Triangle Contact Representations

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Abstract. It is conjectured that every 4-connected plane triangulation has a triangle contact representation with homothetic triangles. We outline a roadmap for a proof of this conjecture and report on partial results and experimental evidence.

1 Introduction

Our interest in this paper are triangle contact representations of planar graphs with homothetic triangles, i.e, vertices are represented by a set of disjoint triangles that are identical up to scalings and translations, two triangles touch exactly if there is an edge between the corresponding vertices. See Figure 1. For brevity we will refer to such a representation as a *htc-representation*. Using an affine map a htc-representation can be transformed into a htc-representation with equilateral triangles. The big conjecture is:

Conjecture 1 *Every 4-connected planar triangulation has a triangle contact representation with homothetic triangles, i.e., a htc-representation.*

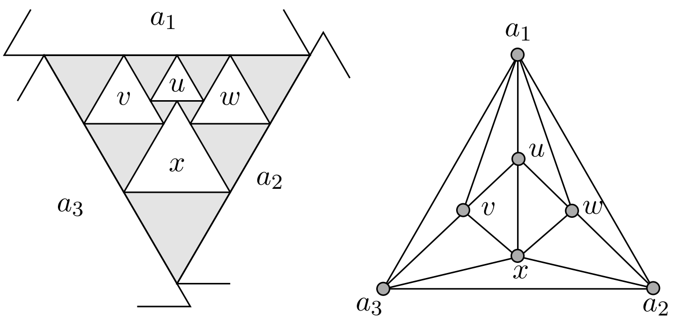


Figure 1: A homothetic triangle contact representation of a planar graph.

The conjecture came up during the Graph Drawing workshop in Bertinoro 2007. In [4] it was shown that max-tolerance graphs are the intersection graphs of homothetic triangles. Lehmann asked whether every planar graph is a max-tolerance graph. Kratochvíl asked for contact representations. A result of the workshop was that planar partial 3-trees (also known as subgraphs of stacked triangulations), and hence also series-parallel graphs, are contact graphs of homothetic triangles, see [1].

De Fraysseix et al. [2] have shown that relaxing the condition on the triangles from equilateral to isosceles allows a contact representation for every planar graph. See Figure 2. Actually, they show that such a representation is possible such that each contact is of the type corner vs. side,

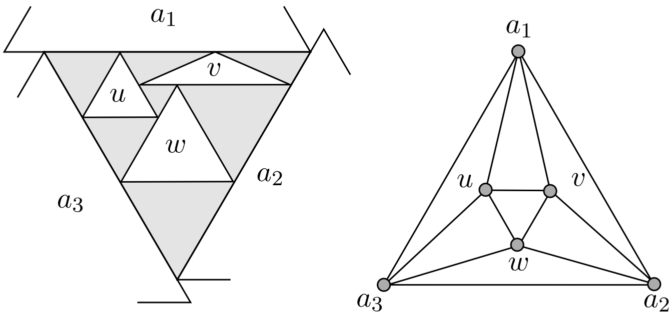


Figure 2: A isocseles triangle contact representation of the octahedron graph.

we call such a contact a *pure contact*. If we ask for a htc-representation of the octahedron graph, then we have to use triangles of equal size for the inner vertices u, v and w. Consequently, there is a point where three corners meet and the 3-face formed by u, v and w is only represented by their mutual contact point, it is degenerated to size 0. This implies that graphs obtained from the octahedron by glueing a triangulation H into the face u, v, w can only have htc-representations where the triangles representing the inner vertices of H are of size 0. We shall not allow this. The kind of degeneracy described with this example of the octahedron graph depend on the existence of separating 3-cycles, i.e., they can only occur if the graph is not 4-connected. This is why we have the restriction in the conjecture.

An essential role in our investigations will be played by Schnyder woods:

Definition 1 *An orientation and coloring of the inner edges of T with colors red, green and blue is a* Schnyder wood *if:*

(1) *All edges incident to a_1 are red, all edges incident to a_2 are green and all edges incident to a_3 are blue.*

(2) *Every inner vertex v has three outgoing edges colored red, green and blue in clockwise order. All the incoming edges in an interval between two outgoing edges are colored with the third color, see Figure 3 (left).*

Figure 3: Left: Schnyder's edge coloring rule.
Right: Triangle contacts induce coloring and orientation of edges.

It was observed by de Fraysseix et al. [2] that a triangle contact representation of a triangulation where all contacts are pure induces a Schnyder wood. The construction is as indicated in Figure 3 (right): Color the corners of the triangles in the representation red, green, blue. Given an edge u, v, look at the contact of the corresponding triangles, if a corner of u's triangle is involved, then color the edge with the color of that corner and orient it from u to v.

The construction of a triangle contact representation of a planar graph, in [2], is as follows[1]: First augment the planar graph H to a triangulation G such that H is an induced subgraph of G. Compute a Schnyder wood of G and use this structure to build a pure triangle contact representation. The consequence is that every Schnyder wood of a triangulation G is induced by some triangle contact representation of G. This is not true for htc-representations.

The steps in our approach for htc-representations of triangulations are as follows:

- Compute a Schnyder wood S of the input graph G.
- Based on S build a system $\mathcal{A}_S$ of linear equations.
- Compute a solution x_S of $\mathcal{A}_S$.

If all entries of x_S are non-negative we are done; based on x_S we can build a htc-representation of G that induces S. If there are negative entries in x_S we use the sign information to transform S into another Schnyder wood S' and iterate. We conjecture that independent of the choice of S the sequence $S \to S' \to S'' \to$ has a finite length, i.e., there is a k such that the solution $x_{S^{(k)}}$ of the system corresponding to $S^{(k)}$ is non-negative.

There is strong experimental evidence that the conjecture is true. We have an implementation of the approach and computed thousands of htc-representations for planar graphs with up to 500 vertices. We have also restarted the computation for a fixed graph with alternate Schnyder woods and compared the result. This suggests that a 4-connected plane triangulation with a prescribed outer face has a *unique* htc-representation.

In the next section we give some details on the system $\mathcal{A}_S$ of linear equations and a sketch of the theoretical results we have so far.

2 Details for the Construction and Partial Results

Let G be a plane triangulation with n vertices and a Schnyder wood S. The system $\mathcal{A}_S$ can be written as $A_S \cdot x = \mathbf{e_1}$ with a $(3n-8) \times (3n-8)$ matrix A_S and the first standard basis vector $\mathbf{e_1}$. The components of x are indexed by the $2n-5$ bounded faces and the $n-3$ inner vertices of G. The first equation is

$$\sum_{f \in \mathcal{F}(a_1)} x_f = 1,$$

where $\mathcal{F}(a_1)$ is the set of bounded faces incident to the special vertex a_1. Every inner vertex induces three equations, one for each color. For $c \in \{\text{red, green, blue}\}$ let $\mathcal{F}_c(v)$ be the set of bounded faces incident to v in the interval where edges of color c are incoming. The equation corresponding to (v, c) is

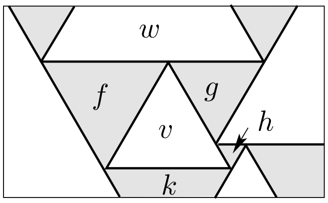
$$-x_v + \sum_{f \in \mathcal{F}_c(v)} x_f = 0.$$

From Figure 3 it is evident that the faces in $\mathcal{F}_c(v)$ are exactly the faces whose triangle has a side contained in the side of v's triangle opposite to the corner of color c. Therefore, the sum of sidelengths of triangles for faces in $\mathcal{F}_c(v)$ has to equal the sidelength of v's triangle. The scheme is illustrated in Figure 4.

The following result implies that the system $\mathcal{A}_S$ has a unique solution.

Fact 1 *The matrix A_S is non-degenerate, i.e.,* $\det(A_S) \neq 0$.

[1] In [2] they speak about *canonical orderings* instead of Schnyder woods, but these are equivalent concepts.



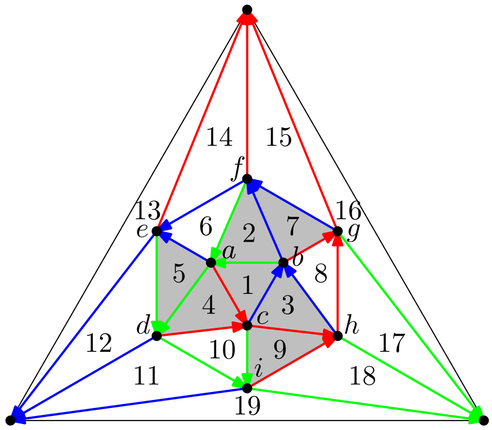
$$x_w = x_f + x_g$$

$$x_v = x_f$$

$$x_v = x_g + x_h$$

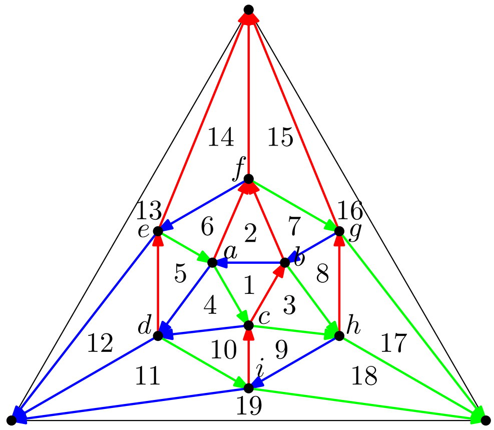
$$x_v = x_k$$

Figure 4: A cutout of a htc-representation and some of the equations it implies. The equations from top to bottom are (w, red), (v, green), (v, blue) and (v, red).



(a) Schnyder wood of the icosahedron. The faces with negative values in the solution vector x are shaded. The boundary of the shaded area is a directed cycle.

Figure 5



(b) Schnyder wood of the icosahedron that results from reverting the cycle in Figure 5a. The new solution vector is non-negative.

The idea for the proof is to show that $(-1)^{n-3}\det(A_S)$ is the number of perfect matchings of an auxiliary graph H_S. Multiplying the columns of A_S corresponding to vertices with -1 yields a 01-matrix $\hat{A}_S$. The graph H_S is the bipartite graph with adjacency matrix $\hat{A}_S$, i.e., it has $6n-16$ vertices, one for each equation of $\mathcal{A}_S$, one for each inner vertex of G and one for each bounded face of G. The non-vanishing summands $\prod_i \hat{a}_{i\,\sigma(i)}$ in the Leibniz-expansion of $\det(\hat{A}_S)$ are in bijection to the perfect matchings M_σ of H_S. The contribution of M_σ to $\det(\hat{A}_S)$ is $\text{sign}(\sigma)$. Define the sign of a matching M_σ as $\text{sign}(M_\sigma) = \text{sign}(\sigma)$. The crucial observations for the proof of Fact 1 are:

(1) If M and M' are perfect matchings of H_S, then $\text{sign}(M) = \text{sign}(M')$.

(2) H_S has a perfect matching.

The proof is based on properties of H_S: The graph H_S is planar and all its bounded faces are of length 6.

Fact 2 *If the unique solution x of $A_S \cdot x = \mathbf{e_1}$ is non-negative, then there is a htc-representation where the triangles of inner vertices and bounded faces have sidelengths as given by the vector x.*

Fact 3 *If the unique solution x of $A_S \cdot x = \mathbf{e_1}$ has negative entries, then we can decompose the boundary between negative and non-negative faces into cycles that are directed in the Schnyder wood.*

From the theory of Schnyder woods it is know that the coloring of edges can be recovered if only the orientation of edges is given and indeed every 3-orientation, i.e., orientation such that every inner vertex has out-degree 3, corresponds to a Schnyder wood. This implies that a directed cycle of a Schnyder wood S can be reverted and appropriate recoloring yields another Schnyder wood S'.

Therefore, Fact 3 implies that whenever the solution x to the system $A_S \cdot x = \mathbf{e_1}$ has negative components, this solution can be used to move to another Schnyder wood S'. Figure 5 shows an example for Fact 3 and the transition $S \to S'$.

Let S and S' be Schnyder woods of a triangulation G. In [3] it is shown that S' can be reached from S via a series of triangle-flips, i.e., via a series of reversals of directed cycles of length three. Moreover if γ is a simple directed cycle in a Schnyder wood S, then $S' = \mathrm{flip}(\mathrm{S}, \gamma)$ can be obtained by flipping the triangles contained in γ.

Therefore it is particularly important to understand the effect of triangle-flips on the solution vectors.

Fact 4 *If Schnyder woods S and S' are related by a triangular-flip at a face f and x, x' are the solutions of the systems $\mathcal{A}_S$ and $\mathcal{A}_{S'}$, then*

$$\mathrm{sign}(x_f) \neq \mathrm{sign}(x'_f).$$

This suggests that starting with some Schnyder wood S and flipping negative faces may lead to Schnyder wood without negative faces, i.e., to a non-negative solution, hence, to a htc-representation. This is what happens in the experiments.

The proof of Fact 4 again uses the correspondence between determinants and matchings that was exploited for Fact 1. Indeed the solution x of $\hat{A}_S \cdot x = \mathbf{e_1}$ is explicitely given as the first column of the inverse of $\hat{A}_S$ wherefore the entry for a vertex or face z can be written in terms of the determinant of a cofactor: $\det(\hat{A}_S)\, x_z = \pm \det([\hat{A}_S]_{1,z})$.

Acknowledgments

First ideas and experiments concerning htc-representations where worked out at the Graph Drawing workshop in Bertinoro 2007. This was joint work with Jan Kratochvíl, Ileana Streinu and Alexander Wolff. Actually the basic idea of using Schnyder woods to generate a system of equations and flipping cycles to get rid of negative variables was born there. Figure 5 is taken from a memo written by Alexander Wolff.

A very good and useful implementation of the approach is due to my student Julia Rucker. I also thank Torsten Ueckerdt for discussions and his continuing interest in the topic.

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