First passage times of Lévy processes over a one-sided moving boundary

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Abstract

We study the one-sided exit problem with a moving boundary for Lévy processes. Our main result states that if the boundary behaves asymptotically as t^γ for some $\gamma < 1/2$ then the probability that the process stays below the boundary behaves as in the case of a constant boundary. This class of boundaries is independent of Spitzer's condition in contrast to previously known results. Both positive and negative boundaries are considered.

These results extend the findings of [17] and are motivated by results in the case of Brownian motion, for which the above result was proved in [36].

Key words and phrases: Lévy processes; moving boundary; one-sided exit problem; one-sided boundary problem; first passage time; survival exponent; boundary crossing probabilities; boundary crossing problem; one-sided small deviations; lower tail probabilities

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1 Introduction

1.1 Statement of the problem

We consider the one-sided exit problem with a moving boundary. In the literature, this problem is known by a variety of names, e.g. *one-sided barrier problem*, *boundary crossing problem*, and *first passage time problem.* For a stochastic process $(X(t))_{t\geq 0}$ and a function $f : \mathbb{R}_+ \to \mathbb{R}$, the question is whether there exists a $\delta > 0$ such that

$$\mathbb{P}\left(X(t) \leq f(t),\ 0 \leq t \leq T\right) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty. \tag{1}$$

If such δ exists it is called the *survival exponent.* The function f serves as a boundary which the process is not allowed to pass. If the function f is constant then we are in the framework of the classical one-sided exit problem.

The one-sided exit problem is a classical question, which is relevant in a number of different applications such as reaction diffusion system or the solution of Burgers' equation. A recent overview of applications is presented in [25].

Let τ_f denote the first passage time across the boundary f for the càdlàg process $(X(t))_{t\geq 0}$, i.e.

$$\tau_f = \inf\{t \geq 0 : X(t) > f(t)\}.$$

Since $\mathbb{P}(\tau_f > T) = \mathbb{P}\left(X(t) \leq f(t),\ 0 \leq t \leq T\right)$, our problem is equivalent to the study of the asymptotic behavior of the distribution of the first passage time.

In this paper we consider Lévy processes $(X(t))_{t\geq 0}$ with triplet (σ^2, b, ν) and focus on the following question: For which functions f does the one-sided exit problem with a constant boundary, i.e.

$$\mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty, \tag{2}$$

imply (1) (and thus the survival exponent is preserved)? Note that results of type (2) can be obtained using classical fluctuation theory.

In order to compare our results to previously known ones for simplicity let us look for a moment at functions $f(t) = 1 \pm t^\gamma$, $\gamma \geq 0$.

Negative boundary $\mathbf{1 - t^\gamma}$: Assuming $\nu(\mathbb{R}_-) > 0$ and (2), we prove (Theorem 1) that

$$\gamma < \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 - t^\gamma,\ 0 \leq t \leq T) = T^{-\delta+o(1)}.$$

Results for Lévy processes so far have included only the case that $X(1)$ satisfies the right-side Cramer condition, i.e. there exists $\lambda > 0$ such that $\mathbb{E}\exp(\lambda X(1)) < \infty$ (see [17], Theorem 2).

Positive boundary $\mathbf{1 + t^\gamma}$: Let us assume for a moment that Spitzer's condition holds with $\rho \in (0, 1)$. Then, [17] proved that

$$\gamma < \rho \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T) \sim T^{-\rho} L(T),$$

where L is a slowly varying function. Our result (Theorem 2) states that assuming $\nu(\mathbb{R}_+) > 0$, $\nu(\mathbb{R}_-) > 0$, and (2)

$$\gamma < \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T) = T^{-\rho+o(1)}.$$

Hence, we improve the result of [17] when $\rho < \frac{1}{2}$.

We stress that in this simplified case, $f(t) = 1 \pm t^\gamma$, our class of boundaries agrees with the class found in the case of Brownian motion (see [36]). Hence, combining our results with [17] we obtain a class of boundaries where the survival exponent is equal to the one in the case of $f \equiv 1$ and which is larger than in the case of Brownian motion. In particular, we have shown that our class of boundaries is independent of Spitzer's condition needed in previous works.

We proceed this paper by introducing our main results in Section 1.2. There, we also present the main idea of the proofs. A brief summary of related work is contained in Section 1.3. The proof of Theorem 1, the case of negative boundaries, is given in Section 3, whereas Section 4 contains the proof for positive boundaries, Theorem 2. For reasons of clarity and readability some auxiliary lemmas are combined in Section 2 and may be of independent interest.

In the following we use the notation for strong and weak asymptotics. We write $f \lesssim g$ if $\limsup_{x\to\infty} f(x)/g(x) < \infty$ and $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. Furthermore, $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to \infty$.

1.2 Main results

We study the one-sided exit problem with moving boundaries for a Lévy process denoted by $(X(t))_{t\geq 0}$. Lévy processes possess stationary and independent increments and almost surely right continuous paths (see [2], [35]). By the Lévy-Khintchine formula, the characteristic function of a marginal of a Lévy process $(X(t))_{t\geq 0}$ is given by

$$\mathbb{E}\left(e^{iuX(t)}\right) = e^{t\Psi(u)}, \quad \text{for every } u \in \mathbb{R},$$

where

$$\Psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - \mathbf{1}_{\{|x|\leq 1\}} iux)\nu(dx), \tag{3}$$

for parameters $\sigma^2 \geq 0$, $b \in \mathbb{R}$, and a positive measure ν concentrated on $\mathbb{R}\backslash\{0\}$, called Lévy measure, satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

For a given triplet (σ^2, b, ν) there exists a Lévy process $(X(t))_{t\geq 0}$ such that (3) holds, and its distribution is uniquely determined by its triplet. We call $(X(t))_{t\geq 0}$ a (σ^2, ν)-Lévy martingale if (3) is equal to

$$\Psi(u) = -\frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx) \tag{4}$$

for a measure ν satisfying $\int |x| \wedge x^2 \nu(dx) < \infty$. It is a martingale in the usual sense.

We can now formulate our first main result, which corresponds to the one-sided exit problem with a *negative boundary.*

Theorem 1. *Let X be a Lévy process with triplet (σ^2, b, ν) where $\nu(\mathbb{R}_-) > 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an almost everywhere differentiable function and continuous in a neighbourhood of zero such that $f(0) < 1$, $\int_1^\infty f'(s)^2 ds < \infty$, and $f'(t) \searrow 0$, for $t \to \infty$. Let $\delta > 0$. If*

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta + o(1)} \tag{5}$$

holds then

$$\mathbb{P}(X(t) \leq 1 - f(t),\ 0 \leq t \leq T) = T^{-\delta + o(1)}. \tag{6}$$

The proof is conducted through a new approach, which is sketched below.

The following theorem corresponds to the one-sided exit problem with a *positive boundary*.

Theorem 2. *Let X be a Lévy process with triplet (σ^2, b, ν) where $\nu(\mathbb{R}_+) > 0$ and $\nu(\mathbb{R}_-) > 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an almost everywhere differentiable function such that $\int_1^\infty f'(s)^2 ds < \infty$ and $\sup_{s\geq 1} |f'(s)| < \infty$. Let $\delta > 0$. If*

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta + o(1)} \tag{7}$$

holds then

$$\mathbb{P}(X(t) \leq 1 + f(t),\ 0 \leq t \leq T) = T^{-\delta + o(1)}.$$

Let us give a few comments on these results.

Remark 3. *In Theorem 1 (Theorem 2, respectively), the negative (positive, respectively) jumps are required in order to compensate the boundary. This is essential for our approach.*

Remark 4. *The assumption of equation (5)/(7) is associated with Spitzer's condition, which is given as follows: X satisfies Spitzer's condition with $\rho \in (0,1)$ if*

$$\frac{1}{t} \int_0^t \mathbb{P}(X(s) > 0) ds \to \rho \in (0,1) \quad \text{as } t \to \infty.$$

Spitzer's condition is equivalent to $\mathbb{P}(X(t) > 0) \to \rho$, as $t \to \infty$ (cf. [11], Theorem 23). Furthermore, according to [2], Theorem 18, Spitzer's condition holds with $\rho \in (0,1)$ if and only if the probability in (5)/(7) is regularly varying with index $-\rho$. Note that the class of Lévy processes satisfying assumption (5)/(7) is strictly larger than the class of Lévy processes satisfying Spitzer's condition (see [31], page 87, or Theorem 1 in [10] for a discrete-time version).

Remark 5. *In both Theorems, the regularity conditions on the function f are for technical purposes. Trivially, both Theorems are also valid for a less regular function g if there is a function f satisfying the conditions in Theorem 1 (Theorem 2, respectively) such that $g(s) \leq f(s)$, for all $s \geq 0$.*

Remark 6. *The assumption of negative jumps in Theorem 2 is required in order to show that (7) implies*

$$\mathbb{P}(X(t) \le 1 + (\ln T)^5,\ 1 \le t \le T) \le T^{-\delta+o(1)}.$$

Alternatively, this can be proved with the help of [23]. In order to use [23] we require – instead of the negative jumps – the assumption that the Lévy process X creeps across some level $x > 0$, i.e. $\mathbb{P}(X(\tau_x) = x) > 0$, for some $x > 0$. A Lévy process creeps across $x > 0$ if and only if the ascending ladder process creeps across x (see [24], page 197).

Let us come back to the question posed in (2), whether necessary and sufficient conditions on the boundary exist for which the survival exponent stays the same compared to the case of a constant boundary. More precisely, let $\delta > 0$, $\beta_+ := \sup\{r \ge 0 : \mathbb{E}\left((X(1)^+)^r\right) < \infty\}$ and $\beta_- := \sup\{r \ge 0 : \mathbb{E}\left((X(1)^-)^r\right) < \infty\}$. Because of the present results and previously known ones (e.g. [26] and [17]) it seems to be reasonable to expect that (5) implies

$$\gamma < \max\left\{\tfrac{1}{2}, \tfrac{1}{\beta_-}\right\} \Longleftrightarrow \mathbb{P}(X(t) \le 1 - t^\gamma,\ 0 \le t \le T) = T^{-\delta+o(1)}.$$

We have shown sufficiency of $\gamma < \frac{1}{2}$. In the same way, one might also expect that (7) implies

$$\gamma < \max\left\{\tfrac{1}{2}, \tfrac{1}{\beta_+}\right\} \Longleftrightarrow \mathbb{P}(X(t) \le 1 + t^\gamma,\ 0 \le t \le T) = T^{-\delta+o(1)}.$$

We have shown sufficiency of $\gamma < \max\left\{\frac{1}{2}, \rho\right\}$. Recall that $\rho \le \frac{1}{\beta_+}$ (cf. [4]).

We conclude this section by presenting a sketch of the proof of Theorem 1. For this purpose, we need the definition of an additive process. This class of processes consists of time-inhomogeneous processes which have independent increments and start at 0 (see [35]). The triplet is given by $(\sigma^2, f_X(t), \Lambda_X(dx, dt))$, $f_X \in C[0,\infty)$ where $f(0) = 0$, $\sigma \ge 0$, and Λ_X is a measure on $\mathbb{R} \times [0, T]$.

Sketch of the proof of Theorem 1: Note that the upper bound is trivial since f is positive.

For the lower bound our main idea is to find an iteration method to reduce the exponent of the boundary in each step such that eventually the boundary turns into a constant boundary. In each iteration step, we start with a change of measure compensating the boundary f by negative jumps. Then, we get an additive process which has the following triplet $\left(\sigma^2, b \cdot s, (1 + f'(s)|x|/m\mathbf{1}_{\{x\in A\}})ds\nu(dx)\right)$, where $A \subseteq [-1, 0)$ and m are suitably chosen. This process can be represented as $X(\cdot) + Z(\cdot)$, where X is the original Lévy process and Z has the triplet $(0, 0, f'(s)|x|/m\mathbf{1}_{\{x\in A\}}ds\nu(dx))$. This approach implies the estimate

$$\mathbb{P}(X(t) \le 1 - f(t),\ t \le T) \ge \mathbb{P}(X(t) + Z(t) \le 1,\ t \le T) \cdot \exp\left(-c \cdot \sqrt{\ln T}\right).$$

The term $\exp\left(-c \cdot \sqrt{\ln T}\right)$ represents the cost of changing the measure. A homogenization yields a Lévy process $\tilde{Z}$ with $Z(\cdot) \stackrel{d}{=} \tilde{Z}(f(\cdot))$ and triplet $(0, 0, |x|/m\mathbf{1}_{\{x\in A\}}\nu(dx))$.

Since $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, we can finally estimate $\mathbb{P}(X(t) + \tilde{Z}(f(t)) \leq 1,\ t \leq T)$ by

$$\mathbb{P}(X(t) \leq 3 - f(t)^{2/3},\ t \leq T).$$

This procedure is repeated until $f(t)^{(2/3)^n} \leq 2$. Then, the asymptotic behavior of $\mathbb{P}(X(t) \leq 3 - f(t)^{(2/3)^n},\ t \leq T)$ follows from (2). Hence, through an n-times iteration of these steps the survival exponent in (1) is obtained with the help of (2) since n is of order $\ln \ln T$. A similar approach is used in the proof of Theorem 2. Here, the upper bound is proved through an iteration method.

1.3 Related work

We summarize some known results on one-sided exit problems with a moving boundary.

In the Brownian motion case the survival exponent for a constant boundary is $1/2$ by the reflection principle. Regarding moving boundaries in the Brownian motion setting, the question was studied by [16, 21, 28, 30, 34, 36] in different ways. Independently of each other [16] and [36] stated an integral condition on the boundary f, for which the survival exponent remains $1/2$. More precisely, they proved under the additional assumption of convexity (concavity) of the function f in case of decreasing (increasing) functions

$$\int_1^\infty |f(t)| t^{-3/2} dt < \infty \Longleftrightarrow \mathbb{P}(X(t) \leq f(t),\ 0 \leq t \leq T) \approx T^{-1/2}, \text{ as } T \to \infty.$$

An alternative proof for this statement can also be found in [30].

A classical area of reasearch concerns exit problems for Lévy processes over a constant boundary. Using fluctuation theory, [4] showed under the assumption that Spitzer's condition holds with $\rho \in (0,1)$ that the survival exponent is equal to ρ. Similar arguments were already used for random walks with zero mean (see [15]).

The first asymptotic relation for the one-sided exit problem with a moving boundary in the case of discrete time, i.e. the case of sums of independent random variables, was achieved by [26] using the technique of factorization identities. Under the additional assumption of $\mathbb{P}(X(1) < y) = |y|^\alpha L(|y|)$, $y < 0$ and L a slowly varying function, they studied boundaries of the form $f(n) \sim -n^\gamma$ with $\gamma > 1/\alpha$. Furthermore, [29] weakened certain restrictions on the boundary f using martingale techniques.

For Lévy processes with jumps bounded from above, [27] started studying moving boundaries using martingale techniques. Similar results as in the case of Brownian motion were obtained. Subsequently, [17] generalized these results for Lévy processes which satisfy Spitzer's condition with $\rho \in (0,1)$. One result was pointed out in the introduction. Furthermore, they proved that if $\mathbb{E}(\chi_1) < \infty$, where $\chi_1 = X(\tau_1) - 1$ with $\tau_1 = \inf\{t \geq 0 : X(t) > 1\}$, then

$$\gamma < \rho \quad \Longleftrightarrow \quad \mathbb{P}(X(t) \leq 1 + t^\gamma,\ t \leq T) \sim c \cdot \mathbb{P}(X(t) \leq 1,\ t \leq T).$$

The condition $\mathbb{E}(\chi_1) < \infty$ is satisfied if X belongs to the domain of attraction of a stable law with index $\alpha \in (1,2)$ and symmetry parameter $\beta = -1$ or $\mathbb{E}\left((X(1)^+)^2\right) < \infty$. In the case of ultimately non-increasing boundaries they stated that if $\mathbb{E}X(1) = 0$ and $(X(t))_{t\geq 0}$ satisfies the right-side Cramer condition then

$$\mathbb{P}(X(t) \leq 1 - t^\gamma,\ t \leq T) \approx \mathbb{P}(X(t) \leq 1,\ t \leq T) \quad \Rightarrow \quad \mathbb{E}(\tau_1^\gamma) < \infty.$$

Furthermore, [37] deals with one-sided stochastic boundaries based on [17].

If the process does not necessarily satisfy Spitzer's condition, various results were obtained for a constant boundary by [1, 3, 5, 6, 9, 10, 23].

Closely related to the asymptotic behavior of the distribution of the first passage time over a boundary is the examination of the moments of this first passage time studied by [12, 19, 33]. Furthermore, [13] gave necessary and sufficient conditions for the first passage time over a power law boundary to be almost surely finite. These conditions depend on the moments of the process and the power of the boundary.

Recently, the stability of the first passage time over a curved boundary for Lévy processes was studied by [18]. Moreover, for diffusion processes [7] and [8] discussed the one-sided exit problem with a moving boundary.

2 Auxiliary results

2.1 Technical tools regarding the boundary

In this section, we show that it is sufficient for the proofs to consider only non-decreasing functions f during the further progress of this work. Furthermore, we indicate some useful properties of the function f.

Remark 7 shows that it is sufficient to prove Theorem 1 and Theorem 2 for functions f which are only non-decreasing.

Remark 7. *We do not require the restriction that the function is non-decreasing to prove the upper bound of Theorem 1 (respectively, the lower bound of Theorem 2). The lower (respectively, upper) bound can be deduced from the case where the function f is non-decreasing through adequate estimates. For this purpose, define $M(t) := \sup_{0\leq s\leq t} f(s)$. We proceed for functions satisfying the assumptions of Theorem 1 as follows:*

$$\mathbb{P}(X(t) \leq 1 + f(t),\ 0 \leq t \leq T) \leq \mathbb{P}(X(t) \leq 1 + M(t),\ 0 \leq t \leq T).$$

Note that M is non-decreasing and satisfies the assumptions of Theorem 1. Thus, it is sufficient to find a lower bound for the one-sided exit problem with the boundary M.

For Theorem 2 we have

$$\mathbb{P}(X(t) \leq 1 - f(t),\ 0 \leq t \leq T) \geq \mathbb{P}(X(t) \leq 1 - M(t),\ 0 \leq t \leq T).$$

Note that M is non-decreasing and satisfies the assumptions of Theorem 2. In this case, it is sufficient to find an upper bound for the one-sided exit problem with the function M.

The following properties which are easy to check will be required for the proofs.

Lemma 8. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-decreasing function satisfying the assumptions of Theorem 2. Then,*

$$f(T) \leq c \cdot T, \text{ for all } T \text{ sufficiently large,} \tag{8}$$

for some constant $c > 0$. Furthermore, if the function f satisfies additionally the assumptions of Theorem 1, then there exists a constant $\tilde{c} > 0$ such that

$$\sqrt{t} f'(s) \leq \tilde{c} \quad \text{a.e. for all } s \geq t \geq 0. \tag{9}$$

2.2 Girsanov transform for additive processes

For the proofs we use the Girsanov transform for additive processes to transform Lévy processes into additive processes. Let us recall that N is a Poisson random measure on $(\mathbb{R}, \mathbb{R}^+)$ with intensity $\Lambda(dx, ds)$. The compensated measure is denoted by $\bar{N}(dx, ds) = N(dx, ds) - \Lambda(dx, ds)$. Furthermore, let $\mathbb{P}_X$ be a probability measure on $(D, \mathcal{F}_D)$ where D is the space of mappings from $[0, \infty)$ into $\mathbb{R}$ right continuous with left limits and $\mathcal{F}_D$ is the smallest σ-algebra that makes $X(t)$, $t \geq 0$, measurable (cf. [35]).

The following theorem needed in the main proofs can be found in [20] (Theorem 3.24) and [35] (Theorems 33.1 and 33.2).

Theorem 9. *Let X and Y be two additive processes with triplets $(\sigma_X^2, f_X(t), \Lambda_X(dx, dt))$ and $(\sigma_Y^2, f_Y(t), \Lambda_Y(dx, dt))$, where Λ_X, Λ_Y are measures concentrated on $\mathbb{R}\backslash\{0\} \times [0, T]$. Then $\mathbb{P}_X|_{\mathcal{F}_T}$ and $\mathbb{P}_Y|_{\mathcal{F}_T}$ are absolutely continuous if and only if $\sigma_X = \sigma_Y$ and there exists $\theta(\cdot, \cdot) : \mathbb{R} \times [0, T] \to \mathbb{R}$ such that*

- $\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)/2} - 1\right)^2 \Lambda_X(dx, ds) < \infty$,
- *Λ_X and Λ_Y are absolutely continuous with $\frac{d\Lambda_Y}{d\Lambda_X}(x, s) = e^{\theta(x,s)}$, and*
- *$f_Y(t) = f_X(t) + \int_0^t \int_{|x|\leq 1} \left(e^{\theta(x,s)} - 1\right) x\Lambda_X(dx, ds)$, for all $t \in [0, T]$.*

The density transformation formula is given by

$$\frac{d\mathbb{P}_Y|_{\mathcal{F}_T}}{d\mathbb{P}_X|_{\mathcal{F}_T}}(X(\cdot)) = \exp\left(-\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)} - 1 - \theta(x, s)\right) \Lambda_X(dx, ds) + \int_0^T \int_{\mathbb{R}} \theta(x, s)\bar{N}_X(dx, ds)(\cdot)\right) \quad \mathbb{P}_X\text{-a.s.} \tag{10}$$

Remark 10. *The density transformation formula can also be expressed by*

$$\frac{d\mathbb{P}_X|_{\mathcal{F}_T}}{d\mathbb{P}_Y|_{\mathcal{F}_T}}(Y(\cdot)) = \exp\left(\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)} - 1 - \theta(x, s)e^{\theta(x,s)}\right) \Lambda_X(dx, ds) - \int_0^T \int_{\mathbb{R}} \theta(x, s)\bar{N}_Y(dx, ds)(\cdot)\right) \quad \mathbb{P}_Y\text{-a.s.} \tag{11}$$

2.3 One-sided exit problem with a moving boundary for Brownian motion

Below, we present a lemma which deals with the one-sided exit problem for Brownian motion including a special kind of boundaries needed in the main proofs.

Lemma 11. *Let $T > 1$ and $c > 0$ be a constant. Let $(B(t))_{t\geq 0}$ be a Brownian motion. Define the function*

$$h_T(t) := \max\left\{(\ln T)^5, t^{3/4}\right\}$$

and the event

$$E := \{B(t) \leq c \cdot h_T(t), \quad t \in [0, T]\}.$$

Then, we have

$$\mathbb{P}\left(E^c\right) \lesssim e^{-(\ln T)^2/4}, \qquad \textit{as } T \to \infty.$$

Proof. First, note that $h_T(t) \geq g_T(t) := (\ln T) t^{6/10}$ for $t \geq 0$.

Define the event $\tilde{E}$ by

$$\tilde{E} := \left\{B(t) \geq c \cdot g_T(t), \quad t \in [0,T]\right\}.$$

Furthermore, denote by Φ the standard normal distribution function. Applying Theorem 4 and Example 7 in [21] it follows that

$$\mathbb{P}\left(E^c\right) \leq \mathbb{P}\left(\tilde{E}^c\right) \lesssim 4\left(\Phi\left((\ln T) T^{\frac{1}{10}}\right) - \Phi(\ln T)\right) \leq \frac{\sqrt{2}}{\sqrt{\pi}} e^{-(\ln T)^2/4},$$

for T sufficiently large, which completes the proof. $\square$

2.4 One-sided exit problem for Lévy processes

First, we study the asymptotic behavior of the first passage time over a constant boundary. If Spitzer's condition holds, then [17], Lemma 2, proves a similar result.

Lemma 12. *Let X be a Lévy process with Lévy triplet (σ^2, b, ν). Let $\delta \geq 0$ and $0 < c < \infty$. We have*

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta + o(1)}$$

if and only if

$$\mathbb{P}(X(t) \leq c,\ 0 \leq t \leq T) = T^{-\delta + o(1)}.$$

Proof. Case 1: Let $c > 1$. On one hand, we have

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) \leq \mathbb{P}(X(t) \leq c,\ 0 \leq t \leq T).$$

On the other hand, let $2 \leq \lceil c \rceil := n \in \mathbb{N}$. Then,

$$p_c(T) := \mathbb{P}(X(t) \leq c,\ 0 \leq t \leq T) \leq \mathbb{P}(X(t) \leq n,\ 0 \leq t \leq T).$$

Define $\tau_n := \inf\{t \geq 0 : X(t) > n-1\}$ and let F_{τ_n} be the associated distribution function. The stationary and independent increments imply, for every $n \geq 2$,

$$\begin{aligned} p_n(T) &\leq p_{n-1}(T) + \int_0^T p_1(T-s) dF_{\tau_n}(s) \\ &\leq p_{n-1}(T) + p_1(T/2) \int_0^{T/2} dF_{\tau_n}(s) + \int_{T/2}^T dF_{\tau_n}(s) \leq 3 p_{n-1}(T/2). \end{aligned}$$

Thus,

$$p_c(T) \leq p_n(T) \leq 3^{n-1} p_1(T/2^{n-1}).$$

Case 2: Now, let $0 < c < 1$. Then, on one hand, we have

$$\mathbb{P}(X(t) \leq c,\ 0 \leq t \leq T) \leq \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T),$$

and, on the other hand, analogously to Case 1 we obtain that

$$p_1(T) = \mathbb{P}\left(\tfrac{1}{c}X(t) \leq \tfrac{1}{c},\ 0 \leq t \leq T\right) \leq d_1 \mathbb{P}\left(\tfrac{1}{c}X(t) \leq 1,\ 0 \leq t \leq d_2 T\right) = d_1 p_c(d_2 T),$$

where $d_1, d_2 > 0$ are dependent of c; and the lemma is proved. □

The following theorem provides a technique to decouple the one-sided boundary problem over different intervals.

Lemma 13. *Let X be a Lévy process with triplet (σ^2, b, ν) and $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Let $0 \leq a < b < c$. Then,*

$$\mathbb{P}\left(X(t) \leq f(t),\ a \leq t \leq c\right) \geq \mathbb{P}\left(X(t) \leq f(t),\ a \leq t \leq b\right) \cdot \mathbb{P}\left(X(t) \leq f(t),\ b \leq t \leq c\right).$$

Proof. For any choice of n and $0 \leq t_1 < ... < t_n$ the random variables $(X(t_i))_{i=1}^n$ are associated (cf. [14]), since they are sums of independent random variables. Hence, the functions $\mathbf{1}_{\{X(t) \leq f(t),\ a \leq t \leq b\}}$ and $\mathbf{1}_{\{X(t) \leq f(t),\ b \leq t \leq c\}}$ can both be written as limits of decreasing functions of associated random variables and are thus also associated. Hence, we obtain the desired assertion. □

Furthermore, we need a result for one-sided exit problem with a boundary that is an increasing function of T.

Lemma 14. *Let X be a Lévy process with Lévy triplet (σ^2, b, ν). Then we have, for T sufficiently large,*

$$\begin{aligned}&\mathbb{P}(X(t) \leq 3,\ 0 \leq t \leq T) \\ &\quad \geq \tfrac{1}{2}\mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \cdot \mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 1 \leq t \leq T\right).\end{aligned}$$

Proof. Due to the stationary and independent increments of $(X(t))_{t \geq 0}$ we have, for $T > 5$,

$$\begin{aligned}&\mathbb{P}(X(t) \leq 3,\ 0 \leq t \leq T) \\ &\quad \geq \mathbb{P}\left(\{X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\} \cap \{X(t) \leq 3,\ (\ln T)^{21} \leq t \leq T\}\right) \\ &\quad \geq \mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \\ &\qquad \cdot \mathbb{P}\left(X(t) - X((\ln T)^{21}) \leq 3 + (\ln T)^6,\ (\ln T)^{21} \leq t \leq T\right) \\ &\quad \geq \mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \cdot \mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq T\right),\end{aligned}$$

where we used that $(\ln T)^7 \geq 3 + (\ln T)^6$, for $T > 5$. Lemma 13 yields

$$\mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq T\right) \geq \tfrac{1}{2}\mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 1 \leq t \leq T\right),$$

since $\mathbb{P}(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq 1) > \frac{1}{2}$, for T sufficiently large. □

Here, we show that, if the boundary is equal to t^α, $\alpha > 1/2$ then the probability of the one-sided exit problem for a Lévy martingale with $\mathbb{E}(|X(1)|^q) < \infty$, for some $q > 4$, over the boundary t^α is larger than a constant.

Lemma 15. *Let X be a Lévy martingale with $\mathbb{E}(|X(1)|^q) < \infty$, for some $q > 4$. Then, for any $\alpha > 1/2$,*

$$\mathbb{P}(X(t) \le t^\alpha,\ 1 \le t \le T) \gtrsim c, \qquad \text{as } T \to \infty,$$

where $c > 0$ is a constant depending only on X and α.

Proof. First note that there exists a $\varepsilon > 0$ such that $q > 2(1+\varepsilon) + 2$. Since $\alpha > 1/2$ there exists $\beta > 0$ such that $\alpha - \beta - \frac{1}{2} > 0$. Choose $K := K(X, \alpha, \beta) > 0$ independent of T such that $K \ge 2^{1/\beta}$ and

$$\sum_{n=K}^{\infty} \left[\frac{\sqrt{2}}{3\sqrt{\pi}} \cdot n^{-(1+\varepsilon)} + 2^{-(1+\varepsilon)/\alpha} \mathbb{E}\left(X(1)^{(1+\varepsilon)/\alpha} \right) \cdot n^{-(1+\varepsilon)} \right] \le \frac{1}{2}. \tag{12}$$

Then, Lemma 13 yields

$$\begin{aligned} g(T) &:= \mathbb{P}(X(t) \le t^\alpha,\ 1 \le t \le T) \\ &\ge g(K) \cdot (1 - \mathbb{P}(\exists\ s \in [K,T] : X(s) > s^\alpha)) \\ &\ge g(K) \cdot \left(1 - \sum_{n=K}^{T} \mathbb{P}(\exists\ s \in (n, n+1] : X(s) > s^\alpha) \right). \end{aligned} \tag{13}$$

On the other hand, due to the stationary and independent increments we obtain, for all $n \ge K$,

$$\begin{aligned} &\mathbb{P}(\exists\ s \in (n, n+1] : X(s) > s^\alpha) \\ &\quad \le \mathbb{P}\left(X(n) \ge n^{\alpha-\beta} \right) + \mathbb{P}\left(\{X(n) < n^{\alpha-\beta}\} \cap \{\exists\ s \in (n, n+1] : X(s) > s^\alpha\} \right) \\ &\quad \le \mathbb{P}\left(X(n)/\sqrt{n} \ge n^{\alpha-\beta-1/2} \right) + \mathbb{P}\left(\exists\ s \in (n, n+1] : X(s) - X(n) > s^\alpha - n^{\alpha-\beta} \right) \\ &\quad \le \mathbb{P}\left(X(n)/\sqrt{n} \ge 3\sqrt{\ln n} \right) + \mathbb{P}\left(\exists\ s \in (n, n+1] : X(s) - X(n) > \tfrac{1}{2} n^\alpha \right) \\ &\quad \le \frac{\sqrt{2}}{3\sqrt{\pi \ln n}} \cdot n^{-(1+\varepsilon)} + \mathbb{P}\left(\exists\ s \in (0,1] : X(s) > \tfrac{1}{2} n^\alpha \right) \\ &\quad \le \frac{\sqrt{2}}{3\sqrt{\pi}} \cdot n^{-(1+\varepsilon)} + 2^{-(1+\varepsilon)/\alpha} \mathbb{E}\left(X(1)^{(1+\varepsilon)/\alpha} \right) \cdot n^{-(1+\varepsilon)}, \end{aligned} \tag{14}$$

where we used in the second last step a result of [32], page 254, and in the last step Doob's martingale inequality. Putting (14) and (12) into (13) yields

$$g(T) \ge g(K)/2 > 0,$$

which proves the lemma. □

2.5 Coupling

With the help of a coupling method we also obtain an upper bound for the one-sided exit problem for a Lévy martingale with some finite exponential moment.

Lemma 16. *Let $c > 0$. Let X_1 and X_2 be two independent Lévy processes, where X_2 is a martingale with some finite exponential moment, i.e. $\mathbb{E}\left(e^{b|X(1)|}\right) < \infty$, for some $b > 0$. Furthermore, let $\mathbb{E}\left(X_2(1)^2\right) = a$. Let B be a Brownian motion and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that there exists a constant $d > 0$ with $f(T) \le d \cdot T$, for T sufficiently large. Then there is a $\kappa_c > 0$ depending on c such that, for T sufficiently large,*

$$\begin{aligned}
&\mathbb{P}\Big(X_1(t) + X_2(f(t)) \le 1,\ 1 \le t \le T\Big) \\
&\le \mathbb{P}\Big(X_1(t) + aB(f(t)) \le 1 + \kappa_c \ln T,\ 1 \le t \le T\Big) + T^{-c}.
\end{aligned}$$

Proof. Since X_2 has some finite exponential moment and $\mathbb{E}X_2(1)^2 = a$, one can couple it with a Brownian motion aB (compare to the Komlós-Major-Tusnády coupling (KMT theorem), [22]) in such a way that, for a suitable $\kappa_c > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T} |X_2(t) - aB(t)| > \kappa_c \log T\right) \le T^{-c/2}.$$

Since $f(T) \le d \cdot T$, for T sufficiently large, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{1\le t\le T} |X_2(f(t)) - aB(f(t))| > \kappa_c \log T\right) \\
&\quad \le \mathbb{P}\left(\sup_{0\le t\le dT} |X_2(t) - aB(t)| > \kappa_c \log T\right) \le d^{-c}T^{-c/2}. \qquad (15)
\end{aligned}$$

Define

$$A := \left\{\sup_{1\le t\le T} |X_2(f(t)) - aB(f(t))| \le \kappa_c \log T\right\}$$

to be the set where the coupling works. Then, by inequality (15), for T sufficiently large,

$$\begin{aligned}
&\mathbb{P}\Big(X_1(t) + X_2(f(t)) \le 1,\ 1 \le t \le T\Big) \\
&\le \mathbb{P}\Big(X_1(t) + X_2(f(t)) \le 1,\ 1 \le t \le T; A\Big) + \mathbb{P}\Big(A^c\Big) \\
&\le \mathbb{P}\Big(X_1(t) + aB(f(t)) \le 1 + \kappa_c \ln T,\ 1 \le t \le T\Big) + T^{-c},
\end{aligned}$$

which completes the proof. □

3 Proof of Theorem 1 (negative boundary)

Since $f(t)$ is positive, our quantity is trivially bounded from above as follows

$$\mathbb{P}(X(t) \le 1 - f(t),\ 0 \le t \le T) \le \mathbb{P}(X(t) \le 1,\ 0 \le t \le T) = T^{-\delta+o(1)}.$$

We have divided the proof of the lower bound into a sequence of subsections.

3.1 External iteration

In this section we provide an iteration method in order to apply the results of Section 3.2. This additional step is required because of technical details in Section 3.2 which contains the main idea of this proof. Therefore, define, for any $T > 1$,

$$G(T) := \mathbb{P}(X(t) \le 1 - f(t),\ \ln T \le t \le T).$$

In Section 3.2 we will prove that

$$G(T) \ge T^{-\delta+o(1)} \cdot G(\ln T), \qquad \text{for all } T > 1. \tag{16}$$

Recall that $\ln^*(T)$ is the number of times the logarithm function must be iteratively applied before the result is less than or equal to one. Denote $\ln^n(T)$ the n-times iteratively applied logarithm and $\ln^0(T) := T$. Moreover, note that $\ln^*(T)$ decays slower than $\ln^k(T)$, for every k.

Lemma 13 yields

$$\begin{aligned}\mathbb{P}(X(t) \le 1 - f(t),\ 1 \le t \le T) &\ge G(1) \cdot G(\ln^{\ln^*(T)-1}(T)) \cdot ... \cdot G(\ln T) \cdot G(T)\\ &= G(1) \prod_{k=0}^{\ln^*(T)-1} G(\ln^k(T)).\end{aligned}$$

Combining this with (16) and the fact that $\ln^*(T) \le \ln^3(T)$, for T sufficiently large we obtain that

$$\begin{aligned}&\mathbb{P}\Big(X(t) \le 1 - f(t),\ 1 \le t \le T\Big)\\ &\ge G(1) \left(\prod_{k=1}^{\ln^*(T)-1} G(\ln^k(T)) \right) \cdot G(\ln T) \cdot T^{-\delta+o(1)}\\ &\ge G(1)^{\ln^*(T)+1} \left(\prod_{k=1}^{\ln^*(T)-1} \prod_{j=k}^{\ln^*(T)-1} \left(\ln^j(T)\right)^{-\delta+o(1)} \right) \cdot T^{-\delta+o(1)}\\ &\ge G(1)^{\ln^*(T)+1} \left(\prod_{k=1}^{\ln^*(T)-1} \left((\ln^k(T))^{-\delta+o(1)}\right)^{\ln^*(T)-k} \right) \cdot T^{-\delta+o(1)}\\ &\ge G(1)^{\ln^*(T)+1} \left(\prod_{k=1}^{\ln^*(T)-1} \left(\ln^k(T)\right)^{-(\ln^*(T)-k)\delta+(\ln^*(T)-k)o(1)} \right) \cdot T^{-\delta+o(1)}\\ &= T^{-\delta+o(1)},\end{aligned}$$

and this is precisely the assertion of the theorem.

3.2 Internal iteration; proof of (16)

3.2.1 Preliminaries

We can assume that $T > 1$ during the further progress of the proof.

Auxiliary function H for the iteration: We define

$$H(x) := x \exp\left(-\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/x)} - c_2 ||f'||^2_{L_2[1,\infty)}\right), \text{ for } x > 0,$$

where $c_1, c_2 > 0$ are constants depending on ν and f specified later. Note that $H'(x) > 0$, for all x sufficiently small. Next, define H^i_β by $H^0_\beta(x) := x$ and, for $i \geq 1$,

$$H^i_\beta(x) := H^{i-1}_\beta\Big(H(x \cdot \beta)\Big)$$

with $0 < \beta < 1$ specified later. Note that H^i_β is well defined since $T > 1$ and $H(x) > 0$, for $x > 0$.

Auxiliary function f_n for the iteration: We define $f_0(t) := \max\{f(\ln T), f(t)\}$ and, for $n \geq 1$,

$$f_n(t) := \max\Big\{1, (f_{n-1}(t) - f_{n-1}(\ln T))^{2/3}\Big\} + f_{n-1}(\ln T).$$

Furthermore, define $\tilde{t}_n = \sup\{s \geq 0 : f_{n-1}(s) - f_{n-1}(\ln T) \leq 1\}$. Note that $f_n'(t) = 0$, for $t \in (0, \tilde{t}_n)$, and

$$f_n'(t) = \tfrac{2}{3}\left(f_{n-1}(t) - f_{n-1}(\ln T)\right)^{-2/3} f_{n-1}'(t) \qquad \text{a.e., for } t > \tilde{t}_n,$$

thus,

$$0 \leq f_n'(t) \leq f'(t) \text{ a.e.,} \tag{17}$$

since $f' \geq 0$. In the following proof we use

$$f_n(t) \leq f(\ln T) + n + \max\{1, f(t)^{(2/3)^n}\}, \qquad \text{for all } t \geq 0, \tag{18}$$

which can be proved by induction.

3.2.2 Iteration rule

3.2.2.1 Goal setting In order to apply the iteration, our next goal is to show that, for every $n \in \mathbb{N}$,

$$g_n(T) \geq H(g_{n+1}(T) \cdot \beta) \tag{19}$$

holds, where

$$g_n(T) := \mathbb{P}(X(t) \leq 1 - f_n(t),\ \ln T \leq t \leq T).$$

3.2.2.2 Change of measure The aim of this subsection is to show the following inequality:

$$\begin{aligned} g_n(T) &\geq \mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T) \\ &\quad \cdot \exp\left(-\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/\mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T))} - c_2 ||f'||^2_{L_2[1,\infty)}\right) \\ &= H\left(\mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T)\right), \end{aligned} \tag{20}$$

where $c_1, c_2 > 0$ are constants depending on ν and f that are chosen later on.

Without loss of generality let $\nu([-1, 0)) > 0$. If $\nu([-1, 0)) = 0$ then we multiply X by $d > 0$ suitably chosen such that $\tilde{\nu}([-1, 0])) > 0$, where $\tilde{\nu}$ is the Lévy measure of $d \cdot X$.

Such a $d > 0$ exists since $\nu(\mathbb{R}_-) > 0$. Due to Lemma 12 we can continue with the process $d \cdot X$ instead of X in the same manner.

Since $\nu([-1,0)) > 0$ we can choose a compact set $A \subseteq [-1,0)$ bounded away from zero such that

$$0 < \int_A x^2 \nu(dx) =: m < \infty.$$

Let $\tilde{X}_n$ and Y_n be two additive processes with triplets $(\sigma^2, f_{\tilde{X}_n}(t), \nu(dx)ds)$ and $(\sigma^2, f_{Y_n}(t), (1 + \frac{f_n'(t)|x|}{m}\mathbf{1}_{\{x\in A\}})\nu(dx)ds)$ respectively, where $f_{Y_n}(t) := b \cdot t + f_n(\ln T)$ and $f_{\tilde{X}_n}(t) := b \cdot t + f_n(t)$.

Then, $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous because of the following considerations. Define $\theta(x,s) := \ln(1 + \frac{f_n'(s)|x|}{m}\mathbf{1}_{\{x\in A\}})$, for all $s \in [0,T]$ and $x \in \mathbb{R}$. We have, for $t > \ln T$,

$$\begin{aligned} f_{Y_n}(t) &= bt + f_n(\ln T) = bt + f_n(t) - \int_{\ln T}^t f_n'(s)ds \\ &= f_{\tilde{X}_n}(t) + \int_0^t \int_{|x|\le 1} \left(e^{\theta(x,s)} - 1\right) x\nu(dx)ds \end{aligned}$$

and since $f_n(t) = f_n(\ln T)$, for $t \in [0, \ln T]$,

$$f_{Y_n}(t) = bt + f_n(\ln T) = bt + f_n(t) = f_{\tilde{X}_n}(t).$$

In this connection, one should point out that $-f_n'(s)x\mathbf{1}_{\{x\in A\}} = f_n'(s)|x|\mathbf{1}_{\{x\in A\}} \ge 0$ almost everywhere.

Define $\Lambda_{Y_n}(dx,ds) := \exp(\theta(x,s))\nu(dx)ds$. According to the choice of the Lévy measures, $\nu(dx)ds$ and $\Lambda_{Y_n}(dx,ds)$ are absolutely continuous with $\frac{d\Lambda_{Y_n}(x,s)}{\nu(dx)ds} = e^{\theta(x,s)}$. In order to apply Theorem 9 we have to check $\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)/2} - 1\right)^2 \nu(dx)ds < \infty$. We know from [35], Remark 33.3, that this condition is equivalent to the following three properties combined

1. $\int_{\{(x,s):\theta(x,s)<-1\}} \nu(dx)ds < \infty$,
2. $\int_{\{(x,s):\theta(x,s)>1\}} e^{\theta(x,s)}\nu(dx)ds < \infty$, and
3. $\int_{\{(x,s):|\theta(x,s)|\le 1\}} \theta^2(x,s)\nu(dx)ds < \infty$.

Since $f_n' \ge 0$, thus $\theta \ge 0$; it is left to prove 2. and 3.

Case 2.: Since $\theta > 1$ and A bounded away from zero, we have

$$\int_{\{(x,s):\theta(x,s)>1\}} e^{\theta(x,s)}\nu(dx)ds \le \int_1^T \int_A (1 + \frac{f_n'(s)|x|}{m})\nu(dx)ds < \infty.$$

Case 3.: Since $\ln(1+z) \le z$, for all $z > -1$, and inequality (17) we get

$$\int_{\{(x,s):|\theta(x,s)|\le 1\}} (\theta(x,s))^2\nu(dx)ds \le \frac{1}{m^2}\int_1^T \int_A (f_n'(s))^2 x^2 \nu(dx)ds = \frac{1}{m}||f_n'||^2_{L_2[1,T]} < \infty.$$

Hence, due to Theorem 9 $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous.

Next, we show inequality (20).

Note that $\theta(x,s) = 0$, for $s \in [0, \ln T)$ and all $x \in \mathbb{R}$. Because of Theorem 9 and the density transformation formula (11) we obtain that

$$\begin{aligned}
&\mathbb{P}(\tilde{X}_n(t) \le 1,\ \ln T \le t \le T) = \mathbb{E}_{\tilde{X}_n}(\mathbf{1}_{\{\tilde{X}_n(t)\le 1,\ \ln T \le t \le T\}}) \\
&= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\le 1,\ \ln T \le t \le T\}} e^{-\int_{\ln T}^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right) \cdot e^{\int_{\ln T}^T \int_{\mathbb{R}} \left(e^{\theta(x,s)} - 1 - \theta(x,s)e^{\theta(x,s)}\right)\nu(dx)ds} \\
&= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\le 1,\ \ln T \le t \le T\}} e^{-\int_{\ln T}^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right) \cdot e^{-\int_{\ln T}^T \int_{\mathbb{R}} g\left(\frac{f_n'(s)|x|}{m}\mathbf{1}_{x\in A}\right)\nu(dx)ds},
\end{aligned} \tag{21}$$

where $g(u) := (1+u)\ln(1+u) - u$, $u > 0$. For $u \ge 0$ bounded away from infinity, we have with a constant $\tilde{c}_1 > 0$, $g(u) \le \tilde{c}_1 u^2$ because of Taylor's expansion. Hence, since f_n' and A is bounded away from $-\infty$, we get

$$\begin{aligned}
&e^{-\int_{\ln T}^T \int_{\mathbb{R}} g\left(\frac{f_n'(s)|x|}{m}\mathbf{1}_{x\in A}\right)\nu(dx)ds} \ge e^{-\tilde{c}_1 \int_{\ln T}^T \int_{\mathbb{R}} \frac{f_n'(s)^2 x^2}{m^2}\mathbf{1}_{x\in A}\nu(dx)ds} \\
&= e^{-\tilde{c}_1 \int_{\ln T}^T f_n'(s)^2 ds \cdot \int_A \frac{x^2}{m^2}\nu(dx)} = e^{-\frac{\tilde{c}_1}{m}||f_n'||^2_{L_2[\ln T, T]}} \ge e^{-\frac{\tilde{c}_1}{m}||f'||^2_{L_2[1,\infty)}},
\end{aligned}$$

having used (17). Let $p > 1$. Using the last estimate and the reverse Hölder inequality in (21) yields that

$$\begin{aligned}
&\mathbb{P}(\tilde{X}_n(t) \le 1,\ \ln T \le t \le T) \\
&\ge \exp\left(-\frac{\tilde{c}_1}{m}||f_n'||^2_{L_2[1,\infty)}\right)(\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T))^p \\
&\cdot \left(\mathbb{E}_{Y_n}\left(e^{\frac{1}{p-1}\int_{\ln T}^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right)\right)^{-(p-1)}.
\end{aligned} \tag{22}$$

Furthermore, we have due to the density transform formula (10)

$$\begin{aligned}
&\left(\mathbb{E}_{Y_n}\left(e^{\frac{1}{p-1}\int_{\ln T}^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right)\right)^{-(p-1)} \\
&= \left(\mathbb{E}_{\tilde{X}_n}\left(e^{\int_{\ln T}^T \int_{\mathbb{R}} \frac{1}{p-1}\theta(x,s)(N(dx,ds) - \Lambda_{Y_n}(dx,ds)) + \theta(x,s)(N(dx,ds) - \nu(dx)ds)}\right)\right)^{-(p-1)} \\
&\cdot \left(e^{-\int_1^T \int_{\mathbb{R}} (e^{\theta(x,s)} - 1 - \theta(x,s))\nu(dx)ds}\right)^{-(p-1)} \\
&= \left(\mathbb{E}_{\tilde{X}_n}\left(e^{\int_{\ln T}^T \int_{\mathbb{R}} (\frac{1}{p-1}+1)\theta(x,s)(N(dx,ds) - \nu(dx)ds)}\right)\right)^{-(p-1)} \\
&\cdot \left(\exp\left(\int_{\ln T}^T \int_{\mathbb{R}} (\frac{\theta(x,s)}{p-1} - \frac{\theta(x,s)}{p-1}e^{\theta(x,s)} - e^{\theta(x,s)} + 1 + \theta(x,s))\nu(dx)ds\right)\right)^{-(p-1)} \\
&= \left(\exp\left(\int_{\ln T}^T \int_{\mathbb{R}} (e^{(\frac{1}{p-1}+1)\theta(x,s)} - 1 - (\tfrac{1}{p-1}+1)\theta(x,s))\nu(dx)ds\right)\right)^{-(p-1)} \\
&\cdot \left(\exp\left(\int_{\ln T}^T \int_{\mathbb{R}} (\frac{\theta(x,s)}{p-1} - \frac{\theta(x,s)}{p-1}e^{\theta(x,s)} - e^{\theta(x,s)} + 1 + \theta(x,s))\nu(dx)ds\right)\right)^{-(p-1)} \\
&= \exp\left((p-1)\int_{\ln T}^T \int_{\mathbb{R}} e^{\theta(x,s)}(-e^{\frac{1}{p-1}\theta(x,s)} + 1 + \tfrac{1}{p-1}\theta(x,s))\nu(dx)ds\right),
\end{aligned}$$

where we used in the third step a modification of Lemma 33.6 of [35]. The difference between [35] and our case consists in the consideration of time-inhomogeneous processes in contrast to time-homogeneous processes used in [35]. Next, define $w(x) := 1 + x - e^x$, for all $x \geq 0$. Assume for a moment that $p > 1$ is chosen such that

$$\tfrac{1}{p-1}\theta(x,s), \qquad \text{for all } x \in \mathbb{R} \text{ and } s \in [\ln T, T], \tag{23}$$

is almost everywhere bounded away from infinity. This boundedness is independent of T and n. Then, there is a constant $\tilde{c}_2 > 0$ such that $w(\frac{1}{p-1}\theta(x,s)) \geq -\tilde{c}_2(\frac{1}{p-1}\theta(x,s))^2$ and hence,

$$\begin{aligned}
&(p-1)\int_{\ln T}^{T}\int_{\mathbb{R}} e^{\theta(x,s)}(-e^{\frac{1}{p-1}\theta(x,s)} + 1 + \tfrac{1}{p-1}\theta(x,s))\nu(dx)ds \\
&\geq -\frac{\tilde{c}_2}{(p-1)}\int_{\ln T}^{T}\int_{\mathbb{R}}(\theta(x,s))^2\nu(dx)ds \geq -\frac{\tilde{c}_2}{(p-1)m^2}\int_{\ln T}^{T}(f_n'(s))^2 ds \cdot \int_A x^2\nu(dx) \\
&\geq -\frac{\tilde{c}_2}{(p-1)m}||f'||^2_{L_2[1,\infty)},
\end{aligned}$$

where we used in the last step again inequality (17). Putting this into (22) implies

$$\begin{aligned}
&\mathbb{P}(\tilde{X}_n(t) \leq 1,\ \ln T \leq t \leq T) \\
&\quad \geq \mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T)^p \cdot \exp\left((-\frac{\tilde{c}_1}{m} - \frac{\tilde{c}_2}{(p-1)m})||f'||^2_{L_2[1,\infty)}\right).
\end{aligned}$$

Optimizing in p shows that the best choice is

$$p := 1 + \sqrt{\frac{\tilde{c}_2||f'||^2_{L_2[1,\infty)}}{2m\ln(1/\mathbb{P}_{Y_n}(Y_n(t) \leq 1,\ \ln T \leq t \leq T))}} > 1.$$

Using this and choosing c_1, c_2 suitably completes the proof of inequality (20).

It is left in (23) to show that $\frac{1}{p-1}\theta(x,s)$ is almost everywhere bounded away from infinity. More precisely, we will prove $\frac{1}{p-1}f_n'(s) \leq c$ a.e., for $s \in [\ln T, T]$, which follows from

$$\mathbb{P}_{Y_n}(Y_n(t) \leq 1,\ \ln T \leq t \leq T) \geq T^{-d} \quad \text{for some } d > 0, \tag{24}$$

for any $n \in \mathbb{N}$. Indeed, if (24) holds then due to the choice of p we obtain

$$\frac{1}{p-1} = \sqrt{\frac{2m\ln(1/\mathbb{P}_{Y_n}(Y_n(t) \leq 1,\ \ln T \leq t \leq T))}{\tilde{c}_2||f'||^2_{L_2[1,\infty)}}} \leq \sqrt{\frac{2m\ln(T^{-d+o(1)})}{\tilde{c}_2||f'||^2_{L_2[1,\infty)}}} \leq \tilde{c}\cdot\sqrt{\ln T}.$$

Combining this with $f_n'(s)(\ln T)^{1/2} \leq f'(s)(\ln T)^{1/2} \leq c$ a.e., for $s \in [\ln T, T]$ (see (9)) we get $\frac{1}{p-1}f_n'(s) \leq \tilde{c}(\ln T)^{1/2}f'(s) \leq c$ a.e. The proof of (24) can be found in Section 3.2.5.

3.2.2.3 Lower bound for the term in (20). Having deduced (20) we will prove the following lower bound, for any $n \in \mathbb{N}$,

$$\mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T) \geq g_{n+1}(T)\cdot\beta,$$

where $\beta > 0$ is a constant specified later.

Homogenization: We represent the process Y_n as a sum of independent processes $Y_n(\cdot) \stackrel{d}{=} X(\cdot) + Z_n(\cdot) + f_n(\ln T)$, where Z_n is an additive process with triplet $(0, 0, \frac{f_n'(s)|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(dx) ds)$. Due to the Lévy-Khintchine formula and

$$f_n(t) - f_n(\ln T) = \int_{\ln T}^{t} f_n'(s) ds = \int_0^t f_n'(s) ds,$$

there exists a Lévy process $\tilde{Z}$ with triplet $(0, 0, \frac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(dx))$ such that $Z_n(\cdot) = \tilde{Z}(f_n(\cdot) - f_n(\ln T))$ in f.d.d. Note that $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, since A is compact in $(-\infty, 0)$ and the characteristic exponent of $\tilde{Z}$ has the following representation

$$\Psi(u) = \int_{\mathbb{R}} (1 - e^{iux} + iux) \frac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(dx)$$

and Lévy measure satisfying $\int (|x| \wedge x^2) \frac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(dx) < \infty$. Thus,

$$\begin{aligned} &\mathbb{P}(Y_n(t) \leq 1, \ \ln T \leq t \leq T) = \mathbb{P}(X(t) + Z_n(t) \leq 1 - f_n(\ln T), \ \ln T \leq t \leq T) \\ &\quad = \mathbb{P}(X(t) + \tilde{Z}(f_n(t) - f_n(\ln T)) \leq 1 - f_n(\ln T), \ \ln T \leq t \leq T). \end{aligned}$$

Independence: Recall that there exists $\kappa > 0$ such that $f(T) \leq \kappa T$, for T sufficiently large (see (8)). Using the independence of X and $\tilde{Z}$ we can write, for T sufficiently large,

$$\begin{aligned} &\mathbb{P}\left(X(t) + \tilde{Z}(f_n(t) - f_n(\ln T)) \leq 1 - f_n(\ln T), \ \ln T \leq t \leq T\right) \\ &\geq \mathbb{P}\left(X(t) \leq 1 - \max\{1, (f_n(t) - f_n(\ln T))^{2/3}\} - f_n(\ln T), \ \ln T \leq t \leq T\right) \\ &\quad \cdot \mathbb{P}\left(\tilde{Z}(f_n(t) - f_n(\ln T)) \leq \max\{1, (f_n(t) - f_n(\ln T))^{2/3}\}, \ \ln T \leq t \leq T\right) \\ &\geq \mathbb{P}\left(X(t) \leq 1 - f_{n+1}(t), \ \ln T \leq t \leq T\right) \cdot \mathbb{P}\left(\tilde{Z}(t) \leq \max\{1, t^{2/3}\}, \ 0 \leq t \leq \kappa T\right) \\ &= g_{n+1}(T) \cdot \mathbb{P}\left(\tilde{Z}(t) \leq \max\{1, t^{2/3}\}, \ 0 \leq t \leq \kappa T\right) \end{aligned} \tag{25}$$

Since $\tilde{Z}$ is a martingale with some exponential moment and using Lemma 15 we have, for $0 < \beta < 1$ suitably chosen and $\beta = \beta(\tilde{Z})$,

$$\mathbb{P}\left(\tilde{Z}(t) \leq \max\{1, t^{2/3}\}, \ 0 \leq t \leq \kappa T\right) \gtrsim \beta. \tag{26}$$

3.2.2.4 Proof of the iteration rule Plugging (26) and (25) into (20) and using that $H' > 0$ in a neighbourhood of zero we obtain, for any $n \in \mathbb{N}$, that

$$\begin{aligned} g_n(T) &\geq \beta \cdot g_{n+1}(T) \cdot \exp\left(-\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln((\ln T)^3 / g_{n+1}(T) c_3)} - c_2 ||f'||^2_{L_2[1,\infty)}\right) \\ &= H(g_{n+1}(T) \cdot \beta), \end{aligned} \tag{27}$$

which shows inequality (19).

3.2.3 End point of the iteration

The aim of this subsection is to find a number $n(T)$ depending on T such that

$$g_{n(T)}(T) \geq T^{-\delta+o(1)} \cdot G(\ln T). \tag{28}$$

3.2.3.1 Number of iteration steps Our next goal is to set, depending on T, the number of iteration steps such that eventually the boundary is larger than $2 - f(\ln T)$. Recall that $f(T) \leq \kappa T$. We choose, for T sufficiently large,

$$n(T) := \left\lceil \frac{\ln(\ln(\kappa T)/\ln(2))}{\ln(3/2)} \right\rceil,$$

and thus, for T sufficiently large,

$$g_{n(T)}(T) \geq \mathbb{P}(X(t) \leq -1 - f(\ln T) - n(T), \ \ln T \leq t \leq T), \tag{29}$$

since f is non-decreasing and inequality (18) holds. Furthermore, we used that $f(\ln T) > 1$, for T sufficiently large. On the other hand, if $f(\ln T) < 1$, for all T, we have $\sup_{t\geq 0} |f(t)| < \infty$ then applying Lemma 12 already proves the theorem.

3.2.3.2 Asymptotic rate of the end point: Here, we show (28). Recall that $f'(t) \searrow 0$, for $t \to \infty$, and $n(T) \leq b_1(\ln(\ln T))$, for $b_1 > 0$ suitably chosen. Define $k(T) := 2 + f'(1) + b_1 \ln(\ln T)$. Since $(X(t))_{t\geq 0}$ has stationary and independent increments we have due to (29)

$$\begin{aligned}
g_{n(T)}(T) &\geq \mathbb{P}\Big(X(t) \leq -1 - f(\ln T) - n(T), \ \ln T \leq t \leq T\Big) \\
&\geq \mathbb{P}\Big(\{X(t) \leq -1 - f(\ln T) - n(T), \ \ln T \leq t \leq T\} \cap \{X(\ln T - 1) \leq 1 - f(\ln T - 1)\}\Big) \\
&\geq \mathbb{P}\Big(\{X(t) - X(\ln T - 1) \leq -2 - f(\ln T) - f(\ln T - 1) - n(T), \ \ln T \leq t \leq T\} \\
&\quad \cap \{X(\ln T - 1) \leq 1 - f(\ln T - 1)\}\Big) \\
&\geq \mathbb{P}\Big(\{X(t) - X(\ln T - 1) \leq -k(T), \ln T \leq t \leq T\} \cap \{X(\ln T - 1) \leq 1 - f(\ln T - 1)\}\Big) \\
&\geq \mathbb{P}\Big(X(t) \leq -k(T), \ 1 \leq t \leq T - \ln T + 1\Big) \\
&\quad \cdot \mathbb{P}\Big(X(t) \leq 1 - f(t), \ \ln(\ln T) \leq t \leq \ln T - 1\Big) \\
&\geq 3^{-k(T)} \cdot \mathbb{P}\Big(X(t) \leq 1, \ 1 \leq t \leq (T - \ln T + 1) \cdot 2^{k(T)}\Big) \\
&\quad \cdot \mathbb{P}\Big(X(t) \leq 1 - f(t), \ \ln(\ln T) \leq t \leq \ln T\Big) \\
&= T^{-\delta+o(1)} \cdot G(\ln T),
\end{aligned}$$

where the second last step follows analogously to Lemma 12 in spite of the negative boundary since $\nu(\mathbb{R}_-) > 0$ and the considered time interval of the one-sided exit problem does not contain zero. In the last step we used assumption (2). Hence, we have (28).

3.2.4 Application of the iteration

In this subsection we combine inequality (19) with (28) to obtain finally inequality (16).

Since $H' > 0$ in the neighbourhood of zero, inequality (19) implies $g_0(T) \geq H_\beta^{n(T)}(g_{n(T)}(T))$. Our first goal is to calculate $H_\beta^{n(T)}(g_{n(T)}(T))$ with the help of (28). First, we show by induction that

$$H_\beta^n(x) \geq W_n(x) \cdot \exp\left(-n\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(W_n(x)^{-1} \cdot Z_n(x)\right)}\right), \tag{30}$$

for all $n \geq 1$ and $x \in (0, 1]$, where

$$W_n(x) := x \cdot \beta^n \cdot \exp\left(-n \cdot c_2||f'||^2_{L_2[1,\infty)}\right),$$

and

$$Z_n(x) := \exp\left((n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)} 2^{n-2} \ln\left(x^{-1}\beta^{-2}\right)} - c_2||f'||^2_{L_2[1,\infty)}\right).$$

Indeed, we have, for $n = 1$, that

$$H_\beta^1(x) = H(x \cdot \beta) = W_1(x) \cdot \exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\Big((W_1(x))^{-1} Z_1(x)\Big)}\right).$$

Assume now that (30) holds, for $n - 1$. Note that, for x sufficiently small, we have

$$H(x) \geq x^2.$$

First, we get

$$W_{n-1}\Big(H(x \cdot \beta)\Big) = W_n(x) \cdot \exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(x^{-1}\beta^{-1}\right)}\right).$$

Hence, we obtain, for $x \in (0, 1]$, that

$$\begin{aligned}
&W_{n-1}\big(H(x \cdot \beta)\big)^{-1} \cdot Z_{n-1}\left(H(x \cdot \beta)\right) \\
&\leq \frac{1}{W_n(x)} \cdot \exp\left(\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(x^{-1}\beta^{-1}\right)} + (n-2)\sqrt{c_1||f'||^2_{L_2[1,\infty)} 2^{n-3} \ln\left(x^{-2}\beta^{-4}\right)}\right) \\
&\leq \big(W_n(x)\big)^{-1} \cdot Z_n(x),
\end{aligned}$$

since $\beta \leq 1$. This implies, for x sufficiently small,

$$\begin{aligned}
H_\beta^n(x) &= H_\beta^{n-1}\Big(H(x \cdot \beta)\Big) \\
&\geq W_{n-1}\Big(H(x \cdot \beta)\Big) \\
&\quad \cdot \exp\left(-(n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(W_{n-1}\Big(H(x \cdot \beta)\Big)^{-1} Z_{n-1}\Big(H(x \cdot \beta)\Big)\right)}\right) \\
&\geq W_n(x) \cdot \exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(x^{-1}\beta^{-1}\right)}\right) \\
&\quad \cdot \exp\left(-(n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(\Big(W_n(x)\Big)^{-1} Z_n(x)\right)}\right) \\
&\geq W_n(x) \cdot \exp\left(-n\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(\Big(W_n(x)\Big)^{-1} Z_n(x)\right)}\right),
\end{aligned}$$

where we used in the last step that, for $n \geq 2$,

$$\Big(W_n(x)\Big)^{-1} Z_n(x)$$
$$= x^{-1}\beta^{-n} \cdot \exp\Big((n-1)c_2||f'||^2_{L_2[1,\infty)}\Big) \cdot \exp\Big((n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)} 2^{n-2}\ln(x^{-1}\beta^{-2})}\Big)$$
$$\geq x^{-1}\beta^{-1}.$$

Recall that $n(T) \leq b_1\big(\ln(\ln T)\big)$ and $g_{n(T)}(T) \leq T^{-\delta+o(1)}$, for $b_1 > 0$ suitably chosen. Then, we obtain that

$$\begin{aligned} Z_{n(T)}(g_{n(T)}(T)) &\leq \exp\Big(b_1\Big(\ln(\ln T)\Big)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\delta \cdot 2^{-1}(\ln T)^{\ln 2}\ln(T\beta^{-2})}\Big) \\ &\leq \exp\Big(\tilde{b}_1 \cdot (\ln T)^{9/10}\Big) \leq \exp\big(\tfrac{\delta}{2} \cdot (\ln T)\big) \leq T^{\delta - o(1)} \\ &\leq \Big(g_{n(T)}(T)\Big)^{-1}, \end{aligned} \tag{31}$$

for T sufficiently large and $\tilde{b}_1 > 0$ suitably chosen, and

$$\exp\Big(-n(T) \cdot c_2||f'||^2_{L_2[1,\infty)}\Big) = T^{o(1)} \geq \exp\Big(-\tfrac{\delta}{2} \cdot (\ln T)\Big) \geq g_{n(T)}(T). \tag{32}$$

Putting (31) and (32) into (30) we obtain, for $b_2 > 0$ suitably chosen, that

$$\begin{aligned} H_\beta^{n(T)}(g_{n(T)}(T)) &\geq g_{n(T)}(T) \cdot \beta^{n(T)} \cdot \exp\Big(-n(T) \cdot c_2||f'||^2_{L_2[1,\infty)}\Big) \\ &\quad \cdot \exp\Big(-n(T)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\big(g_{n(T)}(T)^{-3}\beta^{-n(T)}\big)}\Big) \\ &\geq g_{n(T)}(T) \cdot \exp\Big(-b_2(\ln T)^{6/10}\Big) \\ &\geq g_{n(T)}(T) \cdot T^{o(1)}. \end{aligned}$$

Combining this with (28) and an $n(T)$-times iteration of (27) yields

$$g_0(T) = \mathbb{P}(X(t) \leq 1 - f(t),\ \ln T \leq t \leq T) \geq H_\beta^{n(T)}(g_{n(T)}(T)) = T^{-\delta+o(1)} \cdot G(\ln T),$$

which completes the proof of (16) provided (24) holds.

3.2.5 Proof of (24)

In this section, we prove (24).

For this purpose, we represent the process as a sum of independent processes $Y_n(\cdot) \overset{d}{=} X(\cdot) + S_n(\cdot) + f_n(\ln T)$, where X is the original Lévy process with triplet $(\sigma^2, b, \nu(dx))$, S_n is an additive process with triplet $(0, 0, \frac{f_n'(s)|x|}{m}\mathbf{1}_{\{x\in A\}}\nu(dx)ds)$. Again, by homogenization there exists a Lévy process $\tilde{S}$ with triplet $(0, 0, \frac{|x|}{m}\mathbf{1}_{\{x\in A\}}\nu(dx))$ such that $S_n(\cdot) = \tilde{S}(f_n(\cdot) - f_n(\ln T))$ f.d.d. Note that $\tilde{S}$ is a martingale with some finite exponential moment since A is bounded away from minus infinity.

Since $f_n(\ln T) \leq \kappa \ln T$ then according to Lemma 12 we have, for T sufficiently large,

$$\mathbb{P}(X(t) \leq -f_n(\ln T),\ \ln T \leq t \leq T) \geq T^{-\kappa \ln 3} \cdot \mathbb{P}(X(t) \leq 1,\ \ln T \leq t \leq T^{1+\ln 2}).$$

Combining this with the indendence of X and $\tilde{S}$ yields

$$\begin{aligned}
&\mathbb{P}\Big(Y_n(t) \le 1,\ \ln T \le t \le T\Big) \\
&\quad = \mathbb{P}\Big(X(t) + \tilde{S}(f_n(t) - f_n(\ln T)) + f_n(\ln T) \le 1,\ \ln T \le t \le T\Big) \\
&\quad \ge \mathbb{P}\Big(X(t) \le -f_n(\ln T),\ \ln T \le t \le T\Big) \cdot \mathbb{P}\Big(\tilde{S}(f_n(t) - f_n(\ln T)) \le 1,\ \ln T \le t \le T\Big) \\
&\quad \ge T^{-\kappa \ln 3} \cdot \mathbb{P}\Big(X(t) \le 1,\ \ln T \le t \le T^{1+\ln 2}\Big) \cdot \mathbb{P}\Big(\tilde{S}(t) \le 1,\ 0 \le t \le f_n(T) - f_n(\ln T)\Big) \\
&\quad \ge T^{-\kappa \ln 3} \cdot \mathbb{P}\Big(X(t) \le 1,\ 1 \le t \le T^2\Big) \cdot \mathbb{P}\Big(\tilde{S}(t) \le 1,\ 0 \le t \le \kappa T\Big) \\
&\quad \ge T^{-2\delta - 1/2 - \kappa \ln 3 + o(1)},
\end{aligned}$$

where we used in the last step the fact that the survival exponent of a Lévy martingale with finite variance is equal to 1/2 (see [15], Chapter XII).

4 Proof of Theorem 2 (positive boundaries)

Since f is positive, our quantity is trivially bounded from below as follows

$$\mathbb{P}(X(t) \le 1 + f(t),\ 0 \le t \le T) \ge \mathbb{P}(X(t) \le 1,\ 0 \le t \le T) = T^{-\delta + o(1)}.$$

Our goal is to show

$$\mathbb{P}(X(t) \le 1 + f(t),\ 0 \le t \le T) \le T^{-\delta + o(1)}. \tag{33}$$

The proof of the upper bound is divided into a sequence of subsections.

4.1 Preliminaries

In the following proof we can assume that $T > 1$. We can write

$$\mathbb{P}(X(t) \le 1 + f(t),\ 0 \le t \le T) \le \mathbb{P}(X(t) \le 1 + f(t),\ 1 \le t \le T).$$

Hence, as from now we consider the time interval $[1, T]$.
Auxiliary function H for the iteration: We define

$$H(x) := x \exp\left(\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/x)}\right), \quad x > 0.$$

Note that $H'(x) > 0$, for all x sufficiently small. Furthermore, define $H_2^0(x) := H(2x)$ and, for $i \ge 1$,

$$H_2^i(x) := H\big(2H_2^{i-1}(x)\big).$$

H_2^i is well defined since $H(x) > 0$, for $x > 0$.
Auxiliary function f_n for the iteration: Define $f_0(t) := \max\{f(\ln T), f(t)\}$ and, for $n \ge 1$, $f_n(t) := f(\ln T) + n\kappa_\delta \ln T + n(\ln T)^5$, for $t \le \ln T$, and, for $t > \ln T$,

$$f_n(t) := f_{n-1}(\ln T) + n\kappa_\delta \ln T + \max\left\{(\ln T)^5, (f_{n-1}(t) - f_{n-1}(\ln T))^{3/4}\right\},$$

where $\kappa_\delta > 0$ is constant specified later. By induction it follows, for $t > \ln T$ and $n \geq 0$, that

$$f_n(t) \leq f(\ln T) + n\kappa_\delta \ln T + (n-1)(\ln T)^5 + \max\left\{(\ln T)^5, f(t)^{(3/4)^n}\right\}. \qquad (34)$$

Furthermore, define $\tilde{t}_{T,n} := \inf\{t \geq 0 : (\ln T)^5 < (f_{n-1}(t) - f_{n-1}(\ln T))^{3/4}\}$. Note that, for $n \geq 1$,

$$f_n'(t) = \begin{cases} 0, & t < \tilde{t}_{T,n}, \\ \frac{3}{4}\left(f_{n-1}(t) - f_{n-1}(\ln T)\right)^{-1/4} f_{n-1}'(t), & t > \tilde{t}_{T,n}. \end{cases}$$

Since $(f_{n-1}(t) - f_{n-1}(\ln T))^{3/4} > (\ln T)^5$ we get again by induction

$$f_n'(t) \leq f'(t) \quad \text{a.e.} \qquad (35)$$

Note that $\tilde{t}_{T,n}$ is non-decreasing in n. Without loss of generality we can assume that $\tilde{t}_{T,n} \geq 1$, for all $n > 0$ and T sufficiently large. Otherwise, we choose T sufficiently large such that $(f_{n-1}(1) - f_{n-1}(\ln T))^{3/4} < (\ln T)^5$ and thus, $\tilde{t}_{T,n} \geq 1$.

4.2 Iteration rule

4.2.1 Goal setting

Our first goal is to show that, for every $n \in \mathbb{N}$,

$$g_n(T) \leq H\big(2g_{n+1}(T)\big) \qquad (36)$$

holds, where

$$g_n(T) := \mathbb{P}(X(t) \leq 1 + f_n(t),\ 1 \leq t \leq T).$$

4.2.2 Change of measure

The aim of this subsection is to show the following inequality:

$$\begin{aligned} g_n(T) &\leq \mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T) \\ &\quad \cdot \exp\left(\sqrt{c_1 \|f'\|^2_{L_2[1,\infty)} \ln(1/\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T))}\right) \\ &= H\Big(\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T)\Big), \end{aligned} \qquad (37)$$

where $c_1 > 0$ is a constant depending on ν and f that is chosen later on.

Without loss of generality let $\nu((0,1]) > 0$. If $\nu((0,1]) = 0$ then we multiply X by $d > 0$ suitably chosen such that $\tilde{\nu}((0,1]) > 0$, where $\tilde{\nu}$ is the Lévy measure of $d \cdot X$, such a $d > 0$ exists since $\nu(\mathbb{R}_+) > 0$. Due to Lemma 12 we can continue with the process $d \cdot X$ instead of X.

Since $\nu((0,1]) > 0$, we can choose a compact set $A \subseteq (0,1]$ bounded away from zero such that

$$0 < \int_A x^2 \nu(dx) =: m < \infty.$$

Let $\tilde{X}_n$ and Y_n be two additive processes with triplets $(\sigma^2, f_{\tilde{X}_n}(t), \nu(dx)ds)$ and $(\sigma^2, f_{Y_n}(t), (1+\frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}})\nu(dx)ds)$ respectively, where $f_{Y_n}(t) := b\cdot t - f_n(1)$ and $f_{\tilde{X}_n}(t) := b\cdot t - f_n(t)$.

Then, $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous because of the following considerations. Define $\theta(x,s) := \ln(1+\frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}})$, for all $s\in[0,T]$ and $x\in\mathbb{R}$. This yields, for $t>1$,

$$\begin{aligned} f_{Y_n}(t) &= bt - f_n(1) = bt - f_n(t) + \int_1^t f_n'(s)ds \\ &= f_{\tilde{X}_n}(t) + \int_0^t \int_{|x|\le 1}\left(e^{\theta(x,s)}-1\right)x\nu(dx)ds, \end{aligned}$$

and since $f_n(1) = f_n(t)$, for $t\in[0,1]$,

$$f_{Y_n}(t) = bt - f_n(1) = bt - f_n(t) = f_{\tilde{X}_n}(t),$$

since $\tilde{t}_{T,n}\ge 1$. Define $\Lambda_{Y_n}(dx,ds) := \exp(\theta(x,s))\nu(dx)ds$. According to the choice of the Lévy measures, $\nu(dx)ds$ and
$d\Lambda_{Y_n}(x,s)$ are absolutely continuous with $\frac{d\Lambda_{Y_n}(x,s)}{\nu(dx)ds} = e^{\theta(x,s)}$. Furthermore, in order to apply Theorem 9 we have to check $\int_0^T\int_{\mathbb{R}}\left(e^{\theta(x,s)/2}-1\right)^2\nu(dx)ds<\infty$. We know from [35], Remark 33.3, that this condition is equivalent to the following three properties combined

1. $\int_{\{(x,s):\theta(x,s)<-1\}}\nu(dx)ds<\infty$,
2. $\int_{\{(x,s):\theta(x,s)>1\}}e^{\theta(x,s)}\nu(dx)ds<\infty$, and
3. $\int_{\{(x,s):|\theta(x,s)|\le 1\}}\theta^2(x,s)\nu(dx)ds<\infty$.

Since $f_n'\ge 0$, thus $\theta\ge 0$; it is left to show 2. and 3.

Case 2.: Since $\theta>1$ and A bounded away from zero, we have

$$\int_{\{(x,s):\theta(x,s)>1\}}e^{\theta(x,s)}\nu(dx)ds \le \int_1^T\int_A(1+\frac{f_n'(s)x}{m})\nu(dx)ds<\infty.$$

Case 3.: Since $\ln(1+z)\le z$, for all $z>-1$, and inequality (35) we get

$$\int_{\{(x,s):|\theta(x,s)|\le 1\}}\theta(x,s)^2\nu(dx)ds \le \frac{1}{m^2}\int_1^T\int_A f_n'(s)^2x^2\nu(dx)ds = \frac{1}{m}||f_n'||^2_{L_2[1,\infty)}<\infty.$$

Hence, due to Theorem 9 $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous.

Next, we prove inequality (37).

Note that $\theta(x,s)=0$, for $s\in[0,1]$ and $x\in\mathbb{R}$. Because of Theorem 9 and the density transformation formula (10) we have

$$\mathbb{P}(\tilde{X}_n(t)\le 1,\ 1\le t\le T) = \mathbb{E}_{\tilde{X}_n}(\mathbf{1}_{\{\tilde{X}_n(t)\le 1,\ 1\le t\le T\}}) \tag{38}$$

$$= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\le 1,\ 1\le t\le T\}}e^{-\int_1^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right)\cdot e^{-\int_1^T\int_{\mathbb{R}}g\left(\frac{f_n'(s)x}{m}\mathbf{1}_{x\in A}\right)\nu(dx)ds}, \tag{39}$$

where $g(u) := (1+u)\ln(1+u) - u$, $u \geq 0$. Since $g(u) \geq 0$, for $u \geq 0$, we obtain that

$$e^{-\int_1^T \int_{\mathbb{R}} g(\frac{f_n'(s)x}{m}\mathbf{1}_{x\in A})\nu(dx)ds} \leq 1.$$

Let $p > 1$ and $1/p + 1/q = 1$. Applying Hölder's inequality in (38) yields that

$$\begin{aligned}
&\mathbb{P}(\tilde{X}_n(t) \leq 1,\ 1 \leq t \leq T) \\
&\leq (\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T))^{1/p} \cdot \left(\mathbb{E}_{Y_n}\left(\exp\left(-q\int_1^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)\right)\right)\right)^{1/q}.
\end{aligned} \tag{40}$$

Let us consider the second term in (40). Due to the density transform formula (11) we have

$$\begin{aligned}
&\mathbb{E}_{Y_n}\left(e^{-q\int_1^T \int_{\mathbb{R}} \theta(x,s)\bar{N}_{Y_n}(dx,ds)}\right) \\
&= \mathbb{E}_{\tilde{X}_n}\left(e^{\int_1^T \int_{\mathbb{R}}(-q+1)\theta(x,s)(N(dx,ds)-\nu(dx)ds)}\right) \\
&\quad \cdot \exp\left(\int_1^T \int_{\mathbb{R}}(-q\theta(x,s) + q\theta(x,s)e^{\theta(x,s)} - e^{\theta(x,s)} + 1 + \theta(x,s))\nu(dx)ds\right) \\
&= \exp\left(\int_1^T \int_{\mathbb{R}}(e^{(-q+1)\theta(x,s)} - 1 - (-q+1)\theta(x,s))\nu(dx)ds\right) \\
&\quad \cdot \exp\left(\int_1^T \int_{\mathbb{R}}(-q\theta(x,s) + q\theta(x,s)e^{\theta(x,s)} - e^{\theta(x,s)} + 1 + \theta(x,s))\nu(dx)ds\right) \\
&= \exp\left(\int_1^T \int_{\mathbb{R}} e^{\theta(x,s)}(e^{-q\theta(x,s)} - 1 + q\theta(x,s))\nu(dx)ds\right),
\end{aligned}$$

where we used a modification of Lemma 33.6 of [35] in the second step. The difference between [35] and our case consists in the consideration of time-inhomogeneous processes in contrast to time-homogeneous processes used in [35].

Taylor's expansion implies $e^{-q\theta(x,s)} + q\theta(x,s) - 1 \leq \frac{1}{2}q^2\theta(x,s)^2$, for all $x \in \mathbb{R}$ and $s \in [1,T]$. Since θ is bounded away from infinity we have $\exp(\theta(x,s)) < \tilde{c}_1$, for some $\tilde{c}_1 > 0$, and thus,

$$\begin{aligned}
\frac{1}{q}\int_1^T \int_{\mathbb{R}} e^{\theta(x,s)}(e^{-q\theta(x,s)} + q\theta(x,s) - 1)\nu(dx)ds &\leq q\int_1^T \int_{\mathbb{R}} \frac{\tilde{c}_1}{2}\theta(x,s)^2\nu(dx)ds \\
\leq \frac{q\cdot\tilde{c}_1}{2m^2}\int_1^T f_n'(s)^2 ds \cdot \int_A x^2\nu(dx) &\leq \frac{q\cdot\tilde{c}_1}{2m}||f'||^2_{L_2[1,\infty)},
\end{aligned}$$

having also used (35). Plugging this into (40) yields

$$\begin{aligned}
g_n(T) &= \mathbb{P}(\tilde{X}_n(t) \leq 1,\ 1 \leq t \leq T) \\
&\leq \mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T)^{1/p} \cdot \exp\left(\frac{q\cdot\tilde{c}_1}{2m}||f'||^2_{L_2[1,\infty)}\right).
\end{aligned}$$

Optimizing in p shows that the best choice is

$$1/p := 1 - \sqrt{\frac{\tilde{c}_1 \cdot ||f'||^2_{L_2[1,\infty)}}{2m\ln(1/\mathbb{P}_{Y_n}(Y_n(t) \leq 1,\ 1 \leq t \leq T))}} < 1,$$

which shows inequality (37) with $c_1 > 0$ suitably chosen.

4.2.3 Upper bound for the term in (37)

Having deduced (37) we carry on with the examination of the one-sided exit problem for the process Y_n. More precisely, we will prove the following upper bound, for any $n \in \mathbb{N}$,

$$\mathbb{P}(Y_n(t) \le 1,\ 1 \le t \le T) \le 2 \cdot g_{n+1}(T).$$

4.2.3.1 Homogenization First, we represent the process Y_n as a sum of independent processes $Y_n(\cdot) \overset{d}{=} X(\cdot) + Z_n(\cdot) - f_n(1)$, where Z_n is an additive process with triplet $(0, 0, \frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}}\nu(dx)ds)$. Due to the Lévy-Khintchine formula and

$$f_n(t) - f_n(1) = \int_1^t f_n'(s)ds = \int_0^t f_n'(s)ds$$

there exists a Lévy process $\tilde{Z}$ with triplet $(0, 0, \frac{x}{m}\mathbf{1}_{\{x\in A\}}\nu(dx))$ such that $Z_n(\cdot) = \tilde{Z}(f_n(\cdot) - f_n(1))$ in f.d.d. Note that $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, since A is compact in $(0,\infty)$, and the characteristic exponent of $\tilde{Z}$ has the following representation

$$\Psi(u) = \int_{\mathbb{R}} (1 - e^{iux} + iux)\tfrac{x}{m}\mathbf{1}_{\{x\in A\}}\nu(dx)$$

and the Lévy measure satisfies $\int (|x| \wedge x^2)\frac{x}{m}\mathbf{1}_{\{x\in A\}}\nu(dx) < \infty$. Thus,

$$\mathbb{P}(Y_n(t) \le 1,\ 1 \le t \le T) = \mathbb{P}\Big(X(t) + \tilde{Z}(f_n(t) - f_n(1)) \le 1 + f_n(1),\ 1 \le t \le T\Big).$$

4.2.3.2 Coupling Denote $c_2 := \mathbb{E}\left(\tilde{Z}(1)^2\right) < \infty$. Let B be a Brownian motion. Using Lemma 16 we can write with a suitable constant $\kappa_\delta > 0$

$$\begin{aligned}
&\mathbb{P}\Big(X(t) + \tilde{Z}(f_n(t) - f_n(1)) \le 1 + f_n(1),\ 1 \le t \le T\Big) \\
&\quad \le \mathbb{P}\Big(X(t) \le 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \le t \le T\Big) + T^{-1-\delta}. \qquad (41)
\end{aligned}$$

4.2.3.3 Properties of Brownian motion In order to apply results of one-sided boundary problems for Brownian motion define the sets

$$\begin{aligned}
E_n &:= \Big\{c_2 B(f_n(t) - f_n(1)) \ge -\max\{(\ln T)^5, (f_n(t) - f_n(1))^{3/4}\},\ 1 \le t \le T\Big\} \\
&\ge \Big\{c_2 B(t) \ge -\max\{(\ln T)^5, t^{3/4}\},\ 0 \le t \le \kappa T\Big\} =: \tilde{E}_n,
\end{aligned}$$

since $f(T) \le \kappa T$, for $\kappa > 0$ suitably chosen (see (8)). Then due to Lemma 11 and $f_n(1) = f_n(\ln T)$ we obtain that

$$\begin{aligned}
&\mathbb{P}\Big(X(t) \le 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \le t \le T\Big) \\
&\le \mathbb{P}\Big(X(t) \le 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \le t \le T; E_n\Big) + \mathbb{P}\left(\tilde{E}_n^c\right) \\
&\le \mathbb{P}\Big(X(t) \le 1 + f_n(1) + \kappa_\delta \ln T + \max\{(\ln T)^5, (f_n(t) - f_n(1))^{3/4}\},\ 1 \le t \le T\Big) \\
&\quad + \exp\left(-\kappa(\ln T)^2/4\right) \\
&= g_{n+1}(T) + \exp\left(-\kappa(\ln T)^2/4\right). \qquad (42)
\end{aligned}$$

4.2.4 Proof of the iteration rule (36)

Putting (42) and (41) into (37) and using that $H' > 0$ in a neighbourhood of zero we get

$$\begin{aligned} g_n(T) &\leq \left[g_{n+1}(T) + T^{-1-\delta} + \exp\left(-\kappa(\ln T)^2/4\right)\right] \\ &\quad \cdot \exp\left(\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln(1/\left[g_{n+1}(T) + T^{-1-\delta} + \exp\left(-\kappa(\ln T)^2/4\right)\right])}\right) \\ &\leq H(2g_{n+1}(T)), \end{aligned}$$

where we used in the last step that $g_{n+1}(T) \geq T^{-1-\delta} + \exp\left(-\kappa(\ln T)^2/4\right)$, for sufficiently large $T > 1$, since

$$g_{n+1}(T) \geq \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta+o(1)} \geq T^{-1-\delta} + \exp\left(-\kappa(\ln T)^2/4\right). \tag{43}$$

Hence, we have proved (36).

4.3 End point of the iteration

The aim of this subsection is to find a number $n(T)$ depending on T such that

$$g_{n(T)}(T) \leq T^{-\delta+o(1)}. \tag{44}$$

4.3.1 Number of iteration steps

Our next goal is to set depending on $T > 1$ the number of iteration steps such that eventually the boundary is smaller than $1 + (\ln T)^6$. Due to inequality (8) there exists $\kappa > 0$ such that $f(T) \leq \kappa T$. For this purpose, we choose, for T sufficiently large,

$$n(T) := \left\lceil \frac{\ln(\ln(\kappa T)/\ln(2))}{\ln(4/3)} \right\rceil$$

and thus, for T sufficiently large,

$$\begin{aligned} g_{n(T)}(T) &\leq \mathbb{P}\left(X(t) \leq 1 + f(\ln T) + n(T) \cdot (\ln T)^5,\ 1 \leq t \leq T\right) \\ &\leq \mathbb{P}\left(X(t) \leq 1 + (\ln T)^6,\ 1 \leq t \leq T\right), \end{aligned}$$

where we used inequality (34) combined with $f(t)^{3/4^{n(T)}} < 2$, for $0 \leq t \leq T$, and that $f(T) > 1$ if f is not bounded away from infinity. On the other hand, if $\sup_{t\geq 0} |f(t)| < \infty$, then applying Lemma 12 already proves the theorem.

4.3.2 Asymptotic rate of the end point:

Here, we show (44). Applying Lemma 14 we obtain that

$$\begin{aligned} g_{n(T)}(T) &\leq \mathbb{P}(X(t) \leq 1 + (\ln T)^6,\ 1 \leq t \leq T) \leq \frac{2 \cdot \mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right)}{\mathbb{P}\left(X(t) \leq 1 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right)} \\ &= \mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right) (\ln T)^{21\delta + o(1)}, \end{aligned}$$

where we used Theorem 1 in the last step and with it the assumption $\nu(\mathbb{R}_-) > 0$. Using now the main assumption (2) yields (44).

4.4 Applying of the iteration

In this subsection we combine (36) with (44) to obtain finally inequality (33). For this purpose, we calculate $H_2^{n(T)}(2g_{n(T)}(T))$. First, we show by induction for x sufficiently small that, for any $n \geq 1$,

$$H_2^n(2x) \leq 2^n \cdot x \cdot \exp\left(n\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right). \tag{45}$$

Clearly, we get, for $n = 1$,

$$H_2^1(2x) = H(2x) \leq 2 \cdot x \cdot \exp\left(\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right),$$

since $\ln(1/(2x)) \leq \ln(1/x)$. Now, we assume that (45) holds, for $n-1$. Since H is non-decreasing in a neighbourhood of zero, we have

$$\begin{aligned} H_2^n(2x) = H(2H^{n-1}(2x)) &\leq H\left(2^n x \exp\left((n-1)\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right)\right) \\ &\leq 2^n \cdot x \cdot \exp\left(n\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right), \end{aligned}$$

where we used in the last step that

$$\ln\left(2^{-n} \cdot x^{-1} \exp\left(-(n-1)\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right)\right) \leq \ln(1/x).$$

Combining (45) and (43) with equation (44) and an $n(T)$-times iteration of (36) yields

$$\begin{aligned} &\mathbb{P}(X(t) \leq 1 + f(t),\ 0 \leq t \leq T) = g_0(T) \\ &\leq H_2^{n(T)}\left(2g_{n(T)}(T)\right) \leq g_{n(T)}(T) \cdot 2^{n(T)} \exp\left(n(T)\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/g_{n(T)}(T))}\right) \\ &= T^{-\delta + o(1)}, \end{aligned}$$

which completes the proof.

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