Moduli of Tropical Plane Curves

Michael Joswig

TU Berlin / CNRS-INSMI CMAP & IMJ, Paris

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joint w/ Sarah Brodsky, Ralph Morrison & Bernd Sturmfels

1. Plane Tropical Curves
   - Lattice polygons and height functions
   - Moduli spaces
   - Results

2. Computations
   - Secondary fans
   - The bulk of it

Lattice polygons and height functions

Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with lattice points $A := P \cap \mathbb{Z}^2$.

Any (height) function $h : A \to \mathbb{R}$ yields

- regular subdivision Δ of A and [upper/lower hull]
- tropical polynomial [min/max]

$$F(x,y) \ := \ \bigoplus_{(i,j)\in A} h(i,j) \odot x^{\odot i} \odot y^{\odot j}$$

With $g := \#(\operatorname{int} P \cap \mathbb{Z}^2)$ the tropical hypersurface $\mathcal{C} := \operatorname{trop}(F)$ is a
plane tropical curve of genus g.

A natural length function on the edges turns $\mathcal{C}$ into a planar metric graph, which is dual Δ.

Recall: The Univariate Case

tropical hypersurface $\mathcal{T}(F)$
$:=$ vanishing locus
of tropical polynomial F

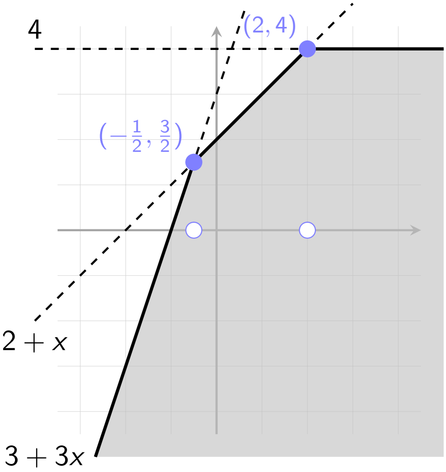
Example

$A = \{0, 1, 3\} \subseteq \mathbb{R}^1$

$F(x) = (3 \odot x^{\odot 3}) \oplus (2 \odot x) \oplus 4$

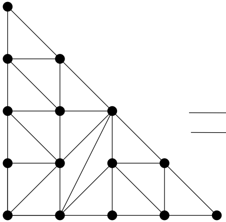
$\mathcal{T}(F) = \{-\frac{1}{2}, 2\} \subset \mathbb{R}^1$

$\oplus = \min$

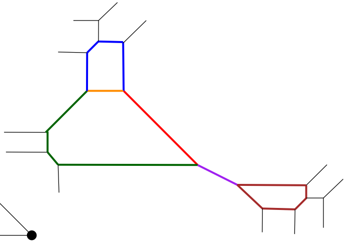


Unimodular Triangulation, Tropical Quartic, and Skeleton

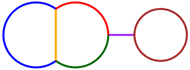
. . . corresponding to a curve of genus $g = 3$



P, A and Δ



$\mathcal{C}$



G

(Berkovich) skeleton G arises from $\mathcal{C}$
by contracting ends and ignoring nodes of degree 2

- $\mathcal{C}$ is smooth $\iff$ Δ is a unimodular triangulation
- in this case: G is a 3-regular plane multigraph of genus g with $2g - 2$ nodes and $3g - 3$ edges

A Zoo of Moduli Spaces

$$\begin{array}{ccccccccc}
 & & & & \mathcal{M}_P & \subseteq & \mathcal{M}_g^{\mathrm{planar}} & \subseteq & \mathcal{M}_g \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \mathrm{trop}(\mathcal{M}_P) & \subseteq & \mathrm{trop}(\mathcal{M}_g^{\mathrm{planar}}) & \subseteq & \mathrm{trop}(\mathcal{M}_g) \\
 & & & & \cup\!\mid & & \cup\!\mid & & \parallel \\
\mathbb{M}_\Delta & \subseteq & \mathbb{M}_{P,G} & \subseteq & \mathbb{M}_P & \subseteq & \mathbb{M}_g^{\mathrm{planar}} & \subseteq & \mathbb{M}_g
\end{array}$$

- Abramovich, Caporaso & Harris 2012+
  - stacky fan $\mathbb{M}_g = \bigcup_G \mathbb{R}_{\geq 0}^{3g-3} / \operatorname{Aut} G$, for $g \geq 2$
- Castryck & Voight 2009
- $\mathbb{M}_g^{\mathrm{planar}} = \bigcup_P \mathbb{M}_P$ is a finite union
  - Scott 1976; Lagarias & Ziegler 1991
  - Koelmann, Haase & Schicho 2009; Castryck 2011: algorithm

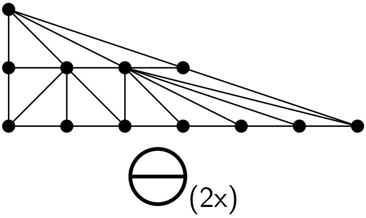
Very Small Genus

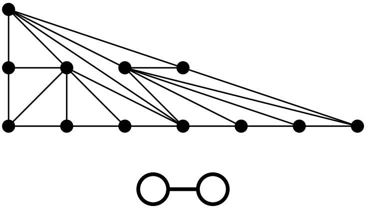
Genus 1

- elliptic curves
- skeleton G = circle; length = tropical j-invariant
- one triangulation Δ of one triangle T suffices:
  - $\mathbb{M}_\Delta = \mathbb{M}_{T,\circ} = \mathbb{M}_T = \mathbb{M}_1^{\text{planar}} = \mathbb{M}_1 = \{*\}$

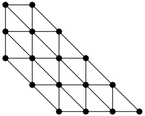
Genus 2

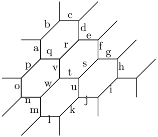
- hyperelliptic curves ... [Chan 2013]
- all metric graphs are realizable as plane tropical curves
  - three triangulations of one triangle suffice

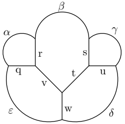




Theoretical Result







Theorem (Brodsky, J., Morrison & Sturmfels 2015)

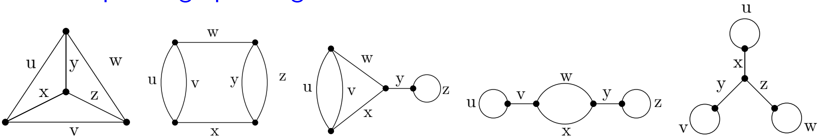
For all $g \geq 2$ there exists a lattice polygon P with g interior lattice points and a unimodular triangulation Δ such that $\mathbb{M}_\Delta$ has the dimension

$$\dim(\mathbb{M}_g^{\text{planar}}) \quad = \quad \dim(\mathbb{M}_\Delta) \quad = \quad \begin{cases} 3 & \textit{if} \quad g = 2, \\ 6 & \textit{if} \quad g = 3, \\ 16 & \textit{if} \quad g = 7, \\ 2g+1 & \textit{otherwise.} \end{cases}$$

- $\dim \mathbb{M}_g^{\text{planar}} = 3g - 3 \iff g \in \{2, 3, 4\}$

Semi-Computational Result

Five trivalent planar graphs of genus 3



(000) (020) (111) (212) (303)

Theorem (Brodsky, J., Morrison & Sturmfels 2015)

A graph in $\mathbb{M}_3$ *arises from a smooth tropical quartic iff it is not of type* (303)*, with edge lengths satisfying, up to symmetry:*

(000) *realizable* $\iff$ $\max\{x,y\} \leq u$, $\max\{x,z\} \leq v$ *and* $\max\{y,z\} \leq w$, *where* ...

(020) *realizable* $\iff$ $v \leq u$, $y \leq z$, *and* ...

(111) *realizable* $\iff$ $w < x$ *and* ...

(212) *realizable* $\iff$ $w < x \leq 2w$.

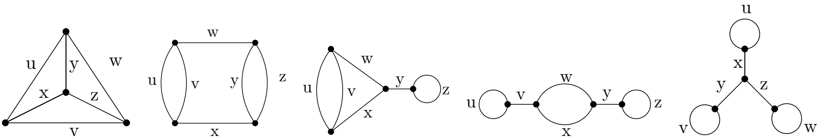
Genus 3 Probabilities

Corollary (Brodsky, J., Morrison & Sturmfels 2015)

The proportion of tropical plane quartics among the metric graphs of genus 3 equals

$$\mathrm{vol}(\mathbb{M}_3^{\text{planar}}) / \mathrm{vol}(\mathbb{M}_3) = 31/105 \approx 29.5\%$$

Graph	(000)	(020)	(111)	(212)	(303)
Probability	4/15	8/15	12/35	1/3	0



(000) (020) (111) (212) (303)

Genus 4 Probabilities (fully computational)

17 trivalent graphs of genus 4

. . . out of which 13 are realizable as plane tropical curves.

Theorem (Brodsky, J., Morrison & Sturmfels 2015)

Less than 0.5% *of all metric graphs of genus* 4 *come from plane tropical curves. More precisely, the fraction is approximately*

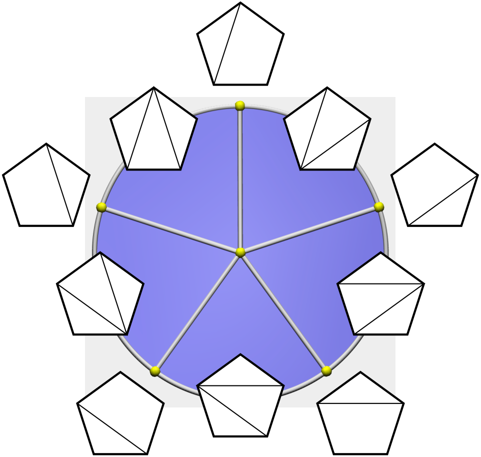
$$\mathrm{vol}(\mathbb{M}_4^{\mathrm{planar}}) / \mathrm{vol}(\mathbb{M}_4) = 0.004788$$

Graph	(000)A	(010)	(020)	(021)	(030)
Probability	0.0101	0.0129	0.0084	0.0164	0.0336

Secondary Fans

Let $A \subset \mathbb{R}^d$ be configuration of n points (affinely spanning).

- height functions inducing fixed subdivision form (relatively open) polyhedral cone
- complete polyhedral fan of dimension n
  - lineality space of dimension $d + 1$



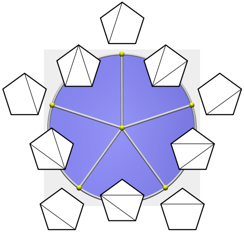
Software for Computing Secondary Fans

`TOPCOM` 0.17.5 [Rambau 2000/2015]

- reduce to oriented matroids
- (breadth-first) search through flip-graph

`Gfan` 0.5 [Jensen 2005/2011]

- (breadth-first) search through dual graph of secondary fan



The Processing Pipeline

After computing the secondary fan

Fix P (with g interior lattice points) and let $A := P \cap \mathbb{Z}^2$.

- $h \in \mathbb{R}^A$ generic yields (regular, unimodular) triangulation Δ
  - E := interior edges of Δ (dual to bounded edges of curve $\mathcal{C}$)
  - $\lambda : h \mapsto$ vector of edge lengths
- κ : edge lengths in $\Delta \mapsto$ edge lengths in skeleton G

$$\mathbb{R}^A \xrightarrow{\lambda} \mathbb{R}^E \xrightarrow{\kappa} \mathbb{R}^{3g-3}$$

Now, for fixed Δ compute secondary cone Σ and ...

$$\kappa(\lambda(\Sigma)) = \mathbb{M}_\Delta$$

polymake Overview

most recent version 2.14 of March 2015

- software for research in mathematics
  - geometric combinatorics:
    convex polytopes, polyhedral fans, matroids, ...
  - linear/combinatorial optimization
  - toric/tropical geometry ⇝ a-tint 2.0beta [Hampe 2015]
  - ...
- open source, GNU Public License
  - interfaces to many other software systems
    - Gfan, normaliz, ppl, Singular, TOPCOM, ...
- co-authored (since 1996) w/ Ewgenij Gawrilow
  - contributions by Benjamin Assarf, Simon Hampe, Katrin Herr, Silke Horn, Lars Kastner, Georg Loho, Benjamin Lorenz, Andreas Paffenholz, Julian Pfeifle, Thomas Rehn, Thilo Rörig, Benjamin Schröter, André Wagner and others

www.polymake.org

Computing Convex Hulls

Open Question

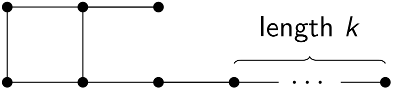
Does there exist a polynomial-time output-sensitive convex hull algorithm?

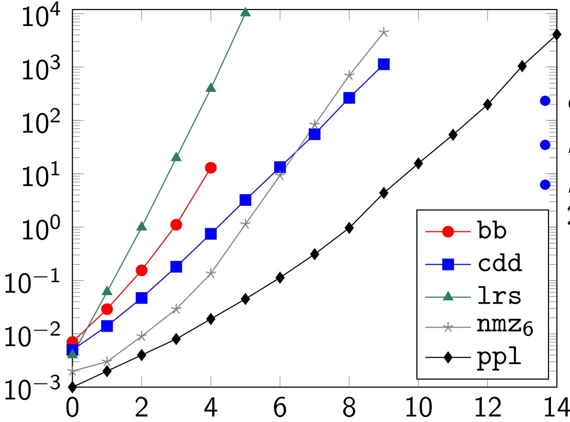
- Bremner 1999: if so, then not incremental
- Khachiyan et al. 2008: probably not at all

Implementations (suitable here)

- Fukuda: `cdd`, Bagnara et al.: `ppl`
- Bruns, Ichim & Söger: `normaliz`
- Avis: `lrs`
- `polymake` team: `bb`

Experiment: Facets of Cut Polytopes





k versus seconds

- $d = k + 6$
- $n = 2^{k+5} = \#$ cuts
- $m = 2d + 8 = 2k + 20$
  - Barahona 1983: facets known if no K_5-minor

timings from summer 2014

Numbers and Dimensions of Moduli Cones

Non-hyperelliptic case

- Genus 3: 1278 (regular unimodular) triangulations (up to symmetry) of triangle T_4

$G \setminus$ dim	3	4	5	6	$\#\Delta$'s
(000)	18	142	269	144	573
(020)		59	216	175	450
(111)		10	120	95	225
(212)			15	15	30
total	18	211	620	429	1278

- Genus 4: three polygons with $5941 + 1278 + 20 = 7239$ triangulations
- Genus 5: four polygons with
  $147{,}908 + 968 + 508 + 162 = 149{,}546$ triangulations
- Genus 6: four polygons, one of which has 561,885 triangulations

Conclusion

- general theoretical results possible
  - good combinatorial model to study some of the classical phenomena
- amenable to computational approach, but results hard to obtain
  - size of secondary fan results in many convex hull computations
- challenge (even for small genus): determine stacky fan structure

1. Brodsky, Morrison, J. & Sturmfels:
   Moduli of tropical plane curves, Res. Math. Sci. (2015)
2. Gawrilow & J.:
   polymake: a framework for analyzing convex polytopes (2000)
3. Assarf et al.:
   polymake in linear and integer programming, `arXiv:1408.4653`