Smooth Fano Polytopes and Their Triangulations

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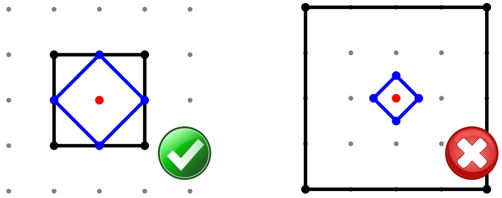
joint w/ Benjamin Assarf, Andreas Paffenholz, & Julian Pfeifle

Reflexivity

Definition

P is reflexive if P and its polar P^* are lattice polytopes

Example

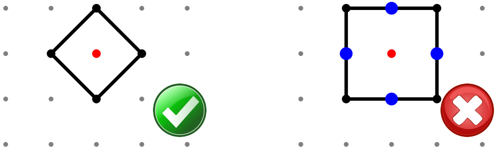


Terminality

Definition

P is terminal if $V(P) \cup \{0\} = P \cap \mathbb{Z}^d$

Example

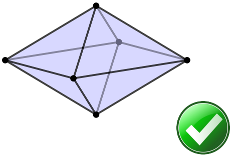


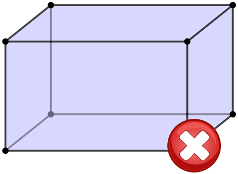
Simpliciality

Definition

P is simplicial if each face is a simplex

Example



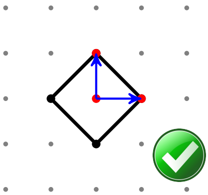


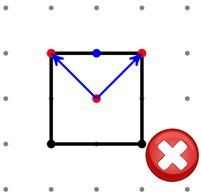
Smoothness

Definition

P is smooth if the vertices of each facet form a lattice basis

Example



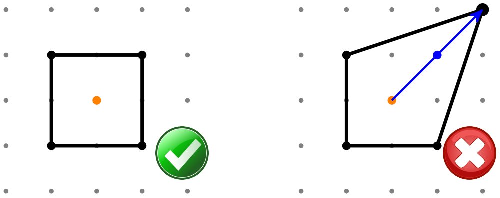


Fano Property

Definition

A lattice polytope P is Fano if the origin is in the interior and all vertices are primitive lattice vectors.

Example



Toric Dictionary

Let $X = X_\Sigma$ be toric Fano variety associated with face fan Σ of Fano polytope P

X is $\mathbb{Q}$-factorial	$\iff$	P simplicial
X has at most terminal singularities	$\iff$	P terminal
X is Gorenstein	$\iff$	P reflexive
X non-singular variety	$\iff$	P smooth

Classification Theorems Yield Smooth Fano Polytopes

Casagrande 2006

The number of vertices of a simplicial, terminal and reflexive d-polytope does not exceed $3d$. Classification of those polytopes.

Øbro 2008

Classification of simplicial, terminal and reflexive d-polytopes with $3d-1$ vertices.

Assarf, J. & Paffenholz 2014

Classification of simplicial, terminal and reflexive d-polytopes with $3d-2$ vertices.

How many?

n	d even	d odd
$3d$	1	0
$3d-1$	1	2
$3d-2$	11	5
$3d-3$	?	?

Finiteness

- Hensley 1983; Lagarias & Ziegler 1992: for $d, m \geq 1$ there are only finitely many d-dimensional lattice polytopes with m interior points (up to lattice equivalence)
- hence: number of terminal/reflexive d-polytopes is finite

Computational classifications

- reflexive: $1, 16, 4319, 473800776$ [Kreuzer & Skarke]
- terminal: $1, 5, 637$ [Kasprzyk]
- smooth reflexive:

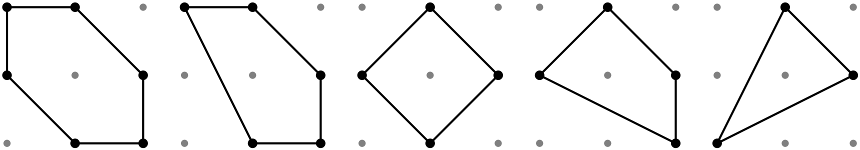
$$\underbrace{1, 5, 18, 124}_{\text{Batyrev}}, \quad \underset{\text{Kreuzer \& Nill}}{866}, \quad \underbrace{7622, 72256, 749892}_{\text{Øbro}}, \quad \underset{\text{Lorenz \& Paffenholz}}{8229721}$$

`polymake` + `PolyDB`

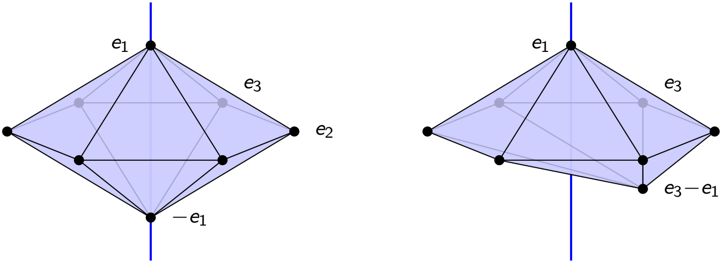
Examples of Smooth Fano Polytopes

Smooth reflexive polygons

[Batyrev 1994]

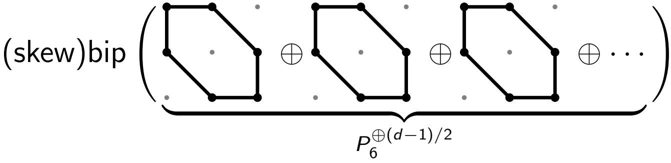


Proper and skew bipyramids over smooth Fano polytope



Smooth Fano Polytopes with $n \geq 3d - 1$ vertices

d evenodd, $n = 3d n = 3d - 1$



- dimension $d = 1 + 2\frac{d-1}{2} = 1 + d - 1 \rightsquigarrow$ odd
- #vertices $n = 2 + 6 \cdot (\frac{d-1}{2}) = 2 + 3d - 3 = 3d - 1$
- Øbro $\implies$ no other cases

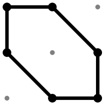
There are at most $3d$ vertices

Let P be smooth Fano d-polytope with n vertices, and let v_P be their sum.

Proof.

- pick special facet F, i.e., such that $v_P \in \operatorname{pos} F$
- up to lattice transformation, assume $F = \operatorname{conv}\{e_1, e_2, \ldots, e_d\}$
- let u_F be (unique) primitive outer facet normal of F
  - $\eta_k = \#\text{vertices with } \langle \cdot, u_F \rangle = k$
- from $\eta_1 = d$ we get $0 \leq \langle v_P, u_F \rangle = d + \sum_{k \leq -1} k \cdot \eta_k$
  - ... and thus $\eta_{-1} + \eta_{-2} + \cdots \leq d$
- Lemma: $\eta_0 \leq d$

$$n = \underbrace{\eta_1}_{=d} + \underbrace{\eta_0}_{\leq d} + \underbrace{\eta_{-1} + \eta_{-2} + \cdots}_{\leq d} \leq 3d$$

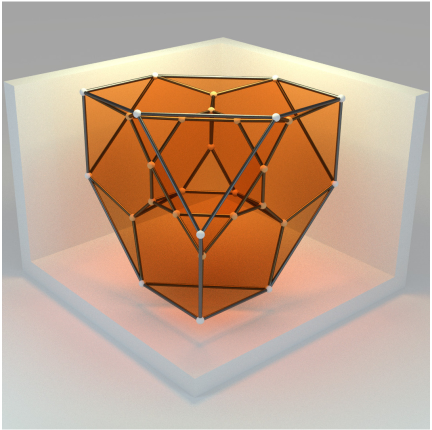


□

One More Smooth Fano Polytope

$$\mathrm{DP}(4) = \mathrm{conv}\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm \mathbf{1}\}$$

- dimension $d = 4 \rightsquigarrow$ even
- #vertices $n = 10 = 3d - 2$



DP(4)$^\vee$

http://gallery.discretization.de/#public,classification_on_fano_polytopes

Main Result

Let P be a simplicial, terminal and reflexive d-polytope with at least $3d - 2$ vertices.

Theorem (Assarf, J. & Paffenholz 2014)

Then P is lattice equivalent to a free sum of copies of P_5, P_6 or $\mathrm{DP}(4)$, or to an iterated (proper or skew) bipyramid over one of the former. In particular, P is smooth Fano.

A Conjecture and One More Result

Let P be smooth Fano d-polytope with $n \geq 3d - k$ vertices.

Conjecture (for $d + k$ even)

If $d \geq 3k$ then P is lattice equivalent to $Q \oplus P_6^{\oplus(\frac{d-3k}{2})}$,
where Q is a $3k$-dimensional smooth Fano polytope.

Theorem (Assarf & Nill 2016)

True if $d > 15k^2 + 37k + 1$.

- for fixed k only finitely many choices for Q!

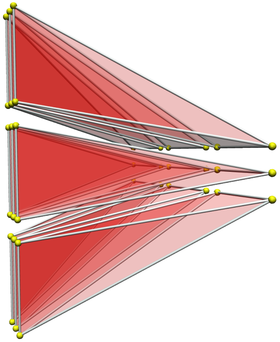
Triangulations of Point Configurations

Let P be a finite set of points in $\mathbb{R}^d$.

Definition

A polytopal decomposition of P is a polyhedral decomposition of $\mathrm{conv}(P)$ whose vertices lie in P.

- partially ordered by refinement
  - triangulation = finest

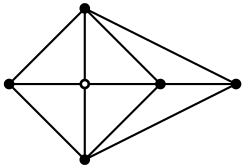


Examples:

- n-gons $\rightsquigarrow$ Catalan combinatorics
- hypersimplices $\rightsquigarrow$ matroid theory
- products of two simplices $\rightsquigarrow$ tropical linear programming
- . . .

General Result, Motivated by Smooth Fano Polytopes

Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$, spanning and such that the (respective) origin is contained in P and lies in the interior.



Theorem (Assarf, J. & Pfeifle)

The triangulations of

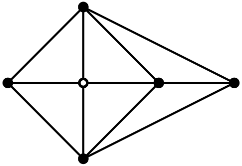
$$P \sqcup_0 Q \quad \subset \; \mathbb{R}^{d+e}$$

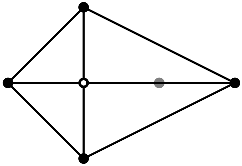
are characterized in terms of the triangulations of P and Q plus additional combinatorial data, which we call webs of spheres.

$$\triangle_P = \langle [-1,0], [0,1], [1,2] \rangle$$
$$\triangle_Q = \langle [-1,0], [0,1] \rangle$$

$$\alpha([-1,0]) = \triangle_Q \quad \beta([-1,0]) = \emptyset$$
$$\alpha([0,1]) = \triangle_Q \quad \beta([0,1]) = \emptyset$$
$$\alpha([1,2]) = \triangle_Q$$

Two Examples





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Fano d-Polytopes by Number of Free Summands

		#summands						
d	total	1	2	3	4	5	6	decomposable
2	5	4	1					20%
3	18	13	4	1				28%
4	124	96	23	4	1			23%
5	866	690	148	23	4	1		20%
6	7622	6261	1165	168	23	4	1	18%

Semi-Regular Triangulations and Webs of Stars

	#triangulations	time	#compatible pairs	time
$DP(2) \sqcup_0 DP(2)$	204	2s	1 157	8s
$DP(2) \sqcup_0 cross(4)$	13	20s	16	11s
$DP(2) \sqcup_0 DP^-(4)$	250 594	2.5h	1 581 647	27s
$DP(2) \sqcup_0 DP(4)$	?		1 677 949 075	10d

timings with `TOPCOM` and `polymake`

Conclusion

- smooth Fano polytopes with many vertices can be classified
- the more vertices, the nicer the polytopes
- finite classification (e.g., by computer) up to fixed dimension translates into classification of an infinite class of smooth Fano polytopes with very many vertices
- analysis lead to general result on triangulations of point configurations

1. Assarf, J. & Paffenholz: Discrete Comput. Geometry 52 (2014)
2. Assarf & Nill: J. Algebraic Combin. 43 (2016)
3. Assarf, J. & Pfeifle: `arXiv:1512.08411`