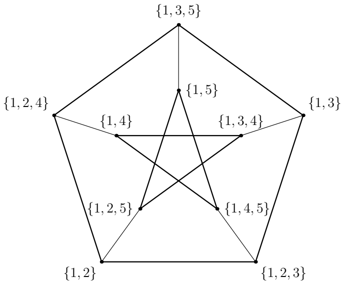
Dressians, Tropical Grassmannians, and Their Rays

Michael Joswig

TECHNISCHE UNIVERSITÄT DARMSTADT

w/ Sven Herrmann

1. Explain the Title
   - Tropical Plücker Vectors
   - Tropical Grassmannians
   - Hypersimplices $\Delta(d, n)$ and Matroid Polytopes
2. Planes and Points
   - Parameterization of Tropical Planes
   - Point Configurations
3. Tight Spans of Rays
   - Tropical Rigidity



Tropical Plücker Vectors and Dressians

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition (Speyer 2005)

$\pi \in \mathbb{R}^{\binom{n}{d}}$ (finite) tropical Plücker vector

:⇔ for every $S \in \binom{[n]}{d-2}$ and every i, j, k, l in $[n] \setminus S$ (pairwise distinct):

$$\min\{\pi_{Sij} + \pi_{Skl}, \pi_{Sik} + \pi_{Sjl}, \pi_{Sil} + \pi_{Sjk}\} \text{ attained at least twice}$$

Definition

Dressian $\mathrm{Dr}(d, n)$: set of all finite tropical Plücker vectors

- tropical pre-variety arising as intersection of all tropical hypersurfaces corresponding to 3-term Plücker relations
- Kapranov 1993: ⇝ Chow quotients of Grassmannians
- Speyer 2005: *tropical pre-Grassmannian*

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition (Speyer 2005)

$\pi \in \mathbb{R}^{\binom{n}{d}}$ (finite) tropical Plücker vector

:$\Leftrightarrow$ for every $S \in \binom{[n]}{d-2}$ and every i, j, k, l in $[n] \setminus S$ (pairwise distinct):

$$\min\{\pi_{Sij} + \pi_{Skl}, \pi_{Sik} + \pi_{Sjl}, \pi_{Sil} + \pi_{Sjk}\} \text{ attained at least twice}$$

Definition

Dressian $\mathrm{Dr}(d, n)$: set of all finite tropical Plücker vectors

- tropical pre-variety arising as intersection of all tropical hypersurfaces corresponding to 3-term Plücker relations
- Kapranov 1993: $\rightsquigarrow$ Chow quotients of Grassmannians
- Speyer 2005: *tropical pre-Grassmannian*

Tropical Plücker Vectors and Dressians

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition (Speyer 2005)

$\pi \in \mathbb{R}^{\binom{n}{d}}$ (finite) tropical Plücker vector

:⇔ for every $S \in \binom{[n]}{d-2}$ and every i, j, k, l in $[n] \setminus S$ (pairwise distinct):

$$\pi \ \in \ \mathcal{T}(p_{Sij}p_{Skl} - p_{Sik}p_{Sjl} + p_{Sil}p_{Sjk})$$

Definition

Dressian $\mathrm{Dr}(d, n)$: set of all finite tropical Plücker vectors

- tropical pre-variety arising as intersection of all tropical hypersurfaces corresponding to 3-term Plücker relations
- Kapranov 1993: $\rightsquigarrow$ Chow quotients of Grassmannians
- Speyer 2005: *tropical pre-Grassmannian*

Tropical Plücker Vectors and Dressians

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition (Speyer 2005)

$\pi \in \mathbb{R}^{\binom{n}{d}}$ (finite) tropical Plücker vector

:$\Leftrightarrow$ for every $S \in \binom{[n]}{d-2}$ and every i, j, k, l in $[n] \setminus S$ (pairwise distinct):

$$\min\{\pi_{Sij} + \pi_{Skl}, \pi_{Sik} + \pi_{Sjl}, \pi_{Sil} + \pi_{Sjk}\} \text{ attained at least twice}$$

Definition

Dressian $\mathrm{Dr}(d, n)$: set of all finite tropical Plücker vectors

- tropical pre-variety arising as intersection of all tropical hypersurfaces corresponding to 3-term Plücker relations
- Kapranov 1993: $\rightsquigarrow$ Chow quotients of Grassmannians
- Speyer 2005: *tropical pre-Grassmannian*

Tropical Grassmannians

$\mathbb{Z}[p] \;:=\; \mathbb{Z}[p_{i_1,\dots,i_d} \,|\, 1 \leq i_1 < i_2 < \cdots < i_d \leq n]$
$p_{i_1,\dots,i_d}$: $d \times d$-minor of generic $d \times n$-matrix with columns $(i_1, i_2, \dots, i_d)$
Plücker ideal $I_{d,n}$: algebraic relations

Definition (Speyer & Sturmfels 2004)

$\mathrm{Gr}_K(d,n) \;:=\; \mathcal{T}(I_{d,n} \otimes K)$ for K an infinite field

- sub-fan of Gröbner fan of $I_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}$
  - contained in $\mathrm{Dr}(d,n)$
- factorize by lineality space / intersect with sphere
  - $\rightsquigarrow$ spherical polytopal complex of dimension $nd - n - d^2$
- points in $\mathrm{Gr}_K(d,n)$ correspond to *realizable* tropical linear spaces

Tropical Grassmannians

$\mathbb{Z}[p] := \mathbb{Z}[p_{i_1,\dots,i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n]$
$p_{i_1,\dots,i_d}$: $d \times d$-minor of generic $d \times n$-matrix with columns $(i_1, i_2, \dots, i_d)$
Plücker ideal $I_{d,n}$: algebraic relations

Definition (Speyer & Sturmfels 2004)

$\mathrm{Gr}_K(d, n) := \mathcal{T}(I_{d,n} \otimes K)$ for K an infinite field

- sub-fan of Gröbner fan of $I_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}$
  - contained in $\mathrm{Dr}(d, n)$
- factorize by lineality space / intersect with sphere
  - $\rightsquigarrow$ spherical polytopal complex of dimension $nd - n - d^2$
- points in $\mathrm{Gr}_K(d, n)$ correspond to *realizable* tropical linear spaces

Tropical Grassmannians

TECHNISCHE UNIVERSITÄT DARMSTADT

$\mathbb{Z}[p] := \mathbb{Z}[p_{i_1,\dots,i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n]$
$p_{i_1,\dots,i_d}$: $d \times d$-minor of generic $d \times n$-matrix with columns $(i_1, i_2, \dots, i_d)$
Plücker ideal $I_{d,n}$: algebraic relations

Definition (Speyer & Sturmfels 2004)

$\mathrm{Gr}_K(d, n) := \mathcal{T}(I_{d,n} \otimes K)$ for K an infinite field

- sub-fan of Gröbner fan of $I_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}$
  - contained in $\mathrm{Dr}(d, n)$
- factorize by lineality space / intersect with sphere
  - $\rightsquigarrow$ spherical polytopal complex of dimension $nd - n - d^2$
- points in $\mathrm{Gr}_K(d, n)$ correspond to *realizable* tropical linear spaces

Tropical Grassmannians

$\mathbb{Z}[p] := \mathbb{Z}[p_{i_1,\dots,i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n]$
$p_{i_1,\dots,i_d}$: $d \times d$-minor of generic $d \times n$-matrix with columns $(i_1, i_2, \dots, i_d)$
Plücker ideal $I_{d,n}$: algebraic relations

Definition (Speyer & Sturmfels 2004)

$\mathrm{Gr}_K(d, n) := \mathcal{T}(I_{d,n} \otimes K)$ for K an infinite field

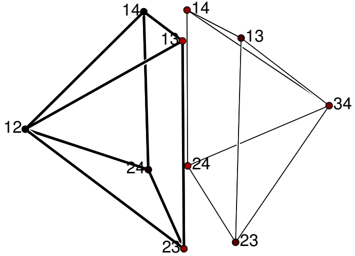
- sub-fan of Gröbner fan of $I_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}$
  - contained in $\mathrm{Dr}(d, n)$
- factorize by lineality space / intersect with sphere
  - $\rightsquigarrow$ spherical polytopal complex of dimension $nd - n - d^2$
- points in $\mathrm{Gr}_K(d, n)$ correspond to *realizable* tropical linear spaces

Matroid Polytopes and Matroid Subdivisions

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem/Definition (Gel′fand et al. 1987)

A (d, n)-matroid polytope is a subpolytope of $\Delta(d, n)$ whose edges are parallel to $e_i - e_j$.



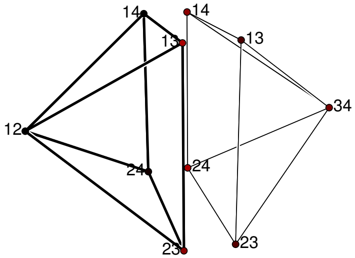
- *hypersimplex* $\Delta(d, n)$
  - convex hull of $0/1$-vectors of length n with exactly d ones
  - $\Delta(2, 4)$ octahedron
  - uniform matroid of rank d on n points
- matroid subdivision
  - polytopal subdivision into matroid polytopes
  - $\Leftrightarrow$ polytopal subdivision without new edges

Matroid Polytopes and Matroid Subdivisions

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem/Definition (Gel′fand et al. 1987)

A (d,n)-matroid polytope is a subpolytope of $\Delta(d,n)$ whose edges are parallel to $e_i - e_j$.



- *hypersimplex* $\Delta(d,n)$
  - convex hull of 0/1-vectors of length n with exactly d ones
  - $\Delta(2,4)$ octahedron
  - uniform matroid of rank d on n points
- matroid subdivision
  - polytopal subdivision into matroid polytopes
  - $\Leftrightarrow$ polytopal subdivision without new edges

Tight Spans
Example: Tropical Polytopes

tight span = cell complex dual to polytopal subdivision

- if subdivision *regular* then tight span *polytopal*
- Develin & Sturmfels 2004: $\text{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{T}^{d-1}$ dual to regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation

Tight Spans

Example: Tropical Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

tight span = cell complex dual to polytopal subdivision

- if subdivision *regular* then tight span *polytopal*
- Develin & Sturmfels 2004: $\operatorname{tconv}\{v_1, \dots, v_n\} \subset \mathbb{T}^{d-1}$ dual to regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation

Tight Spans
Example: Tropical Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

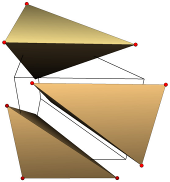
tight span = cell complex dual to polytopal subdivision

- if subdivision *regular* then tight span *polytopal*
- Develin & Sturmfels 2004: $\mathrm{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{T}^{d-1}$ dual to regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation

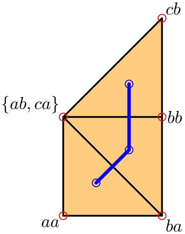
Tight Spans
Example: Tropical Polytopes

tight span = cell complex dual to polytopal subdivision

- if subdivision *regular* then tight span *polytopal*
- Develin & Sturmfels 2004: $\mathrm{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{T}^{d-1}$ dual to regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation



$\Delta_1 \times \Delta_2$



tconv(2 points in $\mathbb{T}^2$)

Lifting Functions on Hypersimplices

TECHNISCHE UNIVERSITÄT DARMSTADT

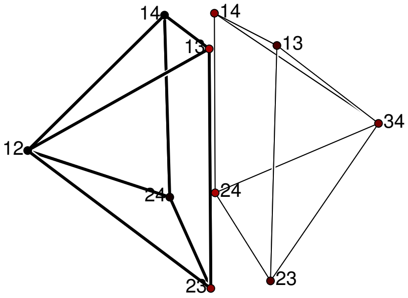
- interpret point in $\mathbb{R}^{\binom{n}{d}}$ as height function on vertices of hypersimplex $\Delta(d, n)$
- tropical Plücker vector gives (regular) matroid decomposition
- imposes fan structure on $\mathrm{Dr}(d, n)$

Example

$d = 2$, $n = 4$, and

$$\pi : S \mapsto \begin{cases} 1 & \text{if } S \in \{12, 13, 14\} \\ 2 & \text{if } S \in \{23, 24\} \\ 3 & \text{if } S = 34 \end{cases}$$

corresponds to a ray of $\mathrm{Dr}(2, 4) = \mathrm{Gr}(2, 4)$
tight span = line segment



Tropical $(d-1)$-Planes in $(n-1)$-Space

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem (Speyer & Sturmfels 2004)

The tropical Grassmannian $\mathrm{Gr}(d,n)$ *parameterizes tropical* $(d-1)$*-planes in* $\mathbb{T}^{n-1}$*.*

Proof.

- fix point $\pi \in \mathrm{Gr}(d,n)$ considered as element of $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\ldots,1)$
- for $J \in \binom{[n]}{d+1}$ consider tropical polynomial

$$F_J(x_1,\ldots,x_n) = \sum_{j \in J} \pi_{J\setminus\{j\}} \cdot x_j$$

- $L_\pi :=$ intersection of all tropical hyperplanes $\mathcal{T}(F_J)$
  - turns out to be tropicalization of a linear space
  - map $\pi \mapsto L_\pi$ bijective

□

Tropical $(d-1)$-Planes in $(n-1)$-Space

TECHNISCHE
UNIVERSITÄT
DARMSTADT

Theorem (Speyer & Sturmfels 2004)

The tropical Grassmannian $\mathrm{Gr}(d,n)$ *parameterizes tropical* $(d-1)$*-planes in* $\mathbb{T}^{n-1}$*.*

Proof.

- fix point $\pi \in \mathrm{Gr}(d,n)$ considered as element of $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\dots,1)$
- for $J \in \binom{[n]}{d+1}$ consider tropical polynomial

$$F_J(x_1,\dots,x_n) = \sum_{j \in J} \pi_{J\setminus\{j\}} \cdot x_j$$

- L_π := intersection of all tropical hyperplanes $\mathcal{T}(F_J)$
  - turns out to be tropicalization of a linear space
  - map $\pi \mapsto L_\pi$ bijective

□

Tropical $(d-1)$-Planes in $(n-1)$-Space

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem (Speyer & Sturmfels 2004)

The tropical Grassmannian $\mathrm{Gr}(d,n)$ *parameterizes tropical* $(d-1)$*-planes in* $\mathbb{T}^{n-1}$*.*

Proof.

- fix point $\pi \in \mathrm{Gr}(d,n)$ considered as element of $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\dots,1)$
- for $J \in \binom{[n]}{d+1}$ consider tropical polynomial

$$F_J(x_1,\dots,x_n) = \sum_{j\in J} \pi_{J\setminus\{j\}} \cdot x_j$$

- L_π := intersection of all tropical hyperplanes $\mathcal{T}(F_J)$
  - turns out to be tropicalization of a linear space
  - map $\pi \mapsto L_\pi$ bijective

□

Example $d = 2$ and $n = 4$, continued

TECHNISCHE UNIVERSITÄT DARMSTADT

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$

Example $d = 2$ and $n = 4$, continued

TECHNISCHE UNIVERSITÄT DARMSTADT

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$

Example $d = 2$ and $n = 4$, continued

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$

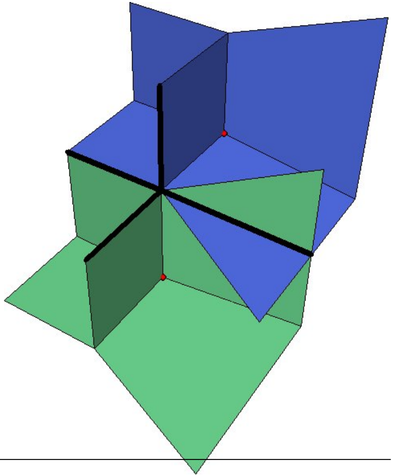
Example $d = 2$ and $n = 4$, continued

TECHNISCHE UNIVERSITÄT DARMSTADT

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$



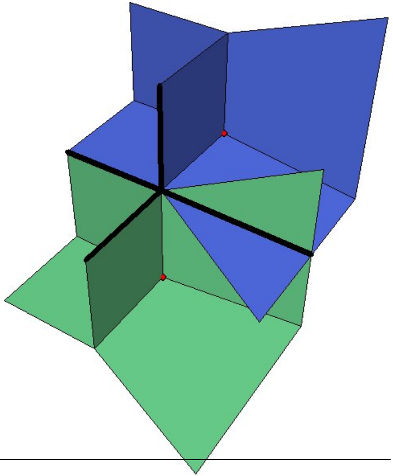
Example $d = 2$ and $n = 4$, continued

TECHNISCHE UNIVERSITÄT DARMSTADT

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$



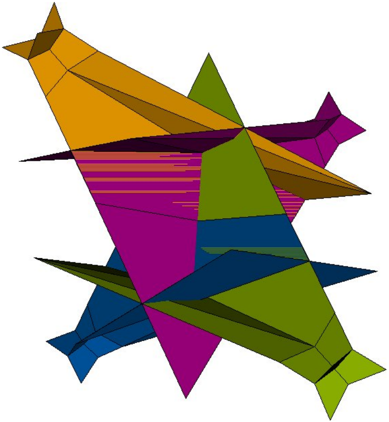
Example $d = 2$ and $n = 4$, continued

TECHNISCHE UNIVERSITÄT DARMSTADT

- consider

$$\pi = \begin{cases} 12 \mapsto 1 \\ 13 \mapsto 1 \\ 14 \mapsto 1 \\ 23 \mapsto 2 \\ 24 \mapsto 2 \\ 34 \mapsto 3 \end{cases}$$

- $F_{123} = 2x_1 + 1x_2 + 1x_3 + \infty x_4$
- $F_{124} = 2x_1 + 1x_2 + \infty x_3 + 1x_4$
- $F_{134} = 3x_1 + \infty x_2 + 1x_3 + 1x_4$
- $F_{234} = \infty x_1 + 3x_2 + 2x_3 + 2x_4$

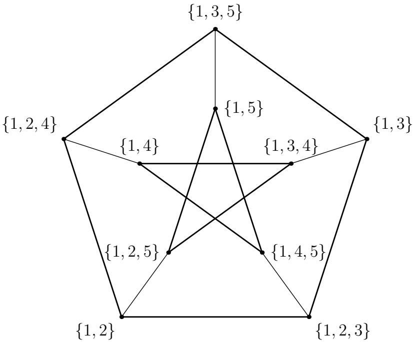


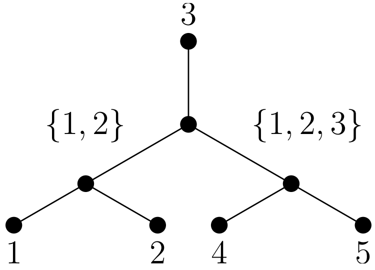
Spaces of Trees

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem (Kapranov 1993; Speyer & Sturmfels 2004)

Dr(2, n) ≅ *space of trivalent metric trees with n marked leaves*





interior edges = splits

- Dr(2, n) = Gr(2, n) as fans

Constructing Points on Gr(d, n)
From Tropical Polytopes to Realizable Tropical Plücker Vectors

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem ($\sim$ Kapranov 1993)

Each regular subdivision Γ of $\Delta_{d-1} \times \Delta_{n-d-1}$ induces a regular matroid subdivision Σ of $\Delta(d,n)$; in fact, this yields a point in Gr(d, n).

- choose arbitrary $V \in \mathbb{R}^{d\times(n-d)}$ as lifting of $\Delta_{d-1} \times \Delta_{n-d-1}$
- concatenate with tropical $d \times d$-unit matrix
- for each set of d columns compute *tropical determinant* to define tropical Plücker vector $\pi : \mathbb{R}^{\binom{n}{d}} \to \mathbb{R}$

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 3 & 5 & 0 & 5 \\ 6 & 2 & 1 & 0 \end{pmatrix}$$

here: $d = 3$ and $n = 7$

Constructing Points on Gr(d, n)

From Tropical Polytopes to Realizable Tropical Plücker Vectors

Theorem ($\sim$ Kapranov 1993)

Each regular subdivision Γ of $\Delta_{d-1} \times \Delta_{n-d-1}$ induces a regular matroid subdivision Σ of $\Delta(d, n)$; in fact, this yields a point in Gr(d, n).

- choose arbitrary $V \in \mathbb{R}^{d\times(n-d)}$ as lifting of $\Delta_{d-1} \times \Delta_{n-d-1}$
- concatenate with tropical $d\times d$-unit matrix
- for each set of d columns compute *tropical determinant* to define tropical Plücker vector $\pi : \mathbb{R}^{\binom{n}{d}} \to \mathbb{R}$

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \infty & \infty \\ 3 & 5 & 0 & 5 & \infty & 0 & \infty \\ 6 & 2 & 1 & 0 & \infty & \infty & 0 \end{pmatrix}$$

e.g., $\pi(245) = \min(0+5+0, 0+5+2) = 5$

here: $d = 3$ and $n = 7$

Constructing Points on Gr(d, n)

From Tropical Polytopes to Realizable Tropical Plücker Vectors

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem ($\sim$ Kapranov 1993)

Each regular subdivision Γ of $\Delta_{d-1} \times \Delta_{n-d-1}$ induces a regular matroid subdivision Σ of $\Delta(d, n)$; in fact, this yields a point in Gr(d, n).

- choose arbitrary $V \in \mathbb{R}^{d\times(n-d)}$ as lifting of $\Delta_{d-1} \times \Delta_{n-d-1}$
- concatenate with tropical $d\times d$-unit matrix
- for each set of d columns compute *tropical determinant* to define tropical Plücker vector $\pi : \mathbb{R}^{\binom{n}{d}} \to \mathbb{R}$

$$\bar{V} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \infty & \infty \\ 3 & 5 & 0 & 5 & \infty & 0 & \infty \\ 6 & 2 & 1 & 0 & \infty & \infty & 0 \end{pmatrix}$$

$$\text{e.g., } \pi(245) = \min(0+5+0, 0+5+2) = 5$$

here: $d = 3$ and $n = 7$

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem ($\sim$ Kapranov 1993)

Each regular subdivision Γ of $\Delta_{d-1} \times \Delta_{n-d-1}$ induces a regular matroid subdivision Σ of $\Delta(d,n)$; in fact, this yields a point in $\mathrm{Gr}(d,n)$.

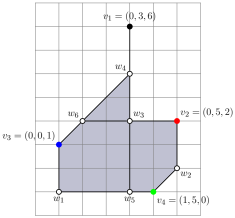
- choose arbitrary $V \in \mathbb{R}^{d\times(n-d)}$ as lifting of $\Delta_{d-1} \times \Delta_{n-d-1}$
- concatenate with tropical $d \times d$-unit matrix
- for each set of d columns compute *tropical determinant* to define tropical Plücker vector $\pi : \mathbb{R}^{\binom{n}{d}} \to \mathbb{R}$

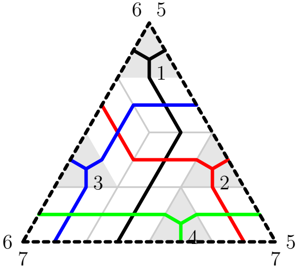
$$\bar{V} = \begin{pmatrix} 0 & 0 & 0 & 1 & \underline{0} & \infty & \infty \\ 3 & \underline{5} & 0 & 5 & \infty & 0 & \infty \\ 6 & 2 & 1 & \underline{0} & \infty & \infty & 0 \end{pmatrix}$$

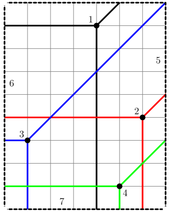
$$\text{e.g., } \pi(245) = \min(0+5+0, 0+5+2) = 5$$

here: $d = 3$ and $n = 7$

A Matroid Subdivision of $\Delta(3, 7)$







label	matroid bases
v_1	125 126 135 136 145 146 156 157 167 256 356 456 567
v_2	124 125 127 145 157 234 235 237 245 246 256 257 267 345 357 456 567
v_3	134 136 137 146 167 234 236 237 246 267 345 346 356 357 367 456 567
v_4	124 127 145 147 157 234 237 246 247 267 345 347 357 456 457 467 567
w_1	134 137 146 167 234 237 246 267 345 346 347 357 367 456 467 567
w_2	124 127 145 157 234 237 245 246 247 257 267 345 357 456 457 567
w_3	123 124 125 126 127 134 137 145 146 157 167 234 235 237 246 256 267 345 357 456 567
w_4	123 125 126 134 135 136 137 145 146 157 167 235 256 345 356 357 456 567
w_5	124 127 134 137 145 146 147 157 167 234 237 246 267 345 347 357 456 467 567
w_6	123 126 134 136 137 146 167 234 235 236 237 246 256 267 345 356 357 456 567

Computational Results for Dr(3, n)

`polymake`

n	dim	f-vector mod Sym(n)	
4	$\underline{0}$	(1)	
5	$\underline{1}$	(1,1)	
6	$\underline{3}$	(9, 8, 3, 1)	SS 04
7	6	(5, 30, 107, 217, 218, 94, 1)	HJJS 09
8	8	(12; 155; 1,149; 5,013; 12,737; 18,802; 14,727; 4,788; 14)	HJ 11+
n	$\sim n^2$		HJJS 09

$$\dim \mathrm{Gr}(3, n) = 2n - 9$$

$$f(\mathrm{Dr}(3,8)) = (15{,}470; 642{,}677; 8{,}892{,}898; 57{,}394{,}505; 194{,}258{,}750; 353{,}149{,}650; 324{,}404{,}880; 117{,}594{,}645; 113{,}400)$$

Computational Results for Dr(3, n)

`polymake`

TECHNISCHE UNIVERSITÄT DARMSTADT

n	dim	f-vector mod Sym(n)	
4	0	(1)	
5	1	(1,1)	
6	3	(9, 8, 3, 1)	SS 04
7	6	(5, 30, 107, 217, 218, 94, 1)	HJJS 09
8	8	(12; 155; 1,149; 5,013; 12,737; 18,802; 14,727; 4,788; 14)	HJ 11+
n	$\sim n^2$		HJJS 09

$$\dim \mathrm{Gr}(3, n) = 2n - 9$$

$$f(\mathrm{Dr}(3,8)) = (15{,}470; 642{,}677; 8{,}892{,}898; 57{,}394{,}505; 194{,}258{,}750; 353{,}149{,}650; 324{,}404{,}880; 117{,}594{,}645; 113{,}400)$$

Tropical Rigidity

Definition

$V \in \mathbb{R}^{d\times(n-d)}$ tropically rigid $:\Leftrightarrow$ regular subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$ induced by V is coarsest

Proposition (Herrmann & J, 2011+)

Let $d = 3$. If V is tropically rigid, then π_V is a ray of $\mathrm{Dr}(3, n)$.

- True for all d.
- Suffices to check *diagonal cases* where $n = 2d$.
  - Original proof for $d = 3$ by induction on n with $n = 6$ as the base case.

Tropical Rigidity

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition

$V \in \mathbb{R}^{d\times(n-d)}$ tropically rigid $:\Leftrightarrow$ regular subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$ induced by V is coarsest

Proposition (Herrmann & J, 2011+)

Let $d = 3$. If V is tropically rigid, then π_V is a ray of $\mathrm{Dr}(3, n)$.

- Conjecture: True for all d.
- Suffices to check *diagonal cases* where $n = 2d$.
  - Original proof for $d = 3$ by induction on n with $n = 6$ as the base case.

Tropical Rigidity

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition

$V \in \mathbb{R}^{d \times (n-d)}$ tropically rigid $:\Leftrightarrow$ regular subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$ induced by V is coarsest

Theorem (Herrmann & J, 2011+; Speyer, EST 13 Dec 2011 13:11:08)

~~*Let $d = 3$.*~~ *If V is tropically rigid, then π_V is a ray of* $\mathrm{Dr}(d, n)$.

- ~~Conjecture:~~ True for all d. [done]
- Suffices to check *diagonal cases* where $n = 2d$.
  - Original proof for $d = 3$ by induction on n with $n = 6$ as the base case.

Tropical Rigidity

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition

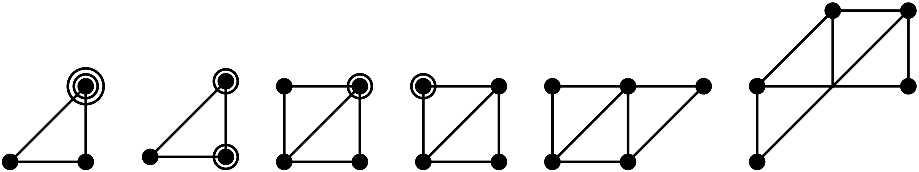
$V \in \mathbb{R}^{d\times(n-d)}$ tropically rigid $:\Leftrightarrow$ regular subdivision of $\Delta_{d-1} \times \Delta_{n-d-1}$ induced by V is coarsest

Theorem (Herrmann & J, 2011+; Speyer, EST 13 Dec 2011 13:11:08)

~~*Let $d = 3$.*~~ *If V is tropically rigid, then π_V is a ray of* $\mathrm{Dr}(d, n)$.

- ~~Conjecture:~~ True for all d. [done]
- Suffices to check *diagonal cases* where $n = 2d$.
  - Original proof for $d = 3$ by induction on n with $n = 6$ as the base case.

Examples
Rays of Dr(3, 8)



- these make up for $2 + 2 + 1 + 1 = 6$ types of rays of Dr(3, 8) (tropically rigid)
- 1 more tropical quadrangle (tropically rigid)
- 4 types of splits (tropically rigid)
- 1 ray with a non-planar tight span $\rightsquigarrow$ gives 12 as the grand total
- all of them contained in Gr(3, 8) [Macaulay2]

Main Results

Theorem (Herrmann & J. 2011+)

For arbitrary $V \in \mathbb{R}^{3\times(n-3)}$ *the tight span of the matroid subdivision of* $\mathrm{Dr}(3, n)$ *induced by* π_V *coincides with (the natural polytopal subdivision of)* $\operatorname{tconv} V$.

Theorem (Herrmann & J. 2011+)

The tight span of a ray of $\mathrm{Dr}(3, n)$ *is either a line segment (and the ray is a split) or a pure two-dimensional simplicial complex which is contractible.*

- splits of $\Delta(d, n)$ were known
  [Herrmann & J. 2008]

Main Results

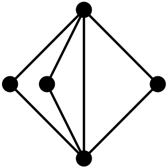
Theorem (Herrmann & J. 2011+)

For arbitrary $V \in \mathbb{R}^{3\times(n-3)}$ *the tight span of the matroid subdivision of* $\mathrm{Dr}(3, n)$ *induced by* π_V *coincides with (the natural polytopal subdivision of)* $\operatorname{tconv} V$*.*

Theorem (Herrmann & J. 2011+)

The tight span of a ray of $\mathrm{Dr}(3, n)$ *is either a line segment (and the ray is a split) or a pure two-dimensional simplicial complex which is contractible.*

- splits of $\Delta(d, n)$ were known
  [Herrmann & J. 2008]



Main Results

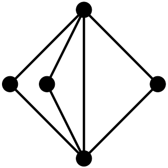
Theorem (Herrmann & J. 2011+)

For arbitrary $V \in \mathbb{R}^{3\times(n-3)}$ *the tight span of the matroid subdivision of* $\mathrm{Dr}(3, n)$ *induced by* π_V *coincides with (the natural polytopal subdivision of)* $\operatorname{tconv} V$.

Theorem (Herrmann & J. 2011+)

The tight span of a ray of $\mathrm{Dr}(3, n)$ *is either a line segment (and the ray is a split) or a pure two-dimensional simplicial complex which is contractible.*

- splits of $\Delta(d, n)$ were known
  [Herrmann & J. 2008]



Open Questions

TECHNISCHE UNIVERSITÄT DARMSTADT

- Does the tropical complex always coincide with the tight span of the induced matroid subdivision, that is, for arbitrary $d \geq 4$?
  - suffices to look at the *diagonal cases* where $n = 2d$
  - would show that tropically rigid point configurations can always be raised to rays of the Grassmannian
- Are all rays with a non-planar tight span induced by tropical point configurations?
- Can you relate the tight span of any ray of the Dressian to a membrane in a Bruhat–Tits-building of type $\widetilde{A}_{d-1}$? [Werner]
  - consider Plücker embedding of (tropical) Grassmannian
  - (weaker) combinatorial version: Is the tight span a flag simplicial complex?

Open Questions

- Does the tropical complex always coincide with the tight span of the induced matroid subdivision, that is, for arbitrary $d \geq 4$? [Speyer: YES!]
  - suffices to look at the *diagonal cases* where $n = 2d$ [irrelevant]
  - does show that tropically rigid point configurations can always be raised to rays of the Grassmannian
- Are all rays with a non-planar tight span induced by tropical point configurations?
- Can you relate the tight span of any ray of the Dressian to a membrane in a Bruhat–Tits-building of type $\widetilde{A}_{d-1}$? [Werner]
  - consider Plücker embedding of (tropical) Grassmannian
  - (weaker) combinatorial version: Is the tight span a flag simplicial complex?

Open Questions

- Does the tropical complex always coincide with the tight span of the induced matroid subdivision, that is, for arbitrary $d \geq 4$? [Speyer: YES!]
  - suffices to look at the *diagonal cases* where $n = 2d$ [irrelevant]
  - does show that tropically rigid point configurations can always be raised to rays of the Grassmannian
- Are all rays with a non-planar tight span induced by tropical point configurations?
- Can you relate the tight span of any ray of the Dressian to a membrane in a Bruhat–Tits-building of type $\widetilde{A}_{d-1}$? [Werner]
  - consider Plücker embedding of (tropical) Grassmannian
  - (weaker) combinatorial version: Is the tight span a flag simplicial complex?

Open Questions

TECHNISCHE UNIVERSITÄT DARMSTADT

- Does the tropical complex always coincide with the tight span of the induced matroid subdivision, that is, for arbitrary $d \geq 4$? [Speyer: YES!]
  - suffices to look at the *diagonal cases* where $n = 2d$ [irrelevant]
  - does show that tropically rigid point configurations can always be raised to rays of the Grassmannian
- Are all rays with a non-planar tight span induced by tropical point configurations?
- Can you relate the tight span of any ray of the Dressian to a membrane in a Bruhat–Tits-building of type $\widetilde{A}_{d-1}$? [Werner]
  - consider Plücker embedding of (tropical) Grassmannian
  - (weaker) combinatorial version: Is the tight span a flag simplicial complex?