Tropical Linear Spaces, Matroid Decompositions, and Tropical Grassmannians

Michael Joswig

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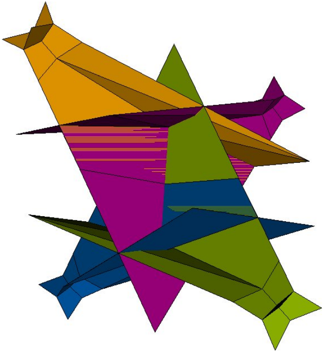
1. Tropicalization of the Grassmannians
   - Tropical Grassmannians
   - Tropical Plücker Vectors and Dressians
2. Matroid Decompositions
   - Hypersimplices and Matroid Polytopes
   - Parameterization of Tropical $(d-1)$-Planes
3. (Not Only) Computational Results



Recalling Tropical Varieties

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Our *tropical semi-ring* is $(\mathbb{R} \cup \{\infty\}, \min, +)$.

I : ideal in $K[x_1, x_2, \dots, x_n]$

tropical variety $\mathcal{T}(I)$ = sub-fan of Gröbner fan in $\mathbb{R}^n$
corresponding to generalized initial ideals without monomials

- Bieri & Groves 1984: if I prime ideal then $\mathcal{T}(I)$ pure fan of dimension r
  - $r = \dim K[x]/I$

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Tropical Grassmannians

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$\mathbb{Z}[p] \; := \; \mathbb{Z}[p_{i_1,\dots,i_d} \,|\, 1 \le i_1 < i_2 < \cdots < i_d \le n]$

$p_{i_1,\dots,i_d}$: $d \times d$-minor of generic $d \times n$-matrix with columns $(i_1, i_2, \dots, i_d)$

Plücker ideal $I_{d,n}$: algebraic relations

Definition (Speyer & Sturmfels 2004)

$\mathrm{Gr}_K(d,n) \; := \; \mathcal{T}(I_{d,n} \otimes K)$

- sub-fan of Gröbner fan of $I_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}$
- here: Krull dimension $r = (n-d)d + 1$
- factorize by lineality space / intersect with sphere
  - $\rightsquigarrow$ spherical polytopal complex of dimension $nd - n - d^2$

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First Example: $d = 2$ and $n = 4$

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running Macaulay2 ...

```
i1 : R = ZZ[x_11..x_24];

i2 : M = matrix {{x_11,x_12,x_13,x_14},{x_21,x_22,x_23,x_24}}

o2 = | x_11 x_12 x_13 x_14 |
     | x_21 x_22 x_23 x_24 |

              2       4
o2 : Matrix R  <--- R

i3 : p_12=det(submatrix(M,{0,1},{0,1}));
...
i9 : print (p_12*p_34-p_13*p_24+p_14*p_23);
0
```

$I_{2,4} = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle$
3-term Plücker relation

$\mathrm{Gr}(2,4)$ = three isolated points
result characteristic-free

Tropical Plücker Vectors and Dressians

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Definition (Speyer 2008)

$\pi \in \mathbb{R}^{\binom{n}{d}}$ (finite) tropical Plücker vector

:⇔ for every $S \in \binom{[n]}{d-2}$ and every i, j, k, l in $[n] \setminus S$ (pairwise distinct):

$$\pi \ \in \ \mathcal{T}(p_{Sij}p_{Skl} - p_{Sik}p_{Sjl} + p_{Sil}p_{Sjk})$$

Definition

Dressian $\mathrm{Dr}(d, n)$: set of all finite tropical Plücker vectors

- tropical pre-variety arising as intersection of all tropical hypersurfaces corresponding to 3-term Plücker relations
- Kapranov 1993: ⇝ Chow quotients of Grassmannians
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Matroid Polytopes and Matroid Subdivisions

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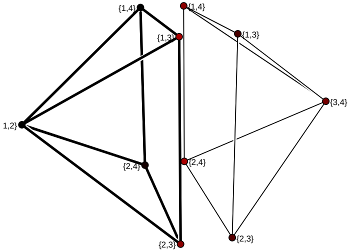
Theorem/Definition (Gel′fand et al. 1987)

A (d, n)-matroid polytope is a subpolytope of $\Delta(d, n)$ whose edges are parallel to $e_i - e_j$.



- *hypersimplex* $\Delta(d, n)$
  - convex hull of $0/1$-vectors of length n with exactly d ones
  - $\Delta(2, 4)$ octahedron
  - uniform matroid of rank d on n points
- matroid subdivision
  - polytopal subdivision into matroid polytopes
  - $\Leftrightarrow$ polytopal subdivision without new edges

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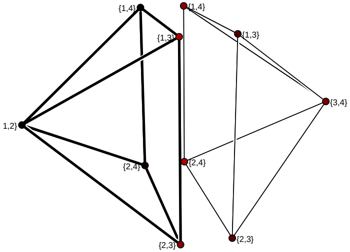

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Lifting Functions

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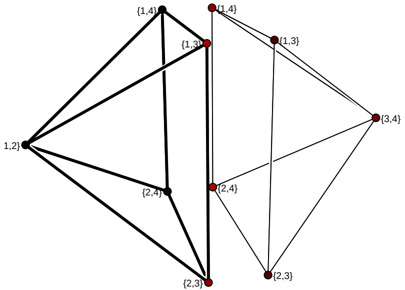
- interpret point in $\mathbb{R}^{\binom{n}{d}}$ as height function on vertices of hypersimplex $\Delta(d, n)$
- tropical Plücker vector gives (regular) matroid decomposition
- imposes fan structure on $\mathrm{Dr}(d, n)$

Example

$d = 2$, $n = 4$, and

$$\pi : S \mapsto \begin{cases} 1 & \text{if } S \in \{12, 13, 14\} \\ 2 & \text{if } S \in \{23, 24\} \\ 3 & \text{if } S = 34 \end{cases}$$



Tropical $(d-1)$-Planes in $(n-1)$-Space

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Theorem (Speyer & Sturmfels 2004)

The tropical Grassmannian $\mathrm{Gr}(d,n)$ *parameterizes tropical* $(d-1)$*-planes in* $\mathbb{T}^{n-1}$.

Proof.

- fix point $\pi \in \mathrm{Gr}(d,n)$ considered as element of $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\dots,1)$
- for $J \in \binom{[n]}{d+1}$ consider tropical polynomial

$$F_J(x_1,\dots,x_n) = \sum_{j\in J} \pi_{J\setminus\{j\}} \cdot x_j$$

- L_π := intersection of all tropical hyperplanes $\mathcal{T}(F_J)$
  - turns out to be tropicalization of a linear space
  - map $\pi \mapsto L_\pi$ bijective

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Example $d = 2$ and $n = 4$, continued

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- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
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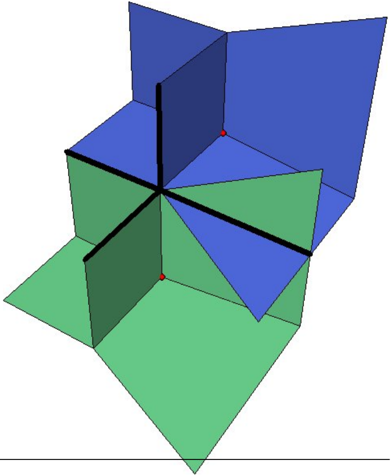

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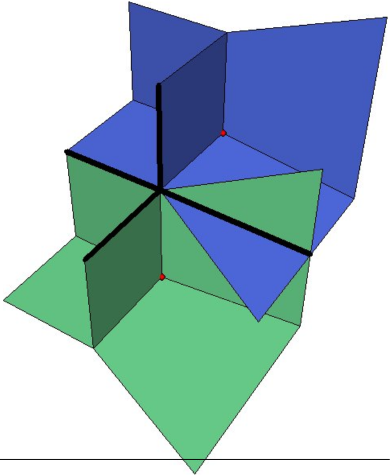

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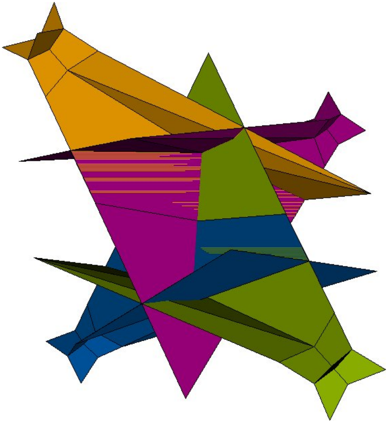

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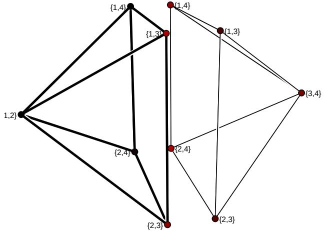
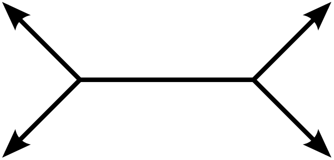
Geometry of Hypersimplices

- two kinds of facets
  - contraction
  - deletion
- tight span := dual of (matroid) decomposition
  - recall: matroid ⇔ no new edges



Lemma

Polytopal decomposition of $\Delta(2, n)$ is matroid iff tight span is tree.



Spaces of Trees

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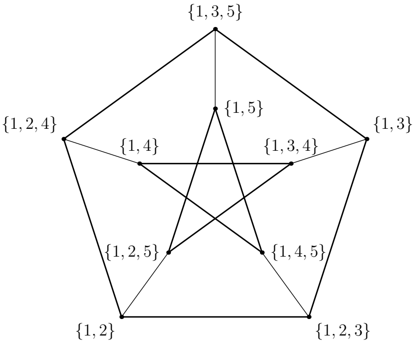
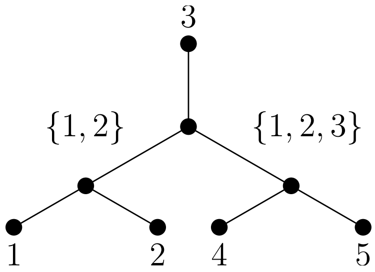
Theorem (Kapranov 1993; Speyer & Sturmfels 2004)

Gr(2, *n*) $\cong$ *space of trivalent metric trees with n marked leaves*

▸ Details





interior edges = splits

- Dr(2, *n*) = Gr(2, *n*) as fans

Constructing a Matroid Subdivision of $\Delta(d, n)$

(From Tropical Polytopes to Tropical Plücker Vectors)

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Theorem (Kapranov 1993; Speyer 2008)

Each regular subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$ induces a regular matroid subdivision of $\Delta(d, n+d)$.

- choose arbitrary $V \in \mathbb{R}^{d \times (n-d)}$ as lifting of $\Delta_{d-1} \times \Delta_{n-d-1}$
- concatenate with tropical $d \times d$ unit matrix
- for each set of d columns compute *tropical determinant* to define tropical Plücker vector $\pi : \mathbb{R}^{\binom{n}{d}} \to \mathbb{R}$

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \infty & \infty \\ 3 & 5 & 0 & 5 & \infty & 0 & \infty \\ 6 & 2 & 1 & 0 & \infty & \infty & 0 \end{pmatrix}$$

$$\text{e.g., } \pi(245) = \min(0+5+0, 0+5+2) = 5$$

here: $d = 3$ and $n = 7$

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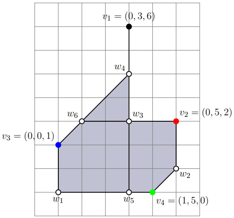
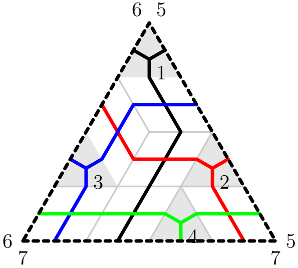
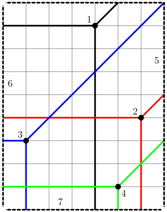
A Matroid Subdivision of $\Delta(3,7)$







label	matroid bases
v_1	125 126 135 136 145 146 156 157 167 256 356 456 567
v_2	124 125 127 145 157 234 235 237 245 246 256 257 267 345 357 456 567
v_3	134 136 137 146 167 234 236 237 246 267 345 346 356 357 367 456 567
v_4	124 127 145 147 157 234 237 246 247 267 345 347 357 456 457 467 567
w_1	134 137 146 167 234 237 246 267 345 346 347 357 367 456 467 567
w_2	124 127 145 157 234 237 245 246 247 257 267 345 357 456 457 567
w_3	123 124 125 126 127 134 137 145 146 157 167 234 235 237 246 256 267 345 357 456 567
w_4	123 125 126 134 135 136 137 145 146 157 167 235 256 345 356 357 456 567
w_5	124 127 134 137 145 146 147 157 167 234 237 246 267 345 347 357 456 467 567
w_6	123 126 134 136 137 146 167 234 235 236 237 246 256 267 345 356 357 456 567

Gr(3, 6)

Theorem (Speyer & Sturmfels 2004)

The tropical Grassmannian Gr(3, 6) *is a* 3*-dimensional simplicial complex with* f*-vector*

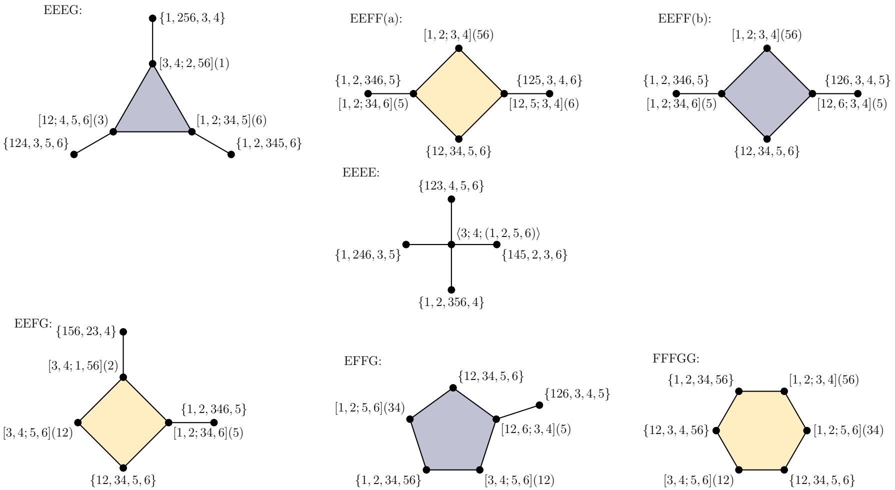
$$(65, 550, 1395, 1035)$$

and integral homology

$$\tilde{H}_* = (0, 0, 0, \mathbb{Z}^{126}) \, .$$

- characteristic-free
- Gr(3, 6) = Dr(3, 6) as sets, but fan structures differ

Tight Spans of Finest Matroid Subdivisions of $\Delta(3,6)$



Arrangements of Trees and the Dressian

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem (Herrmann, Jensen, Sturmfels & J. 2009)

The matroid subdivisions of $\Delta(3, n)$ *bijectively correspond to the equivalence classes of arrangements of n metric trees (on* $n-1$ *labeled leaves).*

- arrangement of metric trees: metrics $\delta_1, \delta_2, \dots, \delta_n$ with

$$\delta_i(j,k) \;=\; \delta_j(k,i) \;=\; \delta_k(i,j)$$

- matroid decomposition of $\Delta(3, n)$ induces (dual) tree on each *contraction facet*
- Dr(3, 7) 6-dimensional non-pure non-simplicial complex
  - *f*-vector = (616, 13860, 101185, 315070, 431025, 211365, 30)
  - homology = $\tilde{H}_*(\mathrm{Dr}(3,7);\mathbb{Z}) = H_5(\mathrm{Dr}(3,7);\mathbb{Z}) = \mathbb{Z}^{7440}$
- complete computation of Dr(3, 8) [Herrmann & J. 2010+]
  - 116,962,265 maximal cones, or 4,748 up to symmetry

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Arrangement of Trees Corresponding to a Ray of Dr(3, 7)

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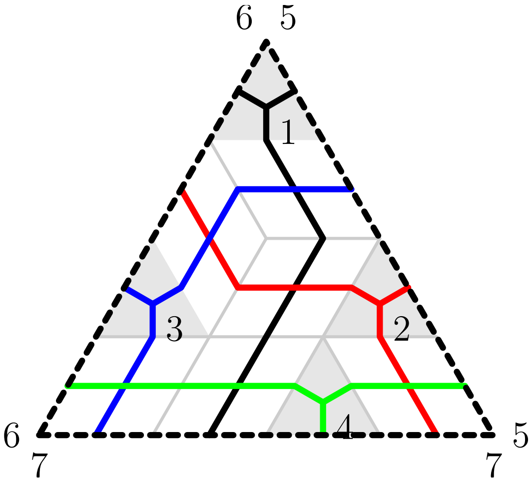

Theorem (Herrmann, Jensen, Sturmfels & J. 2009)

The tropical Grassmannian $\mathrm{Gr}_K(3,7)$ *is a* 5*-dimensional simplicial complex with* f*-vector*

$$(721, 16800, 124180, 386155, 522585, 252000)$$

and integral homology

$$\tilde{H}_*\big(\mathrm{Gr}_K(3,7);\mathbb{Z}\big) = H_5\big(\mathrm{Gr}_K(3,7);\mathbb{Z}\big) = \mathbb{Z}^{7470},$$

unless characteristic equals 2*.*

Advertisements

TECHNISCHE UNIVERSITÄT DARMSTADT

- Book Project “Essentials of Tropical Combinatorics”
  - `www.mathematik.tu-darmstadt.de/~joswig/etc`
- New DFG Priority Programme 1489: “Experimental Methods in Algebra, Geometry, and Number Theory”
- `polymake` meeting in Darmstadt, November 2010

The Space of Trivalent Phylogenetic Trees

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- flag simplicial complex [Billera, Holmes & Vogtmann 2001]
- $2^{n-1} - n - 1$ vertices
- $1 \cdot 3 \cdot 5 \cdot (2n - 5)$ facets
- homotopy equivalent to bouquet of $(n - 2)!$ spheres of dimension $n - 1$
- shellable [Trappmann & Ziegler 1998]

◂ Gr(2, n)