsilon_k\}} \Psi(\varepsilon_{k+1}) \log \frac{2\lambda \varepsilon_k}{\Psi(\varepsilon_{k+1})} = - \sum_{\{k: F(k) \leq r\}} \Psi(\varepsilon_{k+1}) (r + \log 2 - F(k)).$$

This can be re-written as

$$- \sum_{\{k: F(k) \leq r\}} \sum_{F(k) \leq l \leq r} \Psi(\varepsilon_{k+1}) = - \sum_{1 \leq l \leq r} \sum_{1 \leq k \leq F^{-1}(l)} \Psi(\varepsilon_{k+1}).$$

It is clear that $\Psi\left(\frac{x}{2}\right) \geq \Psi(x)^C$, for some $C > 1$. Using only this one can show that the inner sum behaves as the largest term, which means that the double sum can be estimated from below by

$$-c \sum_{1 \leq l \leq r} \Psi(\varepsilon_{F^{-1}(l)+1}) = -c \sum_{1 \leq l \leq r} e^{F(F^{-1}(l)) - F^{-1}(l) \log 2}.$$

Using the same argument, this can be estimated again by the largest term in the sum, i.e. by $-c' \exp \{r - F^{-1}(r) \log 2\}$. Note that $F^{-1}(r) \sim \log_2 r^{1/\gamma} +$

$\log_2(\log r)^{\beta/\gamma}$, which shows that the sum S_1 behaves, up to a constant, as

$$-\lambda(\log \lambda)^{-1/\gamma}(\log \log \lambda)^{-\beta/\gamma}.$$

Evaluation of S_2 : By Lemma 21, S_2 can be estimated from below by

$$-\sum_{\{k:\Psi(\varepsilon_{k+1})\leq 4\lambda\varepsilon_k\leq 4\Psi(\varepsilon_{k+1})\}} \lambda\varepsilon_k \geq -\lambda \sum_{\{k:r\leq F(k)\}} 2^{-k} = -c\lambda 2^{-F^{-1}(r)}.$$

This shows that S_2 is bounded from below by

$$-\lambda(\log \lambda)^{-1/\gamma}(\log \log \lambda)^{-\beta/\gamma}.$$

Evaluation of S_3 : In this case, we can apply the first part of Lemma 19, which implies that the sum can be estimated by

$$-\lambda \sum_{\{k:F(k)\geq r+\log 4\}} 2^{-k} \sqrt{F(k) - (r + \log 2)}.$$

Comparing sum and integral shows that the last term behaves as

$$\begin{aligned} &\approx -\lambda \int_{F^{-1}(r+\log 4)}^{\infty} 2^{-x} \int_0^{F(x)-(r+\log 2)} y^{-1/2} dy dx \\ &= -\lambda \int_0^{\infty} \int_{F^{-1}(y+r+\log 2)}^{\infty} 2^{-x} y^{-1/2} dx dy = -2\lambda \int_0^{\infty} 2^{-F^{-1}(y+r+\log 2)} y^{-1/2} dy. \end{aligned}$$

Recalling that $F^{-1}(y) \sim \log_2 y^{1/\gamma} + \log_2(\log y)^{\beta/\gamma}$ shows that the last term behaves as

$$\approx -\lambda \int_0^{\infty} (y + r + \log 2)^{-1/\gamma} (\log(y + r + \log 2))^{-\beta/\gamma} y^{-1/2} dy.$$

Substituting $rz = y$ and evaluating this finally leads to

$$S_3 \approx \begin{cases} -\lambda(\log \lambda)^{1/2-1/\gamma}(\log \log \lambda)^{-\beta/\gamma} & 0 < \gamma < 2 \\ -\lambda(\log \log \lambda)^{1-\beta/2} & \gamma = 2, \beta > 2. \end{cases}$$

Note that the bound for S_3 is the dominating term. Applying Lemma 18 finishes the proof of Theorem 3. \square

4.3 Proof of Proposition 9

The lower bound for the small deviation probability follows, via the observation in Remark 15 from the proof of Theorem 3. For the upper bound, recall

that the third sum in the proof of Theorem 3 is the dominating term. If we know that $N \approx \Psi$, all the estimates can be reversed. In particular, in order to get an upper bound, we can use the second part of Lemma 19, by keeping only the sum $\sum_{\{k: \Psi(\varepsilon_{k+1}) \geq 4\lambda\varepsilon_k \geq 4\}}$. \square

5 Stable case with critically large entropy

5.1 Proof of Theorem 5

Now the construction of small layers from the proof of Theorem 4 breaks down completely, because the related evaluation was based on $C_2 < 2^\alpha$, which we do not assume anymore. A new construction is as follows. For $k \geq 0$, let $\varepsilon_k = 2^{-k}\varepsilon$ and

$$b_k = S^{-1} (\varepsilon_k^\alpha N(\varepsilon_{k+1}))^{\frac{1}{\alpha+1}} \varepsilon,$$

where

$$S = S(\varepsilon) := \sum_{k=0}^{\infty} (\varepsilon_k^\alpha N(\varepsilon_{k+1}))^{\frac{1}{\alpha+1}}.$$

Note that $b = \sum_{k=0}^{\infty} b_k = \varepsilon$. We use the estimate (32) which holds for all $r > 0$, and obtain

$$\prod_{k=0}^{\infty} \mathbb{P} \{ \varepsilon_k |\xi| \leq b_k \}^{N(\varepsilon_{k+1})} \geq \exp \left\{ -A \sum_{k=0}^{\infty} \left(\frac{\varepsilon_k}{b_k} \right)^\alpha N(\varepsilon_{k+1}) \right\} = e^{-AS^{\alpha+1}\varepsilon^{-\alpha}}.$$

Now we evaluate S . Since Ψ is non-decreasing, we have, for every $k \geq 0$,

$$\int_{\varepsilon_{k+2}}^{\varepsilon_{k+1}} \left(\frac{\Psi(u)}{u} \right)^{\frac{1}{\alpha+1}} du \geq \Psi(\varepsilon_{k+1})^{\frac{1}{\alpha+1}} \int_{\varepsilon_{k+2}}^{\varepsilon_{k+1}} u^{-\frac{1}{\alpha+1}} du = c_\alpha (\Psi(\varepsilon_{k+1}) \varepsilon_k^\alpha)^{\frac{1}{\alpha+1}}.$$

After summing over k , we obtain

$$S \leq \sum_{k=0}^{\infty} (\varepsilon_k^\alpha \Psi(\varepsilon_{k+1}))^{\frac{1}{\alpha+1}} \leq c_\alpha^{-1} \sum_{k=0}^{\infty} \int_{\varepsilon_{k+2}}^{\varepsilon_{k+1}} \left(\frac{\Psi(u)}{u} \right)^{\frac{1}{\alpha+1}} du \leq c_\alpha^{-1} \widehat{\Psi}(\varepsilon).$$

Therefore,

$$\prod_{k=0}^{\infty} \mathbb{P} \{ \varepsilon_k |\xi| \leq b_k \}^{N(\varepsilon_{k+1})} \geq \exp \left\{ -Ac_\alpha^{-\alpha-1} \widehat{\Psi}(\varepsilon)^{\alpha+1} \varepsilon^{-\alpha} \right\}.$$

We do not need to make any changes in the construction and evaluation of higher layers. Therefore, the estimate (29) remains valid. We just show that both terms from this estimate are dominated by that of lower layers' bound.

First, we always have for non-increasing Ψ ,

$$\widehat{\Psi}(\varepsilon)^{\alpha+1} \varepsilon^{-\alpha} \geq \left[\Psi(\varepsilon)^{\frac{1}{\alpha+1}} \cdot \varepsilon^{\frac{-1}{\alpha+1}} \cdot \varepsilon \right]^{\alpha+1} \varepsilon^{-\alpha} = \Psi(\varepsilon).$$

Second, it follows from (8) that under assumption (12) $\widetilde{\Psi}$ is of higher order than Ψ . This is enough to get rid of the higher layers. \square

Proof of Corollary 6 and Remark 8. By Theorem 5 and Theorem 7, respectively, it is already clear that the assumptions imply that

$$\mathbb{P} \left\{ \sup_{s,t \in T} |X(s) - X(t)| \leq K_0 \right\} > 0.$$

Therefore X is bounded with positive probability, which, by the zero-one law in Corollary 9.5.5 in [13], extends to a.s. boundedness. \square

5.2 Proof of Theorem 7

We deal with the stable case of critically large entropy, namely when $N(\varepsilon) \leq C\varepsilon^{-\alpha} |\log \varepsilon|^{-\beta}$. The case $\beta > 1 + \alpha$ is a particular case of Theorem 5. Therefore, let us concentrate on $\max(1, \alpha) < \beta \leq 1 + \alpha$.

We are going to use the Laplace technique, i.e. Lemma 14 instead of Talagrand's idea from Lemma 13 that was the basis for Theorem 5. Since we deal with a symmetric α -stable process we can use the general lower estimate (32). Doing so shows that the term in (20) is bounded from below by

$$\prod_{k=0}^{\infty} \int_0^{\infty} e^{-y} \exp \left\{ -Ay^{-\alpha} (\lambda \varepsilon_k)^{\alpha} N(\varepsilon_{k+1}) \right\} dy.$$

Using $N(\varepsilon) \leq C\varepsilon^{-\alpha} |\log \varepsilon|^{-\beta}$ and the choice $\varepsilon_k = 2^{-k}$, we obtain

$$\prod_{k=1}^{\infty} \int_0^{\infty} \exp \left\{ -(y + By^{-\alpha} \lambda^{\alpha} k^{-\beta}) \right\} dy.$$

We will now need the two following estimates of Laplace integrals, the proofs of which are elementary and therefore omitted.

Lemma 22 *For $L \rightarrow \infty$ we have*

$$\log \int_0^{\infty} e^{-y - Ly^{-\alpha}} dy \sim -C_{\alpha} L^{1/(1+\alpha)}.$$

Lemma 23 For $\delta \rightarrow 0$ we have

$$\log \int_0^\infty e^{-y-\delta y^{-\alpha}} dy \approx \begin{cases} -\delta^{1/\alpha} & \alpha > 1, \\ -\delta \log 1/\delta & \alpha = 1, \\ -\delta & \alpha < 1. \end{cases}$$

By Lemma 22 and Lemma 23 for $\beta > \max(1, \alpha)$, $\alpha \neq 1$,

$$\begin{aligned} \sum_{k=1}^{\infty} \log \int_0^\infty \exp \left\{ -(y + By^{-\alpha} \lambda^\alpha k^{-\beta}) \right\} dy &= \sum_{\lambda^\alpha k^{-\beta} > 1} + \sum_{\lambda^\alpha k^{-\beta} \leq 1} \\ &\geq -C_1 \sum_{k < \lambda^{\alpha/\beta}} \lambda^{\alpha/(1+\alpha)} k^{-\beta/(1+\alpha)} - C_2 \sum_{k \geq \lambda^{\alpha/\beta}} \lambda^{\min(1, \alpha)} k^{-\beta/\max(1, \alpha)}. \end{aligned} \quad (34)$$

For $\max(1, \alpha) < \beta < 1 + \alpha$, both terms are of order $\lambda^{\alpha/\beta}$. This yields that

$$\log \mathbb{E} e^{-\lambda \sup_{t,s \in T} |X(t) - X(s)|} \geq -C \lambda^{\alpha/\beta}.$$

By the usual Tauberian-type argument (the so-called de Bruijn Tauberian Theorem, i.e. Theorem 4.12.9 in [3]), this shows the assertion for the range $\max(1, \alpha) < \beta < \alpha + 1$. The argument for $\alpha = 1$ is similar.

For $\beta = \alpha + 1$, the first term in (34) contains an additional logarithm, whereas the second does not and is thus of lower order. This yields

$$\log \mathbb{E} e^{-\lambda \sup_{t,s \in T} |X(t) - X(s)|} \geq -C \lambda^{\alpha/(1+\alpha)} \log \lambda,$$

and once again the standard Tauberian-type argument proves the theorem's assertion. \square

5.3 Proof of Proposition 10

Recall that we consider the sum of maxima example (Example 3) with $\sigma_n = 2^{-n/\alpha} n^{-\beta/\alpha}$ and $N_n = 2^n$.

The lower bound for the small deviation probability follows, via the observation in Remark 15 applied to $N_n = 2^n$, $\varepsilon_n = 2^{-n/\alpha} n^{-\beta/\alpha}$, from the proof of Theorem 7.

Proof of the upper bound. Consider the corresponding Laplace transform

$$\mathbb{E} e^{-\lambda \sum_{n=1}^N \sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}|} = \prod_{n=1}^N \int_0^\infty e^{-y} \mathbb{P} \{ \lambda \sigma_n |\xi| \leq y \}^{N_n} dy. \quad (35)$$

We estimate this term using that $t := \mathbb{P}\{|\xi| \leq 1\} < 1$ and the analog to (32) for large arguments as follows

$$\int_0^\infty e^{-y} \mathbb{P}\left\{|\xi| \leq \frac{y}{\lambda \sigma_n}\right\}^{N_n} dy \leq \int_0^{\lambda \sigma_n} e^{-y} t^{N_n} dy + \int_{\lambda \sigma_n}^\infty e^{-y - A \lambda^\alpha n^{-\beta} y^{-\alpha}} dy. \quad (36)$$

The case $\beta \geq 1 + \alpha$. Let

$$\begin{aligned} n \in A_\lambda &:= \left\{k : \lambda \sigma_k \leq \lambda^{\alpha/(1+\alpha)} k^{-\beta/(1+\alpha)}, k \leq \lambda^{\alpha/\beta}\right\} \\ &\supseteq \left\{k : \frac{\alpha}{1+\alpha} \log \lambda \leq k \leq \lambda^{\alpha/\beta}\right\} =: A'_\lambda. \end{aligned}$$

Then the first term in (36) can be estimated by $t^{N_n} \leq e^{-C' \lambda^{\alpha/(1+\alpha)} n^{-\beta/(1+\alpha)}}$. On the other hand, the second term in (36) is less than

$$\int_0^\infty e^{-y - A \lambda^\alpha n^{-\beta} y^{-\alpha}} dy \leq e^{-C \lambda^{\alpha/(1+\alpha)} n^{-\beta/(1+\alpha)}},$$

by Lemma 22 and the fact that $n \leq \lambda^{\alpha/\beta}$. Using these estimates, (35), and letting N tend to infinity, we obtain

$$\log \mathbb{E} e^{-\lambda^S} \leq \sum_{n \in A_\lambda} -C \lambda^{\alpha/(1+\alpha)} n^{-\beta/(1+\alpha)} \leq -C' \lambda^{\alpha/(1+\alpha)} \sum_{n \in A'_\lambda} n^{-\beta/(1+\alpha)}.$$

Note that this term is less or equal to

$$\begin{cases} -C'' \lambda^{\alpha/(1+\alpha)} (\log \lambda)^{1-\beta/(1+\alpha)} & \beta > 1 + \alpha, \\ -C'' \lambda^{\alpha/(1+\alpha)} (\log \lambda) & \beta = 1 + \alpha. \end{cases}$$

Hence, de Bruijn's Tauberian Theorem (Theorem 4.12.9 in [3]), implies the assertion.

The case $\max(1, \alpha) < \beta < 1 + \alpha$. We let $n \in B_\lambda$, where $B_\lambda := \{k : \lambda^{\alpha/\beta} \leq k \leq 2\lambda^{\alpha/\beta}\}$. Then the second term in (36) is bounded by

$$\int_0^\infty e^{-y - A \lambda^\alpha n^{-\beta} y^{-\alpha}} dy.$$

Since $A \lambda^\alpha n^{-\beta} \leq A$, we have by Lemma 23 that the last term is bounded by

$$\begin{cases} e^{-C \lambda n^{-\beta/\alpha}} & \alpha > 1, \\ e^{-C \lambda n^{-\beta} \log(\lambda^{-\alpha} n^\beta)} & \alpha = 1, \\ e^{-C \lambda^\alpha n^{-\beta}} & \alpha < 1. \end{cases} \quad (37)$$

On the other hand, the first term in (36) is bounded by e^{-C2^n} , which is certainly smaller than (37). Using this, (37), and (35), and letting N tend to

infinity we obtain

$$\log \mathbb{E} e^{-\lambda S} \sum_{n=1}^{\infty} \sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| \leq -C \lambda^{\alpha/\beta},$$

in all three cases. Again, de Bruijn's Tauberian Theorem implies the assertion.

The case $\beta \leq \max(\mathbf{1}, \alpha)$. Here we use Kolmogorov's Three Series Theorem to show that S is infinite a.s. On the one hand, it is necessary for the convergence of S that

$$\sum_n \mathbb{P} \left\{ \sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| > 1 \right\} < \infty.$$

Using the tail estimate (32), it is easy to see that this is true if and only if $\sum_n \sigma_n^\alpha 2^n < \infty$, which is violated for $\beta \leq 1$. Thus we are finished for $0 < \alpha \leq 1$.

On the other hand, it is necessary for S to be a.s. finite that

$$\sum_n \mathbb{E} \sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| \mathbb{1}_{\{\sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| \leq 1\}} < \infty. \quad (38)$$

Let $\alpha > 1$. Note that

$$\sigma_n \max_k |\xi_{n,k}| \mathbb{1}_{\{\sigma_n \max_k |\xi_{n,k}| \leq 1\}} = \sigma_n \max_k |\xi_{n,k}| - \sigma_n \max_k |\xi_{n,k}| \mathbb{1}_{\{\sigma_n \max_k |\xi_{n,k}| > 1\}}.$$

It is easy to show using the tail estimate (32) that

$$\sum_n \mathbb{E} \sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| \mathbb{1}_{\{\sigma_n \max_{k=1, \dots, N_n} |\xi_{n,k}| > 1\}} < \infty \quad \Leftrightarrow \quad \beta > 1 \text{ and } \alpha > 1.$$

However, for $\alpha > 1$, $\mathbb{E} \max_{k=1, \dots, N_n} |\xi_{n,k}| \approx N_n^{1/\alpha}$ (cf. e.g. [11], p. 271), which shows that

$$\sum_n \mathbb{E} \sigma_n \max_k |\xi_{n,k}| < \infty \quad \Leftrightarrow \quad \beta > \alpha.$$

It follows that the series (38) diverges when $1 < \beta \leq \alpha$. This finishes the proof of Proposition 10. \square

6 Concluding remarks

1. There is another type of processes with slowly vanishing small deviation probabilities. Take for example a stationary Gaussian process $X(t), t \in \mathbb{R}$, with quickly decreasing spectral density f , say

$$f(\lambda) = \exp\{-\lambda^2\}, \quad \lambda \in \mathbb{R}.$$

Then the small deviation probability is vanishing too slowly, e.g.

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^h \log \mathbb{P} \left\{ \sup_{s, t \in [0, 1]} |X(s) - X(t)| \leq \varepsilon \right\} = 0, \quad \forall h > 0,$$

while the covering numbers grow polynomially. Namely, $N(\varepsilon) \approx \varepsilon^{-1}$, due to the smoothness of X . Such kind of small deviation behaviour can not be obtained from our results. It is rather related with extremely good approximation of the analytical process X by finite rank processes. See [16], for more details and statistical applications.

2. There exists a surprising relation between the small deviations in the critical stable and critical Gaussian case, as the following example shows. Let (ξ_n) be i.i.d. standard normal and let A_n be i.i.d. totally skewed positive $\alpha/2$ -stable random variables. Then $\theta_n = A_n^{1/2} \xi_n$ are i.i.d. symmetric α -stable random variables. The studies of small deviation probabilities $\mathbb{P} \{ \sum_n |\sigma_n \xi_n|^\alpha \leq \varepsilon^\alpha \}$ and $\mathbb{P} \{ \sum_n |\sigma_n \theta_n|^2 \leq \varepsilon^2 \}$ can be completely reduced to each other. In particular, the critical stable case, with $\sigma_n \sim n^{-1/\alpha} (\log n)^{-\beta/\alpha}$ considered in (13) with entropy $N(\varepsilon) \approx \varepsilon^{-\alpha} |\log \varepsilon|^{-\beta}$, corresponds to the Gaussian case with large entropy $\log N(\varepsilon) \approx \varepsilon^{-2} |\log \varepsilon|^{-2\beta/\alpha}$. Both, the stable and the Gaussian process, are bounded if and only if $\beta > 1$.

3. \mathbb{R}^d -valued Processes. Let us consider $(X(t))_{t \in T}$ to be a Gaussian or symmetric α -stable process with values in \mathbb{R}^d . Then we define for a Gaussian process the analogue to the Dudley metric by

$$\rho(t, s) := \left(\mathbb{E} \|X(t) - X(s)\|^2 \right)^{1/2},$$

replaced by the r -th moment for the stable case. Here, $\|\cdot\|$ denotes any norm on \mathbb{R}^d . As above we consider the covering numbers $N(\varepsilon)$ of the quasi-metric space (T, ρ) , which we assume to be relatively compact.

Proposition 24 *All the above theorems and corollaries hold literally for the case of an \mathbb{R}^d -valued Gaussian or symmetric α -stable process, respectively.*

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