Equiangular polygon contact representations

Stefan Felsner*
felsner@math.tu-berlin.de
Institut für Mathematik
Technische Universität Berlin

Hendrik Schrezenmaier*
schrezen@math.tu-berlin.de
Institut für Mathematik
Technische Universität Berlin

Raphael Steiner
steiner.raphael@gmx.de
Fakultät für Mathematik und Informatik
FernUniversität in Hagen

December 20, 2017

Abstract. Planar graphs are known to have contact representations of various types. The most prominent example is Koebe's 'kissing coins theorem'. Its rediscovery by Thurston lead to effective versions of the Riemann Mapping Theorem and motivated Schramm's Monster Packing Theorem. Monster Packing implies the existence of contact representations of planar triangulations where each vertex v is represented by a homothetic copy of some smooth strictly-convex prototype P_v.

With this work we aim at computable approximations of Schramm representations. For fixed K approximate P_v by an equiangular K-gon Q_v with horizontal basis. From Schramm's work it follows that the given triangulation also has a contact representation with homothetic copies of these K-gons. Our approach starts by guessing a *K-contact-structure*, i.e., the combinatorial structure of a contact representation. From the combinatorial data, we build a system of linear equations whose variables correspond to lengths of boundary segments of the K-gons. If the system has a non-negative solution, this yields the intended contact representation. If the solution of the system contains negative variables, these can be used as sign-posts indicating how to change the K-contact-structure for another try.

In the case $K = 3$ the K-contact-structures are Schnyder woods, and in the case $K = 4$ they are transversal structures. As in these cases, for $K \geq 5$ the K-contact-structures of a fixed graph are in bijection to certain integral flows, and can be viewed as elements of a distributive lattice.

The procedure has been implemented, it computes the solution with few iterations. The experiments involved graphs with up to one hundered vertices.

*Partially supported by DFG grant FE-340/11-1.

1 Introduction

Representations of graphs by contacts of geometric objects are actively studied in graph theory and geometry. An early result in this direction is Koebe's circle packing theorem from 1936, it states that every planar graph can be represented as the contact system of a set of interiourly disjoint circles. Koebe arrived at this result in the context of conformal mapping of 'contact domains'. Unaware of Koebe's work Thurston reproved the circle packing theorem and connected it to the Riemann Mapping Theorem. This line of research resulted in discretizations of conformal mapping and has strong impact in the area of discrete differential geometry. We refer to [21] and [2] for further details on those connections.

A very strong generalization of Koebe's theorem is Schramm's Convex Packing Theorem (Theorem 2) from 1990 [16]. The theorem states that if each vertex v of a planar triangulation G has a prescribed convex prototype P_v, then there is a contact representation of G where each vertex is represented by a (possibly degenerate) homothet of its prototype. When the prototypes have a smooth boundary there are no degeneracies. With this work we aim at computable approximations of Schramm representations. The idea is to approximate the prototypes P_v with simpler shapes, we use equiangular K-gons. Clearly, a sequence of approximating contact representations with K-gons, one for each positive integer K and each of them confined to the unit square, will contain a subsequence converging to a representation with the prototypes P_v.

Contact representations of graphs with polygons have also been studied widely. Triangle contact representations have been investigated by De Fraysseix et al. [6]. They observed that Schnyder woods can be considered as combinatorial encodings of triangle contact representations of triangulations and that any Schnyder wood can be used to construct a corresponding triangle contact system. Gonçalves et al. [12] observed that Schramm's Convex Packing Theorem can be used to prove the existence of contact representations with homothetic triangles for all 4-connected triangulations. A more combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations which are based on a Schnyder wood was described by Felsner [8]. On the basis of this approach Schrezenmaier [19] reproved the existence of homothetic triangle contact representations.

Representations of graphs with side contacts of rectangles have applications in architecture and VLSI design. For links into the extensive literature we recommend [4] and [9]. Representations of graphs using squares or, more precisely, graphs as a tool to model packings of squares already appear in classical work of Brooks et al. [3] from 1940. Schramm [17] proved that every 5-connected inner triangulation of a 4-gon admits a square contact representation. Again there is a combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations, see Felsner [9]. In this context *transversal structures* play the role of Schnyder woods. As in the case of homothetic triangles this approach is based on an iterative procedure, however, a proof that the iteration terminates is still missing. On the basis of the approach Schrezenmaier [18] reproved Schramm's Squaring Theorem.

Before stating our results we introduce some precise terminology. A *K-gon contact system* $\mathcal{S}$ is a finite system of convex K-gons in the plane such that the interiors of any two K-gons are disjoint. If all K-gons of $\mathcal{S}$ are equiangular K-gons (i.e., all interior angles are $\frac{K-2}{K}\pi$) with a horizontal segment at the bottom, we call $\mathcal{S}$ an *equiangular K-gon contact representation.* The contact system has an *exceptional touching* if there is a point where two corners of K-gons meet. The *contact graph* $G^*(\mathcal{S})$ of $\mathcal{S}$ is the graph that has a vertex for every K-gon and an edge for every contact of two K-gons in $\mathcal{S}$. Note that $G^*(\mathcal{S})$ inherits a crossing-free embedding from $\mathcal{S}$. For a given plane graph G and a K-gon contact system $\mathcal{S}$ with $G^*(\mathcal{S}) = G$ we say that $\mathcal{S}$ is a *K-gon contact representation* of G.

We will only consider the case that G is an *inner triangulation of a K-gon*, i.e., the outer face

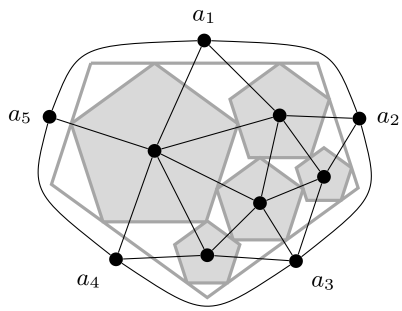

Figure 1: An equiangular pentagon contact representation of the graph shown in black where each inner vertex is represented by a regular pentagon.

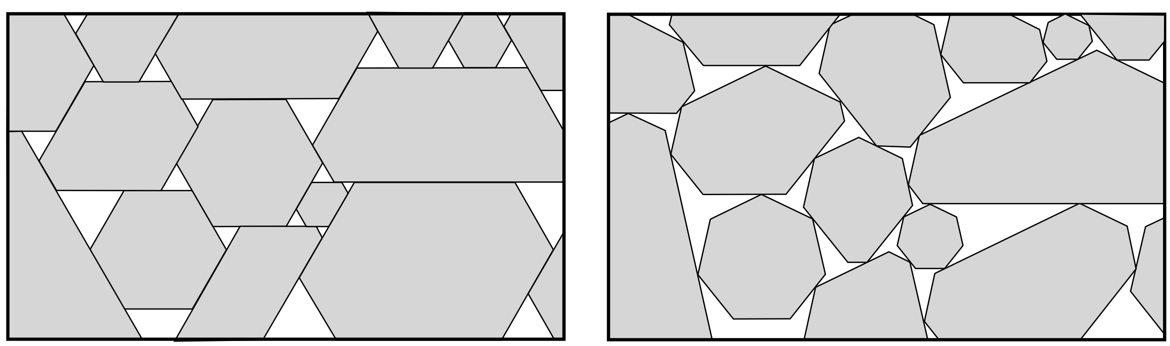

Figure 2: Parts of equiangular 6-gon and 7-gon contact representations of the same graph.

of G is a K-gon with vertices $a_1, \ldots, a_K$ in clockwise order, all inner faces are triangles, there are no loops or multiple edges, and there are no additional edges between the outer vertices. Our interest lies in regular K-gon contact representations of G with the additional property that $a_1, \ldots, a_K$ are represented by line segments $s_1, \ldots, s_K$ which together form an equiangular K-gon. The line segment s_1 is always horizontal and at the top, and $s_1, \ldots, s_K$ is the clockwise order of the segments of the K-gon. Figure 2 shows contact systems of 6-gons and 7-gons respectively.

Let G be an inner triangulation of a K-gon and for each inner vertex v of G let P_v be a prescribed equiangular K-gon. From Schramm's Convex Packing Theorem it follows that G has a representation as contact graph of homothets of the prototypes (see Section 2). The representation is non-degenerate whenever $K \geq 5$ and odd, or $K \geq 8$ and even. For $K = 3$ and $K = 6$ the graph needs to be 4-connected to guarantee a non-degenerate representation, this is because the three K-gons corresponding to a triangle in G can touch in a single point such that there is no space left for the K-gons of vertices in the interior of this triangle.

We propose a new method for computing equiangular K-gon contact representations. The idea is to guess the combinatorial structure of the representation of G, i.e., for each edge uv of G guess whether the contact involves a corner of P_u or a corner of P_v and also guess which corner of the respective prototype is involved. The guess is encoded in a K-contact-structure. The K-contact-structure leads to a system of linear equations whose variables correspond to lengths of boundary segments of the K-gons. The system is non-singular. If it has a non-negative solution, the values of the variables determine the geometry of a K-gon contact representation. If the solution of the system contains negative values, then it is possible to locally modify the K-contact-structure in the local neighborhood of negative variables. The modified K-contact-structure corresponds to a new system of equations which has a new solution. This yields an iterative procedure which *hopefully* stops with a positive solution, i.e., with a K-gon contact representation.

We cannot prove that the above iterative procedure stops. However the algorithm has been implemented and was used for extensive experiments. These have always been successful. Similar algorithms for the computation of contact representations by homothetic triangles or squares have been described by Felsner [8, 9]. These have also been implemented and successfully tested, c.f. Rucker [15] and Piccetti [13] respectively. We therefore conjecture that the proposed algorithm for computing equiangular K-gon contact representations always terminates with a solution.

In Section 3 we introduce *K-contact-structures* of G, these are certain weighted orientations of a supergraph of G. In Section 4 we enhance K-contact-structures with a K-coloring of the edges. The color classes are directed forests, they somehow resemble the trees of a Schnyder wood. In Section 5 we show that there is a distributive lattice on the set of K-contact-structures of a fixed graph G and describe the combinatorial change in K-contact-structures that form a cover pair. In Section 6 we discuss the system of linear equations and prove that it is non-singular. Section 7 describes the iteration which is proposed as a heuristic for computing equiangular K-gon contact representations. In Section 8 we describe the experiments we made to test the heuristic.

In this paper we focus on odd $K \geq 5$. The case $K = 3$ is well-studied and the case $K \geq 6$ and even will be added in a later version of this paper. The case $K = 5$ was first studied in the bachelor thesis of Steiner [20] and further elaborated by the present team of authors [11].

In the main part we skip most of the proofs and some lemmas. They can be found in the Appendix. The results of the experiments described in Section 8 can also be found in the Appendix.

2 The existence of equiangular K-gon contact representations

In this section let G be an inner triangulation of a K-gon and let V_{inner} be the set of inner vertices of G. Further, for each $v \in V_{\text{inner}}$, let $\mathcal{P}_v$ be an equiangular K-gon with a horizontal segment at the bottom. We call $\mathcal{P}_v$ the *prototype* of v. A *(positive) homothetic* copy of a prototype $\mathcal{P}_v$ is a set in the plane that can be obtained from $\mathcal{P}_v$ by scaling (with a positive factor) and translation. By a homothetic copy we always mean a positive homothetic copy.

Theorem 1. *For odd $K \geq 5$ there exists an equiangular K-gon contact representation of G in which each $v \in V_{\text{inner}}$ is represented by a homothetic copy of its prototype $\mathcal{P}_v$.*

The proof of this theorem is based on the following general result about contact representations by Schramm.

Theorem 2 (Convex Packing Theorem [16])**.** *Let H be an inner triangulation of the triangle abc. Further let C be a simple closed curve in the plane partitioned into three arcs $\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_c$, and for each interior vertex v of H let $\mathcal{Q}_v$ be a convex set in the plane containing more than one point. Then there exists a contact representation of a supergraph of H (on the same vertex set, but possibly with more edges) where each interior vertex v is represented by a single point or a homothetic copy of its prototype $\mathcal{Q}_v$ and each outer vertex w by the arc $\mathcal{Q}_w$.*

Proof of Theorem 1. By adding edges from $a_{\frac{K+1}{2}}$ to $a_{\frac{K-3}{2}}, a_{\frac{K-5}{2}}, \ldots, a_1$ and edges from $a_{\frac{K+3}{2}}$ to $a_{\frac{K+7}{2}}, a_{\frac{K+9}{2}}, \ldots, a_1$ in this order in the outer face of G, it becomes a triangulation G' with outer face $a_1 a_{\frac{K+1}{2}} a_{\frac{K+3}{2}}$. We define the arcs $\mathcal{Q}_{a_1}, \mathcal{Q}_{a_{\frac{K+1}{2}}}, \mathcal{Q}_{a_{\frac{K+3}{2}}}$ to be elongations of the edges s_1, $s_{\frac{K+1}{2}}$ and $s_{\frac{K+3}{2}}$ of an equiangular K-gon A with horizontal segment s_1 at the top, such that $\mathcal{Q}_{a_1} \cup \mathcal{Q}_{a_{\frac{K+1}{2}}} \cup \mathcal{Q}_{a_{\frac{K+3}{2}}}$ is a triangle and therefore a simple closed curve. We define the convex sets $\mathcal{Q}_{a_i}$ for all $i \in \{2, \ldots, \frac{K-1}{2}, \frac{K+5}{2}, \ldots, K\}$ to be line segments parallel to the edge s_i of the pentagon A (see Fig. 3). Finally, for each interior vertex v of G let $\mathcal{Q}_v := \mathcal{P}_v$ be the given prototype of v.

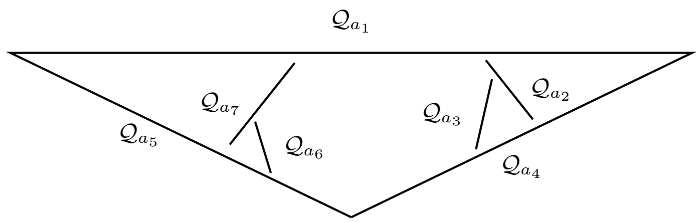

Figure 3: Prototypes for the outer vertices of G (for $K = 7$).

Now we can apply Theorem 2. Therefore there exists a contact representation of a supergraph of G' where $a_1, a_{\frac{K+1}{2}}, a_{\frac{K+3}{2}}$ are represented by $\mathcal{Q}_{a_1}, \mathcal{Q}_{a_{\frac{K+1}{2}}}, \mathcal{Q}_{a_{\frac{K+3}{2}}}$ and the other vertices v by a homothetic copy of $\mathcal{Q}_v$ or a single point.

Next we will prove that in this contact representation of G' none of the homothetic copies of the prototypes is degenerate to a single point. So assume there is a degenerate copy in this contact representation. Let H be a maximal connected component of the subgraph of G' induced by the vertices whose K-gons are degenerate to a single point. Since the line segments corresponding to the three outer vertices are not degenerate, H has to be bounded by a cycle C of vertices whose K-gons respectively line segments are not degenerate. In the contact representation all vertices of H are represented by the same point and therefore all K-gons respectively line segments representing the vertices of C meet in this point. But since the interior angles of equiangular K-gons are too large for $K \geq 5$, at most two of these can meet in a single point. Thus C is a 2-cycle, in contradiction to our definition of inner triangulations that does not allow multiple edges.

After cutting the segments $\mathcal{Q}_{a_i}$, the vertices $a_1, \ldots, a_K$ are represented by an equiangular K-gon with a horizontal segment at the top and we obtain a regular K-gon contact representation of G. $\square$

Similar proofs have been given for the case $K = 3$ in [12] and $K = 5$ in [20] and [11].

3 The combinatorial structure of equiangular polygon contact representations

For the entire section let G be an inner triangulation of a K-gon, $K \geq 3$ odd. We call an inner face of G a *completely inner face* if it is only incident to inner edges. We denote the set of inner edges of a planar graph H by $E_{\text{inner}}(H)$. For the directed graphs used later in this section we denote the sets of incoming and outgoing edges of a vertex v by $E_{\text{in}}(v)$ and $E_{\text{out}}(v)$.

Definition 1. The *stack extension* $G^\star$ of G is the extension of G that contains an extra vertex in every completely inner face. These new vertices are connected to all three vertices of the respective face. We call the new vertices *stack vertices* and the vertices of G *normal vertices*. Analogously, we call the new edges *stack edges* and the edges of G *normal edges*.

Definition 2. A *K-contact-structure* on G is an orientation and weighting $w : E_{\text{inner}}(G^\star) \to \mathbb{N}$ of the inner edges of $G^\star$ such that

(P1) $w(e) = 1$ for each normal edge e,

(P2) each stack edge is oriented towards its incident stack vertex,

(P3) the out-flow of each normal vertex u is K, i.e., $\sum_{e \in E_{\text{out}}(u)} w(e) = K$,

(P4) the in-flow of each stack vertex v is $\frac{K-3}{2}$, i.e., $\sum_{e \in E_{\text{in}}(v)} w(e) = \frac{K-3}{2}$.

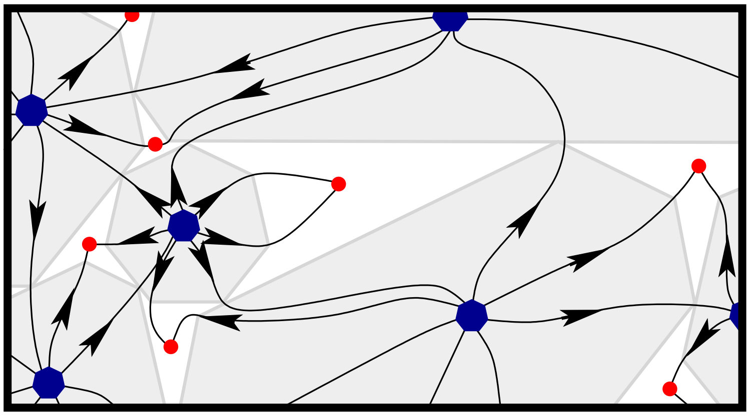

Figure 4: The graph $G^\star_+(\mathcal{A})$ for the 7-contact-structure $\mathcal{A}$ induced by the shown contact representation.

Definition 3. Let $\mathcal{A}$ be a K-contact-structure on G. Then we can associate with $\mathcal{A}$ a modified version of $G^\star$ where each inner edge e is replaced by $w(e)$ parallel edges and all edges are oriented as in $\mathcal{A}$. We denote this graph by $G^\star_+(\mathcal{A})$.

The following theorem shows the key correspondence between K-contact-structures and equiangular K-gon contact representations.

Theorem 3. *Every equiangular K-gon contact representation induces a K-contact-structure on its contact graph (see Fig. 4 for an illustration).*

Proof. Let $\mathcal{S}$ be an equiangular K-gon contact representation of $G = G^*(\mathcal{S})$ without exceptional touchings. Let e be an inner normal edge of $G^\star$. Then e corresponds to the contact of a corner of a K-gon A and a segment of a K-gon B in $\mathcal{S}$. We orient the edge e from the vertex corresponding to A to the vertex corresponding to B. Now let $e = uv$ be a stack edge with normal vertex u and stack vertex v. Then u corresponds to a K-gon A of $\mathcal{S}$ and v to an area F in $\mathcal{S}$ which is enclosed by A and two more K-gons or outer segments s_i. Note that F is a pseudotriangle, i.e., a polygon with exactly three convex corners and arbitrarily many concave corners. We define $w(e)$ to be the number of concave corners of F which are also corners of A.

Let u be a normal vertex of $G^\star$ and A the corresponding K-gon in $\mathcal{S}$. Since each corner of A either corresponds to an outgoing normal edge of u or contributes exactly one amount of weight to an outgoing stack edge of u, the out-flow of u is exactly K. Thus, property (P3) is fulfilled.

For showing property (P4) consider the following labeling of the segments and corners of the K-gons of $\mathcal{S}$: The segments of a K-gon get the labels $1, \dots, K$ in clockwise order, starting with label 1 at the horizontal segment at the bottom. Further also the corners get the labels $1, \dots, K$ in clockwise order, starting with label 1 at the corner at the top. Then for geometric reasons each corner-segment contact of two K-gons involves a corner and a segment with the same labels. Let F be the pseudotriangle in $\mathcal{S}$ corresponding to a stack vertex of $G^\star$. Let a be the number of segments involved in F and let b be the number of corners involved in F. Then $a = b$ since the border of F is an alternating sequence of segments and corners. Further $a + b = K + 3$ since for each label l either a segment or a corner with label l is involved in F, except for the three convex corners of F where both, a segment and a corner of the same label, are involved (see Fig.5). Therefore, the number of concave corners involved in F is $b - 3 = \frac{K+3}{2} - 3 = \frac{K-3}{2}$.

In the case that $\mathcal{S}$ has exceptional touchings, each exceptional touching of two K-gon corners can be interpreted in two ways as a corner-segment contact with infinitesimal distance to the

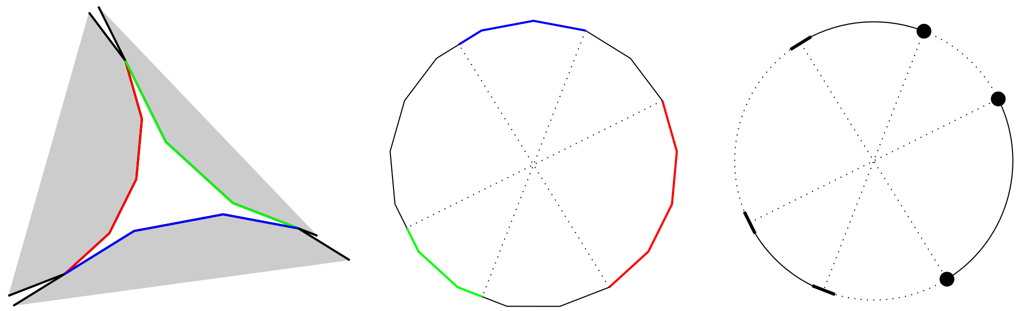

Figure 5: A pseudotriangle F in an equiangular K-gon contact representation. In the middle the edges involved in F have been transferred to a regular K-gon. On the right there is an abstract representation we will use later.

other corner. We choose one of these interpretations and proceed as before. Hence, the K-contact-structure induced by an equiangular K-gon contact representation with exceptional touchings is not unique. $\square$

Theorem 4. *Let G be an inner triangulation of a K-gon. Then there exists a K-contact-structure on G.*

Theorem 4 immediately follows from Theorem 1 and Theorem 3. Since we aim for a theory independent from the Monster Packing Theorem by Schramm, we give another elementary proof of Theorem 4.

For a plane graph H and a subset A of the vertices of H we denote the set of edges of H incident to a vertex of A by $E_{\text{inc}}(H, A)$ and the set of faces of H incident to a vertex of A by $F_{\text{inc}}(H, A)$. If the graph is clear from the context, we also use the shorter notations $E_{\text{inc}}(A)$ and $F_{\text{inc}}(A)$.

Lemma 1. *Let T be a triangulation and let A be a set of $k \geq 1$ vertices of T. Then*

$$|E_{\text{inc}}(T, A)| \geq 3k \ , \ |F_{\text{inc}}(T, A)| \geq 2k + 1 \ .$$

Proof. Let $\overline{A} := V(T) \setminus A$. Then $E_{\text{inc}}(A) = E(T) \setminus E(T[\overline{A}])$. Since T is a triangulation, we have $|E(T)| = 3|V(T)| - 6$. As $T[\overline{A}]$ is a simple plane graph, we also have $|E(T[\overline{A}])| \leq 3(|V(T)| - k) - 6$. Putting this together we obtain the inequality $|E_{\text{inc}}(A)| \geq 3k$.

The faces of T contained in $F(T) \setminus F_{\text{inc}}(A)$ are also faces of $T[\overline{A}]$. But at least one face of $T[\overline{A}]$ is not a face of T since it has a vertex of A in its interior. Thus $|F_{\text{inc}}(A)| \geq |F(T)| - (|F(T[\overline{A}])| - 1)$. As T is a triangulation, $|F(T)| = 2|V(T)| - 4$. As $T[\overline{A}]$ is planar, $|F(T[\overline{A}])| \leq 2(|V(T)| - k) - 4$. Putting this together we get $|F_{\text{inc}}(A)| \geq 2k + 1$. $\square$

Proof of Theorem 4. Let H be the graph obtained from $G^\star$ by replacing each stack edge by $\frac{K-3}{2}$ parallel edges. We will show that there exists an orientation of the inner edges of H such that each normal vertex has out-degree K and each stack vertex has in-degree $\frac{K-3}{2}$. Then we obtain a K-contact-structure on G by giving each normal edge the orientation from H and by setting the weight of each stack edge $e = uv$ with normal vertex u and stack vertex v to the number of edges in H oriented from u to v.

Note that, instead of requiring the in-degree of a stack vertex v to be $\frac{K-3}{2}$, we can require its out-degree to be $\deg(v) - \frac{K-3}{2} = K - 3$. Orientations with prescribed out-degrees for all vertices have been studied in [7] under the name of *α-orientations*. From there we take the following sufficient condition for the existence of the orientation we seek for: The orientation exists if for every subset W of vertices of H, if W consists of k normal inner vertices and l stack vertices, then $|E_{\text{inc}}(H,W)| \geq kK + l(K-3)$ with equality if W is the set of all inner vertices of H.

Let W be a set of inner vertices of $G^\star$ and let X and Y be the sets of normal vertices and stack vertices from W. We have to show that $|E_{\text{inc}}(H,W)| \geq |X|K + |Y|(K-3)$. To show the inequality we count edges incident to W in $G^\star$. Let $E^{\text{old}}_{\text{inc}}(G^\star,W) = E_{\text{inc}}(G,X)$ be the edges of G which are incident to W. From Lemma 1 we obtain that $|E^{\text{old}}_{\text{inc}}(G^\star,W) \geq 3|X|$. Let $E^{\text{new}}_{\text{inc}}(G^\star,W) = E_{\text{inc}}(G^\star,W) \setminus E^{\text{old}}_{\text{inc}}(G^\star,W)$. Note that $|E^{\text{new}}_{\text{inc}}(G^\star,W)| \geq 2|W|$ implies $|E_{\text{inc}}(H,W)| \geq |W|(K-3) + 3|X| = |X|K + |Y|(K-3)$.

To estimate $|E^{\text{new}}_{\text{inc}}(G^\star,W)|$ we look at X and Y independently. First we define G° to be G minus the edges of the outer cycle. Note that all the inner faces of G° are completly inner faces of G, i.e., in $G^\star$ all these faces contain stack vertices. Let $\mathcal{F}$ be the set of inner faces of G°. From Lemma 1 it follows that every subset X' of X is incident to at least $2|X'|$ faces in $\mathcal{F}$. This is the Hall condition for the bipartite graph whose vertices are two copies of X on one side and $\mathcal{F}$ on the other side, and edges are given by incidences. A maximal matching M on this graph is an assignment of two faces to every element of X. This yields $2|X|$ stack edges incident to vertices of X in $G^\star$. Now consider a vertex $y \in Y$, the face of G containing y has been assigned to at most one vertex from X through M. Therefore, we can assign the remaining two edges to y. Doing this for all $y \in Y$ we get $2|Y|$ stack edges incident to vertices of Y which have not been counted for X. This shows $|E^{\text{new}}_{\text{inc}}(G^\star,W)| \geq 2|W|$.

If G has n vertices, then it has $n-K$ inner vertices and $2n-K-2$ inner faces. Hence, the number of stack vertices in $G^\star$ is $2(n-K-1)$. It remains to verify that $|E_{\text{inc}}(H, V_{\text{inner}}(H))| = (n-K)K + 2(n-K-1)(K-3)$. Each stack vertex is incident to 3 stack edges in $G^\star$, this yields $3\frac{K-3}{2}2(n-K-1)$ edges of H. In addition there are $3n-2K-3$ edges which are incident to inner vertices in G. Since $3(K-3)(n-K-1) + (3n-2K-3) = (n-K)K + 2(n-K-1)(K-3)$, this completes the proof. $\square$

4 Coloring K-contact-structures

In this section let G be an inner triangulation of a K-gon, let $\mathcal{A}$ be a K-contact-structure on G and let $G^\star_+ := G^\star_+(\mathcal{A})$. In the following, the set of colors $1,\ldots,K$ is to be understood as representatives modulo K, i.e., colors c and $c+zK$ are the same for any $z \in \mathbb{Z}$.

Definition 4. A *proper coloring* of $G^\star_+$ is a coloring of the inner edges of $G^\star_+$ in the colors $1,\ldots,K$ such that

(C1) for $i = 1,\ldots,K$ all edges incident to the outer vertex a_i have color i,

(C2) each normal vertex has exactly one outgoing edge in each color and the clockwise order of the colors is $1,\ldots,K$,

(C3) incoming edges of a normal vertex, which are located between the outgoing edges of colors c and $c+1$, have color $c - \frac{K-1}{2}$.

An equiangular K-gon contact representation $\mathcal{S}$ induces a K-contact-structure together with a proper coloring. To see this, recall the construction in the proof of Theorem 3. Each inner edge of $G^\star_+$ corresponds to a corner of a K-gon of $\mathcal{S}$. We color the corners of each K-gon of $\mathcal{S}$ in the colors $1,\ldots,K$ in clockwise order, starting with color 1 at the corner at the top (opposite to the horizontal segment at the bottom). Then each inner edge of $G^\star_+$ gets the color of the corner it corresponds to.

We want to show that this coloring is a property of the K-contact-structure itself, i.e., each K-contact-structure has a unique proper coloring. The idea of the construction of the colors will be as follows: We start with an inner edge e of $G_+^\star$ and follow a properly defined path that at some point reaches one of the outer vertices. Then the color of this outer vertex will be the color of e. This approach is similar to the proof of the bijection of Schnyder Woods and 3-orientations in [5].

Now we will define the paths starting with an inner edge e and ending at an outer vertex that allow us to define the color of e. In the definition of these paths we aim at continuing with the outgoing edge on the opposite side of a vertex. This is motivated by the following geometric idea: If we are already given an equiangular contact representation, such paths keep a constant slope and therefore run into an outer segment with corresponding slope. If we run into a stack vertex, there is no unique opposite edge. Therefore, the path of e is not unique, but we can associate a unique outer vertex with e.

Definition 5. Let $e = uv$ be an outgoing inner edge of $G_+^\star$ such that u is a normal vertex. We will recursively define a set $\mathcal{P}(e)$ of walks starting with e by distinguishing several cases concerning v.

- If v is an outer vertex, i.e., $v = a_i$ for some i, the set $\mathcal{P}(e)$ contains only one path, the path only consisting of the edge e.
- If v is an inner normal vertex, let e' be the opposite outgoing edge of e at v, i.e., the $\frac{K+1}{2}$th outgoing edge in clockwise or counterclockwise direction, and we define $\mathcal{P}(e) := \{e + P : P \in \mathcal{P}(e')\}$.
- If v is a stack vertex, let v_1 and v_2 be the two vertices of G which follow u in the clockwise traversal of the facial cycle of G corresponding to v. Further let n_1 be the number of edges from u to v to the left of e and n_2 be the number of edges from u to v to the right of e. Let e_1 be the $(\frac{K-1}{2} - n_1)$th outgoing edge of v_1 in counterclockwise order after the edge v_1u and let e_2 be the $(\frac{K-1}{2} - n_2)$th outgoing edge of v_2 in clockwise order after the edge v_2u. Note that e_i is well defined if v_i is not an outer vertex, and that not both of v_1 and v_2 can be outer vertices. If both of e_1 and e_2 are well defined, we set $\mathcal{P}(e) := \{e+vv_1+P : P \in \mathcal{P}(e_1)\} \cup \{e+vv_2+P : P \in \mathcal{P}(e_2)\}$. If only e_i is well defined, we set $\mathcal{P}(e) := \{e + vv_i + P : P \in \mathcal{P}(e_i)\}$.

It is not clear that these walks are paths. If they do not cycle, they have to end in an outer vertex. But we have to prove that they indeed do not cycle.

Lemma 2. *Let C be a simple cycle of length ℓ in $G_+^\star$ and let all edges of C be inner normal edges. Then there are exactly $\frac{K-1}{2}\ell - K$ edges pointing from C into the interior of C.*

Proof. First we view C as a cycle in G. Let k be the number of vertices inside C. Since G is a triangulation, Euler's formula implies that there are exactly $2k + \ell - 2$ faces and $3k + \ell - 3$ edges strictly inside C.

Now we view C as a cycle in $G_+^\star$. Then the number of edges inside C is

$$\frac{K-3}{2}(2k + \ell - 2) + (3k + \ell - 3) = K(k-1) + \frac{K-1}{2}\ell \ .$$

The number of edges starting at a vertex inside C is kK. Therefore the number of edges pointing from a vertex of C into the interior is

$$K(k-1) + \frac{K-1}{2}\ell - kK = \frac{K-1}{2}\ell - K \ .$$

□

To be able to apply Lemma 2 to the walks defined in Definition 5, we introduce the *shortcut* of such a walk that avoids the stack vertices.

Definition 6. Let P be a subwalk of a walk in $\mathcal{P}(e)$ for some edge e that starts and ends with a normal vertex. Then the *shortcut* P' of P is obtained from P by replacing every consecutive pair uv, vw of edges, where v is a stack vertex, by the edge uw. We call uw a *shortcut edge*.

Lemma 3. *Let P' be a shortcut walk of length ℓ that does not start and does not end with a shortcut edge. Then the number of edges pointing from the interior vertices of P' to the right (left) of P' is $\frac{K-1}{2}(\ell-1)$.*

Proof. We prove this by induction on the length ℓ of the walk. Let $P' = v_0, e_1, v_1, \ldots, e_\ell, v_\ell$. If $e_{\ell-1}$ is not a shortcut edge, the statement immediately follows by induction. So assume that $e_{\ell-1}$ is a shortcut edge and let $e' = v_{\ell-2}w, e'' = wv_{\ell-1}$ be the corresponding edges of the original path P. Let Q' be the subwalk of P' starting at v_0 and ending at w. Note that we can apply the induction hypothesis to Q' by assuming that w is a normal vertex. We distinguish three cases.

Case 1: In G the edge $v_{\ell-2}v_{\ell-1}$ (in this orientation) has the face corresponding to w to its left and in $G_+^\star$ it is oriented from $v_{\ell-2}$ to $v_{\ell-1}$. Then let n_2 be the number of parallel edges of $e' = v_{\ell-2}w$ (in this orientation) to its right in $G_+^\star$. Then at $v_{\ell-2}$ the path P' has exactly n_2+1 less edges pointing to its right than Q'. The number of edges pointing from P' to its right at $v_{\ell-1}$ is $K - (\frac{K-1}{2} - n_2) = \frac{K-1}{2} + n_2 + 1$. Therefore the total number of edges pointing from P' to its right is by induction $\frac{K-1}{2}(\ell-1)$.

Case 2: In G the edge $v_{\ell-2}v_{\ell-1}$ (in this orientation) has the face corresponding to w to its left and in $G_+^\star$ it is oriented from $v_{\ell-1}$ to $v_{\ell-2}$. Then let n_2 be defined as in Case 1. Then at $v_{\ell-2}$ the path P' has exactly n_2 less edges pointing to its right than Q'. The number of edges pointing from P' to its right at $v_{\ell-1}$ is $\frac{K-1}{2} + n_2$. Therefore the total number of edges pointing from P' to its right is by induction $\frac{K-1}{2}(\ell-1)$.

Case 3: In G the edge $v_{\ell-2}v_{\ell-1}$ (in this orientation) has the face corresponding to w to its right. Then let n_1 be the number of parallel edges of $e' = v_{\ell-2}w$ (in this orientation) to its left in $G_+^\star$. Then P' has at $v_{\ell-2}$ exactly n_1+1 more edges pointing to its right than Q'. The number of edges pointing from P' to its right at $v_{\ell-1}$ is $\frac{K-3}{2} - n_1 = \frac{K-1}{2} - n_1 - 1$. Therefore the total number of edges pointing from P' to its right is by induction $\frac{K-1}{2}(\ell-1)$. $\square$

Lemma 4. *(i) The walks $P \in \mathcal{P}(e)$ are paths, i.e., there are no vertex repetitions in P.*

(ii) Let $P_1, P_2 \in \mathcal{P}(e)$ be two paths starting with the same edge e. Then P_1 and P_2 end in the same outer vertex.

(iii) Let v be a normal vertex and let $e_1 = vv_1, e_2 = vv_2$ be two different outgoing edges. Further let $P_1 \in \mathcal{P}(e_1)$ and $P_2 \in \mathcal{P}(e_2)$ be two paths. Then P_1 and P_2 do not cross and they end in different outer vertices.

Proof. For (i) assume that P cycles. Let C be a subcycle of the shortcut of P and let ℓ be the length of C. Then, according to Lemma 3, there are at least $\frac{K-1}{2}(\ell-1)$ edges pointing into the interior of C, in contradiction to Lemma 2 which states that there are only $\frac{K-1}{2}\ell - K$ edges pointing into the interior of C.

For (ii) assume that P_1 and P_2 coincide up to a vertex v, then P_1 goes to the left and P_2 to the right, and at a normal vertex w they meet again (they might already meet at a stack vertex immediately before w). Note that v has to be a stack vertex and let v' be its predecessor in P_1 and P_2. Let Q_1 and Q_2 be the subpaths of P_1 and P_2 starting at v' and ending at w, extended

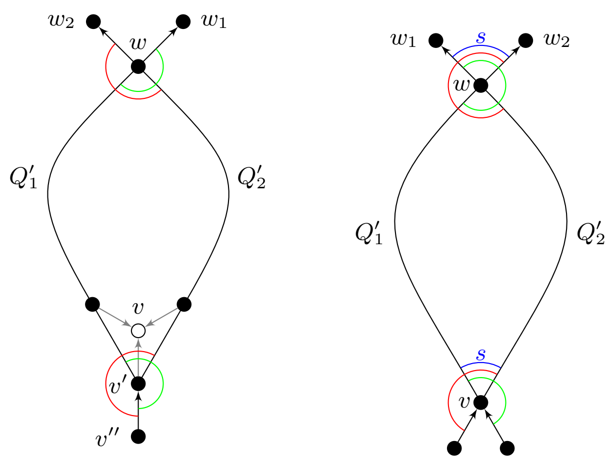

Figure 6: Illustration of the proof of Lemma 4. The edges counted by Q'_1 (Q'_2) are indicated by green (red) angles.

by a common artificial normal edge $v''v'$ and individual artificial normal edges ww_1 and ww_2 (see Fig. 6 (left) for an illustration). For $i = 1, 2$ let Q'_i be the shortcut of Q_i and let ℓ_i be the length of Q'_i. Due to Lemma 3 there are exactly $\frac{K-1}{2}(\ell_1 - 1)$ edges pointing from Q'_1 to the right and exactly $\frac{K-1}{2}(\ell_2 - 1)$ edges pointing from Q'_2 to the left. Let C be the cycle formed by the shortcuts Q'_1 and Q'_2 without their artificial edges. Then the length of C is $\ell_1 + \ell_2 - 4$ and therefore, due to Lemma 2, there are exactly $\frac{K-1}{2}(\ell_1 + \ell_2 - 4) - K$ edges pointing into the interior of C. If we add the number of edges pointing from Q'_1 to the right and the number of edges pointing from Q'_2 to the left, we count exactly K too many edges at v'. Hence, the number of edges we count too many at w is

$$\left(\frac{K-1}{2}(\ell_1 - 1) + \frac{K-1}{2}(\ell_2 - 1)\right) - \left(\frac{K-1}{2}(\ell_1 + \ell_2 - 4) - K\right) - K = K - 1 \ .$$

This means that the artificial edges ww_1 and ww_2 have to be identical and that this artificial edge is the the only outgoing edge of w we do not count too often. Therefore P_1 and P_2 continue with the same edge at w.

Now assume that P_1 and P_2 end at different outer vertices a_i and a_j. Let v be the last common vertex of P_1 and P_2. Because of the above observation v has to be a stack vertex. Assume that P_1 goes to the left at v and P_2 goes to the right. Let v' be the predecessor of v in P_1 and P_2. Further let Q_1 and Q_2 be the subpaths of P_1 and P_2 starting at v' and ending at a_i and a_j, extended by a common artificial normal edge $v''v'$. For $i = 1, 2$ let Q'_i be the shortcut of Q_i and let ℓ_i be the length of Q'_i. Let $\ell_3 \geq 2$ be the length of the path P_3 between a_i and a_j that alternates between outer and inner normal vertices. Let C be the cycle formed by Q'_1, Q'_2 and P_3, and let ℓ be the length of C. Then, due to Lemma 2, exactly $\frac{K-1}{2}\ell - K$ edges are pointing into the interior of C. Now we distinguish three cases concerning the intersection of P_3 and the two paths Q'_1 and Q'_2. If P_3 has no edge in common with Q'_1 and Q'_2, the length of C is $\ell = (\ell_1 - 1) + (\ell_2 - 1) + \ell_3 = \ell_1 + \ell_2 + \ell_3 - 2$. But the number of edges pointing into the interior of C is at most

$$\begin{aligned} &\frac{K-1}{2}(\ell_1 - 1) + \frac{K-1}{2}(\ell_2 - 1) - K + \frac{\ell_3}{2}(K-2) \\ &\quad = \frac{K-1}{2}(\ell_1 + \ell_2 + \ell_3 - 2) - K - \frac{\ell_3}{2} < \frac{K-1}{2}\ell - K \ . \end{aligned}$$

If P_3 shares an edge with Q'_1, but not with Q'_2, the length of C is $\ell = (\ell_1 - 2) + (\ell_2 - 1) + (\ell_3 - 1) =$

$\ell_1+\ell_2+\ell_3-4$. But the number of edges pointing into the interior of C is at most

$$\begin{aligned}&\frac{K-1}{2}(\ell_1-1)+\frac{K-1}{2}(\ell_2-1)-K-1+\frac{\ell_3-2}{2}(K-2)\\&\quad=\frac{K-1}{2}(\ell_1+\ell_2+\ell_3-4)-K-1-\frac{\ell_3-2}{2}<\frac{K-1}{2}\ell-K\ .\end{aligned}$$

If P_3 shares an edge with Q_1' and with Q_2', the length of C is $\ell=(\ell_1-2)+(\ell_2-2)+(\ell_3-2)=\ell_1+\ell_2+\ell_3-6$. But the number of edges pointing into the interior of C is at most

$$\begin{aligned}&\frac{K-1}{2}(\ell_1-1)+\frac{K-1}{2}(\ell_2-1)-K-2+\frac{\ell_3-4}{2}(K-2)\\&\quad=\frac{K-1}{2}(\ell_1+\ell_2+\ell_3-6)-K-2-\frac{\ell_3-4}{2}<\frac{K-1}{2}\ell-K\ .\end{aligned}$$

In all three cases we have a contradiction and therefore the assumption, that P_1 and P_2 end in different outer vertices, was wrong.

For (iii) assume that P_1 and P_2 have a common vertex different than v and let w be the first normal vertex of this kind (note that P_1 and P_2 might already meet at a stack vertex immediately before w). Let Q_1 and Q_2 be the subpaths of P_1 and P_2 that end at w and that are extended by artificial normal edges at v and at w. We denote the artificial edges at w by ww_1 and ww_2. Further let Q_1' and Q_2' be the corresponding shortcuts, and let ℓ_1 and ℓ_2 be its lengths. Let s be the number of outgoing edges of v between Q_1' and Q_2' (see Fig. 6 (right) for an illustration). Then because of the construction of the artificial edges there are exactly $s+1$ outgoing edges of v between the two artificial edges. Let C be the cycle we get by gluing Q_1' and Q_2' without the artificial edges together. Then the length of C is $\ell_1+\ell_2-4$ and therefore, due to Lemma 2, exactly $\frac{K-1}{2}(\ell_1+\ell_2-4)-K$ edges are pointing into the interior of C. If we add the number of edges pointing from Q_1' to the right and the number of edges pointing from Q_2' to the left, we count exactly $K-(s+1)$ too many edges at v. Hence, the number of edges we count too many at w is

$$\left(\frac{K-1}{2}(\ell_1-1)+\frac{K-1}{2}(\ell_2-1)\right)-\left(\frac{K-1}{2}(\ell_1+\ell_2-4)-K\right)-(K-s-1)=K+s\ .$$

Thus, ww_1 is to the left of ww_2 and therefore the edges of Q_1' and Q_2' ending in w cannot be both normal edges. It follows that w is not an outer vertex. Further there are exactly s outgoing edges of w between ww_1 and ww_2. Therefore we can inductively repeat the argument for the subpaths of P_1 and P_2 starting at w. □

Theorem 5. *The graph $G_+^\star$ has a unique proper coloring.*

Proof. We begin with the proof of the existence of a proper coloring. We claim that coloring each inner edge e in color i if the paths in $\mathcal{P}(e)$ end in the outer vertex a_i yields a proper coloring. Due to Lemma 4 (i) and (ii) this coloring is well defined. It immediately follows from the definition of the paths that properties (C1) and (C3) are fulfilled. Let v be an inner normal vertex. Then, due to Lemma 4 (iii), the outgoing edges of v have pairwise different colors. Since the paths of two different outgoing edges of v do not cross, the order of the colors of the outgoing edges of v coincides with the order of the colors of the outer vertices. Therefore, also property (C2) is fulfilled.

Now we show the uniqueness of the proper coloring. Because of properties (C2) and (C3) the knowledge of the color of an edge incident to an inner normal vertex v implies the knowledge of the colors of all edges incident to v. Since, due to property (C1), the colors of the edges incident to the outer vertices are fixed and G is connected, this implies the uniqueness of the colors of all edges. □

5 The distributive lattice of K-contact-structures

Let G be an inner triangulation of a K-gon. The following definitions give us a formalism how to change a K-contact-structure of G to obtain a new one.

Definition 7. Let $\mathcal{A}$ be a K-contact-structure of G. We call a multiset E of oriented edges of $G^\star$ *flippable* in $\mathcal{A}$ if

- E is Eulerian,
- each normal edge is contained at most once in E and only in the orientation of $\mathcal{A}$,
- each stack edge $e = uv$ with stack vertex v is contained at most $w_{\mathcal{A}}(e)$ times in E in the orientation from u to v, there is no restriction for the opposite orientation.

Definition 8. Let $\mathcal{A}$ be a K-contact-structure of G and let E be a flippable set of edges in $\mathcal{A}$. Then we can perform a *flip* on $\mathcal{A}$ and obtain a new K-contact-structure $\mathcal{A}'$ by changing the orientation of all normal edges in E, and by setting $w_{\mathcal{A}'}(e) := w_{\mathcal{A}}(e) - a + b$ for each stack edge $e = uv$ with normal vertex u and stack vertex v if e is contained a times in E oriented from u to v and b times oriented from v to u.

It can easily be seen that a flip indeed yields a new K-contact-structure. We can even reach every K-contact-structure $\mathcal{A}'$ from $\mathcal{A}$ by flipping a suitable flippable set of edges.

These flipping operations already show the close relation between K-contact-structures and integral flows on $G^\star$. We now want to formalize this relation and thereby obtain the structure of a distributive lattice on the set of K-contact-structures of G. In particular, K-contact-structures can be equivalently modeled as flows $f : E_{\text{inner}}(\overrightarrow{G^\star}) \to \mathbb{Z}$ on a fixed orientation $\overrightarrow{G^\star}$ of $G^\star$ where each stack edge is oriented towards the incident stack vertex and each normal edge obtains an arbitrary fixed orientation. In such a flow the *excess* of a vertex v is defined as $\omega(v) := \sum_{e \in E_{\text{in}}(v)} f(e) - \sum_{e \in E_{\text{out}}(v)} f(e)$.

Definition 9. A flow $f : E_{\text{inner}}(\overrightarrow{G^\star}) \to \mathbb{Z}$ is called a K-*contact-flow* if

- $f(e) \in \{0, 1\}$ for each normal edge e,
- $f(e') \in \{0, \ldots, \frac{K-3}{2}\}$ for each stack edge e',
- $\omega(u) = \operatorname{indeg}(u) - K$ for each normal vertex u,
- $\omega(v) = \frac{K-3}{2}$ for each stack vertex v.

For each normal edge e we set $c_l(e) := 0$ and $c_u(e) := 1$. For each stack edge e' we set $c_l(e') := 0$ and $c_u(e') := \frac{K-3}{2}$. Then the first two conditions can also be formulated as $c_l(e'') \leq f(e'') \leq c_u(e'')$ for each edge e''. The set of integral flows $\mathcal{F}(H, \omega, c_l, c_u)$ of a directed planar graph H fulfilling such constraints (bounds c_l, c_u on the flow values and prescribed excesses ω) has been studied in [10].

The following describes a bijection between the set of K-contact-structures and the set of K-contact-flows of G. Let $\mathcal{A}$ be a K-contact-structure on G. If a normal edge e has the same orientation in $\overrightarrow{G^\star}$ and in $\mathcal{A}$, we set $f(e) = 1$, otherwise $f(e) = 0$. For a stack edge e' we set $f(e') = w_{\mathcal{A}}(e')$.

It has been shown in [10] that the set $\mathcal{F}(H, \omega, c_l, c_u)$ carries the structure of a distributive lattice. We need some definitions to be able to describe the cover relation of this lattice. For a flow $f \in \mathcal{F}(H, \omega, c_l, c_u)$ let H_f be the following reorientation of H: An edge vw of H is oriented from v to w in H_f if $f(vw) > c_l(vw)$ and it is oriented from w to v in H_f if $f(vw) < c_u(vw)$. Note that in H_f an edge can have no orientation, one orientation, or two orientations. If we decrease the flow f by one on an Eulerian subgraph of H_f, we obtain a new flow $f' \in \mathcal{F}(H, \omega, c_l, c_u)$. This operation corresponds to a flip in the K-contact-structure.

Definition 10. A *chordal path* of a simple cycle C is a directed path consisting of edges interior to C whose first and last vertex are vertices of C. These two vertices are allowed to coincide.

Definition 11. A simple cycle C is an *essential cycle* if there is a flow f such that C is a directed cycle in H_f and has no chordal path in H_f.

Theorem 6 ([10])**.** *The following relation on the set $\mathcal{F}(H, \omega, c_l, c_u)$ of flows of a planar graph H is the cover relation of a distributive lattice: A flow f' covers a flow f if and only if f' can be obtained from f by subtracting one unit of flow on a counterclockwise oriented essential cycle in H_f.*

Now we can apply this to the set of K-contact-flows of G.

Theorem 7. *The set of all K-contact-structures of G carries the structure of a distributive lattice. In this lattice a K-contact-structure $\mathcal{A}'$ covers a K-contact-structure $\mathcal{A}$ if there is a flippable counterclockwise oriented facial cycle in $G^\star$ such that $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by flipping this cycle.*

Proof. We need to show that for a flow $f \in \mathcal{F}(\overrightarrow{G^\star}, \omega, c_l, c_u)$ the essential cycles in $\overrightarrow{G^\star}_f$ are exactly the directed facial cycles in $\overrightarrow{G^\star}_f$. Let C be such an essential cycle.

Claim 1. *There is no edge pointing into the interior of C.*

Proof. Assume there is an edge $e = vw$ pointing into the interior of C. If v is a normal vertex, let $P \in \mathcal{P}(e)$ be a directed path starting with the edge e and ending in an outer vertex of $G^\star$. Then P has to cross C at some point and the subpath of P that ends at the first crossing vertex with C is a chordal path of C, contradicting that C is essential. Notice that we used the fact that all edges of P also appear as edges in $\overrightarrow{G^\star}$ with the same orientation which follows from $P \in \mathcal{P}(e)$ and the bijection between K-contact-structures and flows $f \in \mathcal{F}(\overrightarrow{G^\star}, \omega, c_l, c_u)$. If v is a stack vertex, the vertex w is a normal vertex in the interior of C. Otherwise, e itself would be a chord of C. Let $e' \neq e$ be an outgoing edge of w and let $P' \in \mathcal{P}(e')$ be a directed path starting with the edge e' and ending in an outer vertex of $G^\star$. Then P' has to cross C at some point and again we have a chordal path of C. $\triangle$

Claim 2. *The cycle C contains at least one stack vertex.*

Proof. Assume that C contains only normal vertices. Then according to Lemma 2 there are exactly $\frac{K-1}{2}\ell(C) - K \neq 0$ edges pointing into the interior of C, in contradiction to Claim 1. $\triangle$

Now let v be a stack vertex of C. Let w_1v and vw_2 be the two incident edges on C. Since w_1v is an incoming edge of v in $\overrightarrow{G^\star}_f$, we have $f(w_1v) > 0$ and therefore for each neighbor $w' \neq w_1$ of v we have $f(w'v) < \frac{K-3}{2}$. Thus these edges are oriented from v to w' in $\overrightarrow{G^\star}_f$. Since there is no edge pointing from v into the interior of C, the vertices w_1 and w_2 have to be consecutive neighbors of v in the cyclic order of the neighbors of v. Then $G^\star$ contains the edge w_1w_2 which has (as every normal edge) exactly one orientation in $\overrightarrow{G^\star}_f$. If w_1w_2 does not belong to C, it is a chord of C. Hence, it has to belong to C and C is a facial cycle. $\square$

6 System of linear equations

In this section let G be an inner triangulation of a K-gon and let $\mathcal{A}$ be a K-contact-structure of G. Let $G_+^\star := G_+^\star(\mathcal{A})$. We will propose a system of linear equations that allows us to compute an equiangular K-gon contact representation of G with induced K-contact-structure $\mathcal{A}$ if such a representation exists. If such a representation does not exist, the solution of the system will have negative variables.

Every inner vertex v gets a variable x_v representing the scaling factor of the prototype P_v of v. Further every segment e in the skeleton of the contact representation, we want to compute, (they are in bijection to the angles of the normal vertices in $G_+^\star$) gets a variable x_e representing its length.

We introduce equations which ensure that the scaling factor x_v of each normal vertex fits together with the edge lengths x_e of the K-gon corresponding to v. For $i = 1, \ldots, K$ let $\ell_i(P_v)$ be the length of the ith segment of P_v, starting with the horizontal segment and then proceeding in clockwise direction. Further let $E_i(v)$ be the edges of the skeleton corresponding to the angles of v between the outgoing edges of colors $i + \frac{K-1}{2}$ and $i + \frac{K+1}{2}$. Then the sum of the lengths of the edges in $E_i(v)$ has to be equal to $x_v \ell_i(P_v)$, the scaled segment length of the prototype:

$$\sum_{e \in E_i(v)} x_e - \ell_i(P_v) x_v = 0 \ .$$

We still need to ensure that the faces of G are represented by pseudotriangles, in particular that the edge lengths x_e of such an pseudotriangle make it a closed curve. For that we do the following construction: Let B_f be the pseudotriangle corresponding to the face f of G. Then we add the angle bisectors of all three convex corners of B_f and cut B_f into three pseudotriangles and a small triangle in the center like in Fig. 7 (left). Afterwards each of the three pseudotriangles is cut into triangles by elongating the edges at the concave corners (see Fig. 7 (right)). Note that we know the slopes of all edges of the cut pseudotriangle. Therefore, for each triangle t in this cut pseudotriangle, we know a prototype P_t such that t is a homothetic copy of P_t. We introduce a variable for each edge of t representing its length and a variable representing the scaling factor of the prototype. Further we introduce a variable for each of the three angle bisectors representing its length. Then we can introduce three equations for each triangle and two equations for each angle bisector ensuring that the scaling factors and edge lengths fit together.

Additionally, each inner face f of G incident to one or two outer vertices corresponds to a pseudotriangle with one straight line segment or a corner-triangle respectively. For the first case, variables are defined just as in the setting of an inner pseudotriangle whereas the segment belonging to the outer polygon is treated as if arising from a K-gon. In the second case,we again have a triangle with known slopes and therefore we describe it by a variable x_f representing the scaling factor of a corresponding prototype and three variables for the side lengths which are linearly related to x_f.

Finally, we add one more equation to our system stating that the sum of the lengths of the edges building the line segment corresponding to the outer vertex a_1 of G is exactly 1. This equation is the only inhomogeneous equation and will ensure that the solution of the system is unique.

We denote the entire system of linear equations by $A_{\mathcal{A}} x = \mathbf{e_1}$ where $A_{\mathcal{A}}$ is a matrix depending on the K-contact-structure $\mathcal{A}$ and where $\mathbf{e_1} = (1, 0, 0, \ldots, 0)$.

We will show that the system of linear equations $A_{\mathcal{A}} x = \mathbf{e_1}$ is uniquely solvable. For this purpose we need a technical lemma about perfect matchings in plane bipartite graphs. So let H be a bipartite graph with vertex classes $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$. Then a perfect matching of H induces a permutation $\sigma \in \mathcal{S}_k$ by $\sigma(i) = j :\Leftrightarrow \{v_i, w_j\} \in M$. We define the *sign* $\operatorname{sgn}(M)$ of a perfect matching M as the sign of the corresponding permutation.

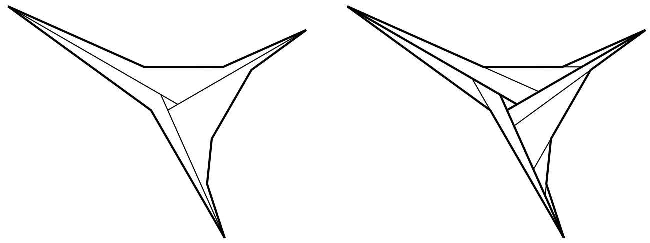

Figure 7: Cutting a pseudotriangle into three pseudotriangles and a small triangle in the center, and cutting the smaller pseudotriangles into triangles.

Lemma 5. *Let H be a bipartite graph and let M, M' be two perfect matchings of H. If the symmetric difference of M and M' is the disjoint union of simple cycles $C_1, \ldots, C_m$ such that, for $i = 1, \ldots, m$, the length ℓ_i of C_i fulfills $\ell_i \equiv 2 \mod 4$, then $\operatorname{sgn}(M) = \operatorname{sgn}(M')$.*

If H is a plane graph such that every inner face f of H is bounded by a simple cycle of length $\ell_f \equiv 2 \mod 4$, this property is fulfilled for any two perfect matchings of H.

Proof. For $i = 1, \ldots, m$, there is an $n_i \in \mathbb{N}$ with $\ell_i = 4n_i + 2$. Then on the vertices of C_i the permutation σ corresponding to M and the permutation σ' corresponding to M' differ in a cyclic permutation τ_i of length $2n_i + 1$. Hence we have $\sigma' = \sigma \circ \tau_1 \circ \cdots \circ \tau_m$ and therefore

$$\begin{aligned} \operatorname{sgn}(\sigma') &= \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau_1) \cdots \operatorname{sgn}(\tau_m) \\ &= \operatorname{sgn}(\sigma) \cdot (-1)^{2n_1} \cdots (-1)^{2n_m} = \operatorname{sgn}(\sigma) \ . \end{aligned}$$

In the case that H is a plane graph such that each inner face f of H is bounded by a simple cycle of length $\ell_f \equiv 2 \mod 4$, for each cycle of length ℓ with k' vertices in its interior the formula $\ell + 2k' \equiv 2 \mod 4$ is valid. This can be shown by induction on the number of faces enclosed by the cycle. Since each of the cycles $C_1, \ldots, C_m$ contains an even number of vertices in its interior, this implies $\ell_i \equiv 2 \mod 4$ for $i = 1, \ldots, m$. □

Theorem 8. *The system $A_{\mathcal{A}}x = \mathbf{e_1}$ has a unique solvution.*

Proof. We show that $\det(A_{\mathcal{A}}) \neq 0$. The matrix is a square matrix, as is implicitly shown later in Claim 1. Let $\hat{A}_{\mathcal{A}}$ be the matrix obtained from $A_{\mathcal{A}}$ by multiplying all columns corresponding to variables which represent scaling factors (of K-gons, triangles, or angle bisectors) with -1. Since in $A_{\mathcal{A}}$ all entries in these columns are nonpositive and the entries in all other columns are nonnegative, all entries of $\hat{A}_{\mathcal{A}}$ are nonnegative. Further we have $\det(A_F) = (-1)^N \det(\hat{A}_{\mathcal{A}})$ where N is the number of columns we multiplied with -1.

We want to interpret the Leibniz formula of $\det(\hat{A}_{\mathcal{A}})$ as the sum over the perfect matchings of a plane auxiliary graph $H_{\mathcal{A}}$. Let $H_{\mathcal{A}}$ be the bipartite graph whose first vertex class $v_1, \ldots, v_k$ consists of the variables of the equation system and whose second vertex class $w_1, \ldots, w_k$ consists of the equations of the equation system. There is an edge v_iw_j in $H_{\mathcal{A}}$ if and only if $(\hat{A}_{\mathcal{A}})_{ij} > 0$. Then we have

$$\det(\hat{A}_{\mathcal{A}}) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i} (\hat{A}_{\mathcal{A}})_{i\sigma(i)} = \sum_{M} \operatorname{sgn}(M) P_M \ ,$$

where the second sum goes over all perfect matchings M of $H_{\mathcal{A}}$ and where $P_M > 0$ for all perfect matchings M.

Next we will define a crossing-free embedding of $H_{\mathcal{A}}$ into the plane. We start with a crossing-free drawing of $G_+^\star$ in which single edges are straight lines and multiple edges are allowed to have one bend. Then we put pairwise disjoint disks around all normal vertices. We put a second, smaller

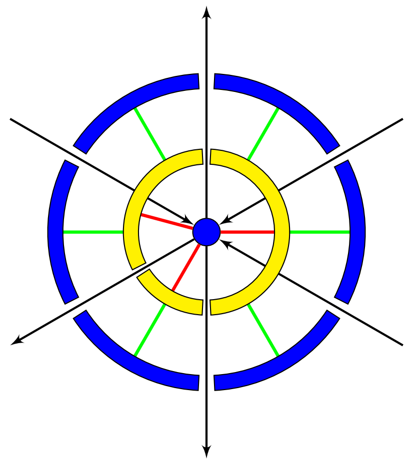

Figure 8: Gadget for embedding $H_{\mathcal{A}}$ into the plane. In the center there is a normal vertex, the incident edges of $G_+^\star$ are drawn in black, the incident equations in yellow, and the incident edge-variables in blue. Red (green) edges are edges of $H_{\mathcal{A}}$ corresponding to a negative (positive) coefficient in the equation.

circle around the normal inner vertices. The smaller circle around a normal vertex u is cut at the outgoing edges of u into K arcs. These K arcs are the drawings of the K equation-vertices incident to u. The larger circle around u is cut into arcs at all incident edges of u. These arcs are the drawings of edge-variables. Then we add edges between u and all arcs on the smaller circle around u, and we add edges between an arc on the smaller and an arc on the larger circle around u if there is a straight line crossing u and both arcs (see Fig. 8 for an illustration). In each inner face f of the induced drawing of G there are contained the drawings of exactly those edge-variables that are part of the boundary of the corresponding pseudotriangle B_f. Figure 9 shows a gadget that allows us to embed the remaining variables and equations corresponding to B_f inside this face.

Claim 1. *There exists a perfect matching of $H_{\mathcal{A}}$.*

Proof. We describe an explicit construction of a perfect matching of $H_{\mathcal{A}}$. We look at the unique proper coloring of $G_+^\star$. The K equation-vertices adjacent to a normal vertex u of $G_+^\star$ are corresponding to the K colors: An equation-vertex has color c if it is opposite to the outgoing edge of color c. We always match the vertex u with the equation-vertex of color $\frac{K-1}{2}$. The equation-vertices of colors $2, \ldots, \frac{K-3}{2}$ are matched with the last incident edge-vertex in clockwise order, and the equation-vertices of colors $\frac{K+1}{2}, \ldots, K, 1$ are matched with the last incident edge-vertex in counterclockwise order.

Now we want to show that exactly two edge-vertices of every stack vertex remain unmatched, and that these two edge-vertices belong to different concave parts of the corresponding pseudotriangle. Figure 10 verifies this for the two possible cases. All other cases are symmetric. In the first case the pseudotriangle contains an edge of color $\frac{K-1}{2}$, in the second case it does not. In both cases the drawn pairs of opposite segments and vertices can be exchanged. Since in each case the segment of such a pair contributes an unmatched edge-variable, the statement remains true in all these cases.

Figure 11 shows a way to match the variables and equations inside a pseudotriangle such that only all edge-variables on the boundary of the pseudotriangle, the central triangle-variable and its three adjacent equations remain unmatched. Figure 11 also shows augmenting paths connecting an edge-variable on the boundary with one of the equations of the central triangle. We increase our matching by using these augmenting paths for the two unmatched edge-variables on the boundary. Then we match the central triangle-variable with the last unmatched equation.

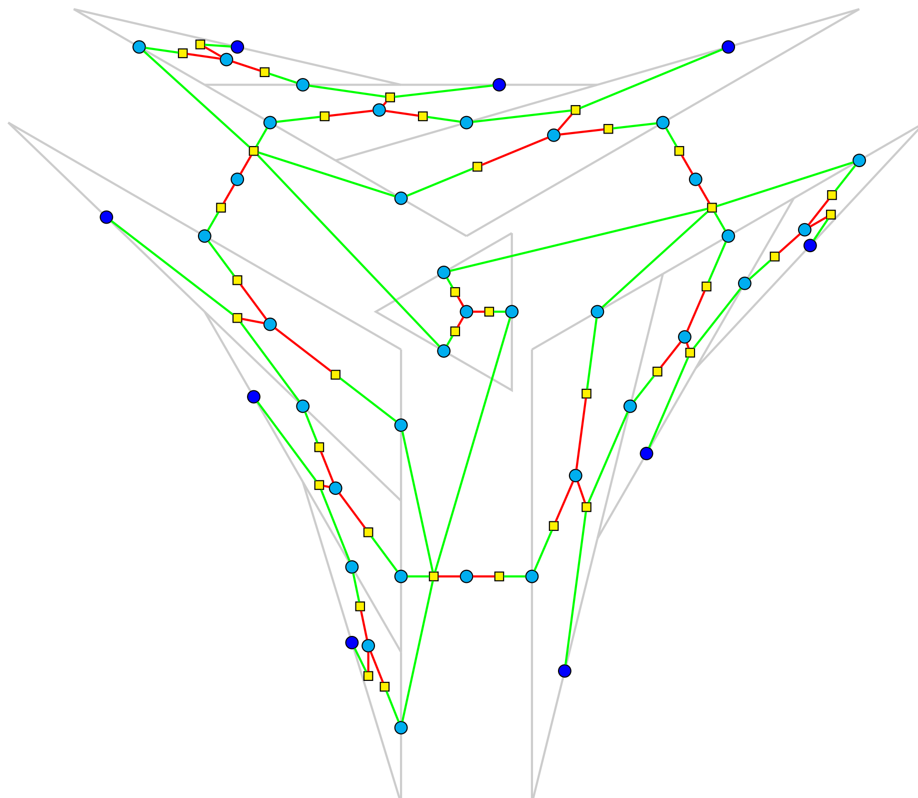

Figure 9: Face-gadget for embedding $H_{\mathcal{A}}$ into the plane. The dark blue vertices are the edge-variables coming from the gadgets of Fig. 8. All other vertices can be placed freely. Variables are drawn in cyan and equations in yellow. Red (green) edges are edges of $H_{\mathcal{A}}$ corresponding to a negative (positive) coefficient in the equation.

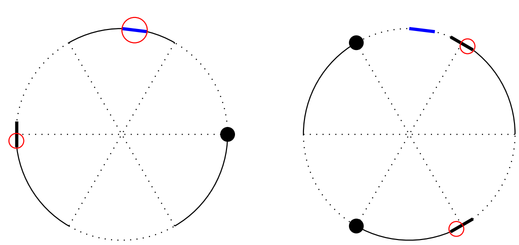

Figure 10: Case distinction for showing that exactly two edge-variables of each pseudotriangle are not matched (compare Fig. 5). The edges of color $\frac{K-1}{2}$ are drawn in blue. The unmatched edges are highlighted in red.

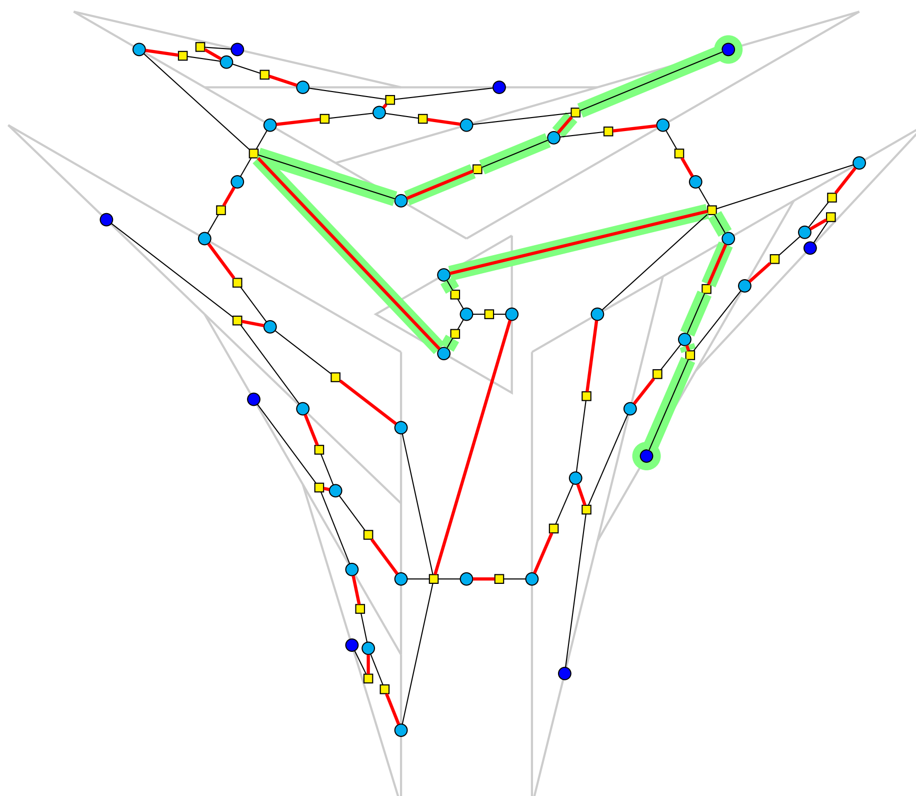

Figure 11: A matching (red) which lets only the outer edge-variables, the central variable and its incident equations unmatched. In green there are two examples of augmenting paths connecting an outer edge-variable and a central equation.

All the variables corresponding to partial segments of the bounding K-gon are matched properly by this procedure, except for the rightmost variable of the upper horizontal segment. Finally, this variable can be matched to the inhomogeneous equation, and we obtain a perfect matching. $\triangle$

Claim 2. *Let M_1, M_2 be perfect matchings. Then* $\operatorname{sgn}(M_1) = \operatorname{sgn}(M_2)$.

Proof. Due to Lemma 5 it suffices to show that each inner face f of $H_{\mathcal{A}}$ is bounded by a simple cycle of length $\ell_f \equiv 2 \mod 4$. This can be verified by distinguishing all different types of faces of $H_{\mathcal{A}}$. $\triangle$

From Claims 1 and 2 we immediately get

$$\det(\hat{A}_{\mathcal{A}}) = \sum_M \operatorname{sgn}(M) P_M \neq 0 \ .$$

Therefore, we also have $\det(A_F) = (-1)^N \det(\hat{A}_{\mathcal{A}}) \neq 0$. $\square$

Definition 12. The solution x of $A_{\mathcal{A}} x = \mathbf{e_1}$ is *nearly nonnegative* if all negative variables of x are scaling factors of the central triangle in the triangle decomposition of a pseudotriangle.

The following lemma helps us to prove that a nearly nonnegative solution of the system $A_{\mathcal{A}} x = \mathbf{e_1}$ leads to an equiangular K-gon contact representation of G.

Lemma 6. *Let H be an inner triangulation of a polygon. For every inner face f of H with vertices v_1, v_2, v_3 in clockwise order let T_f be a triangle in the plane with vertices $p(f, v_1), p(f, v_2), p(f, v_3)$ in clockwise order such that the following conditions are satisfied:*

(i) Let v be an inner vertex of H and let $f_1, \ldots, f_k$ be its incident faces. Then

$$\sum_{i=1}^{k} \beta(f_i, v) = 2\pi$$

where $\beta(f, v)$ denotes the interior angle of T_f at $p(f, v)$.

(ii) Let v be an outer vertex of H and let $f_1, \ldots, f_k$ be its incident inner faces. Then

$$\sum_{i=1}^{k} \beta(f_i, v) \leq \pi \ .$$

(iii) Let vw be an inner edge of H and let f_1, f_2 be its incident faces. Then

$$p(f_1, v) - p(f_1, w) = p(f_2, v) - p(f_2, w) \ ,$$

i.e., the vector between v and w is the same in T_{f_1} and T_{f_2}.

Then there exists a crossing-free straight line drawing of H in which the drawing of every inner face f can be obtained from T_f by translation.

Proof. Let H^* be the dual graph of H without the vertex corresponding to the outer face of H. Further let S be a spanning tree of H^*. Then we can glue the triangles T_f of all inner faces f of H together along the edges of S. We need to show that the resulting shape has no holes or overlappings. For the edges of S we already know that the triangles of the two incident faces are touching in the right way. For the edges of the complement $\overline{S}$ of S we still need to show this. We consider $\overline{S}$ as a

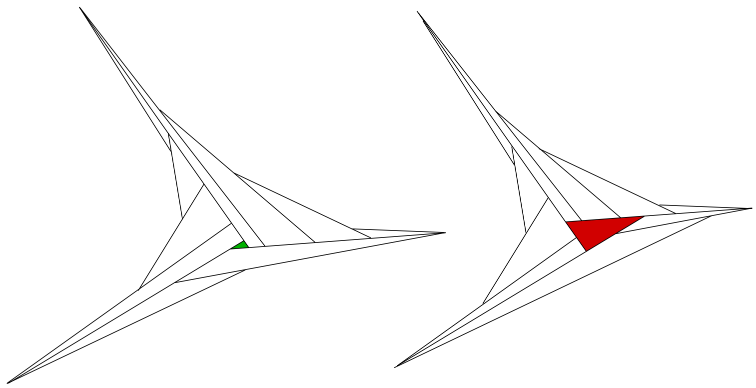

Figure 12: Flipping negative triangles in the center of a pseudotriangle.

subset of the edges of H. Note that $\overline{S}$ is a forest in H. Let e be an edge of $\overline{S}$ incident to a leaf v of this forest that is an inner vertex of H. Then for all incident edges $e' \neq e$ of v we already know that the triangles of the two incident faces of e are touching in the right way. But then also the two triangles of the two incident faces of e are touching in the right way because v fulfills property (i). Since the set of edges we still need to check is still a forest, we can iterate this process until all inner edges of H are checked.

We have to exclude that the resulting polygon has overlappings. The existence of such overlappings would imply that the drawings of two non-neighboring edges of the outer face of H intersect. But because of property (ii) the drawings of the edges of the outer face of H form the boundary of a convex polygon, and therefore such an intersection cannot exist. □

Theorem 9. *The unique solution of the system $A_{\mathcal{A}}x = \mathbf{e_1}$ is nearly nonnegative if and only if the K-contact-structure $\mathcal{A}$ is induced by an equiangular K-gon contact representation of G with the given prototypes.*

Proof. Assume there is an equiangular K-gon contact representation $\mathcal{S}$ of G with the given prototypes that induces the K-contact-structure $\mathcal{A}$. Then the edge lengths given by $\mathcal{S}$ define a nearly nonnegative solution of $A_{\mathcal{A}}x = \mathbf{e_1}$.

For the opposite direction, assume the solution of $A_{\mathcal{A}}x = \mathbf{e_1}$ is nearly nonnegative. To be able to apply Lemma 6 we first construct an internally triangulated extension of the skeleton graph of a hypothetical equiangular K-gon contact representation with induced K-contact-structure $\mathcal{A}$. This triangulation should extend the cutting of the skeleton we produced to define the equation system. For example, the K-gons are split into K triangles.

In the case that a variable corresponding to a small triangle in the middle of a pseudotriangle is negative, we have to flip the order of intersection of the three angle bisectors defining the triangle in the abstract layout we used to define the equation system, see Fig. 12. The variable of the small triangle thereby changes the sign in the equation corresponding to each of the three bisector segments. This means, that using the original layout with this eventual local modifications yields an (abstract) dissection into triangles, now each variable being nonnegative, since the negative small triangles changed their sign. We can now use this triangulated dissection together with Lemma 6 and obtain an equiangular contact representation of G. □

7 A heuristic

In this section we propose a heuristic to compute an equiangular K-gon contact representation of a given triangulation G of a K-gon. The basic idea of our heuristic is to start with an arbitrary

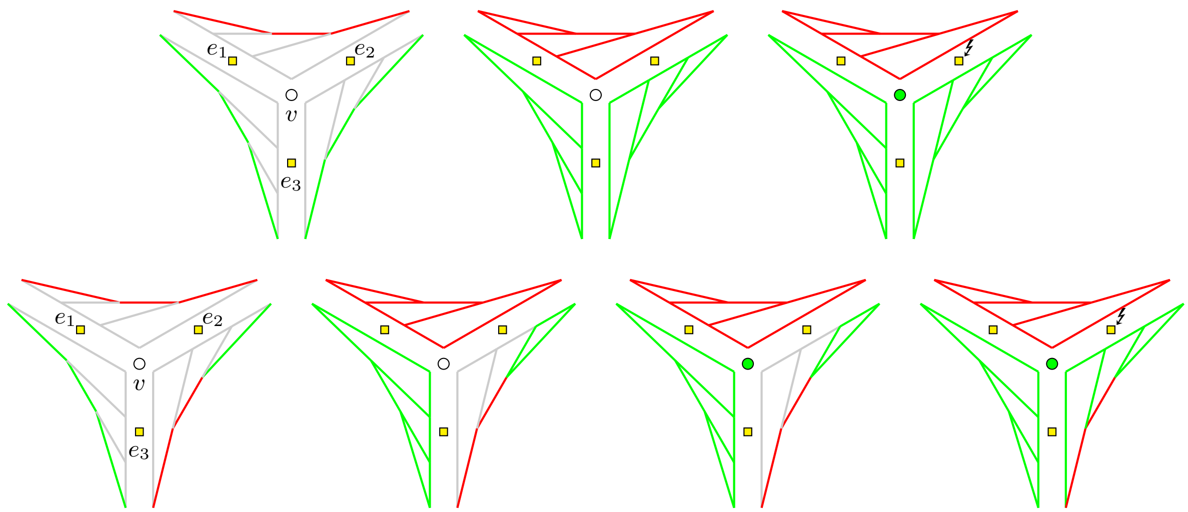

Figure 13: Illustration of the proof that a pseudotriangle cannot have two convex and no concave sign-change (first row), or three convex and one concave sign-change (second row).

K-contact-structure $\mathcal{A}$ of G and to solve the system $A_{\mathcal{A}}x = \mathbf{e_1}$. If the solution is nearly nonnegative, we can construct the contact representation from the edge lengths given by the solution and are done. Otherwise, we can use the negative variables of the solution as sign-posts indicating how to change the K-contact-structure for another try.

We begin with studying the structure of solutions of $A_{\mathcal{A}}x = \mathbf{e_1}$ which are not nearly nonnegative. Consider a pseudotriangle and the signs of the surrounding edge-variables. In the cyclic traversal of these edge-variables the sign can change at the three convex corners (we call these *convex sign-changes*) and at the last concave corner before a convex corner and at the first concave corner after a convex corner (we call these *concave sign-changes*). At the intermediate concave corners the sign cannot change because the involved edge-variables correspond to entire segments of the same K-gon and therefore have the same sign.

Lemma 7. *A pseudotriangle cannot have exactly two convex sign-changes and no concave sign-change, or exactly three convex sign-changes and exactly one concave sign-change.*

Proof. The proof is illustrated by Fig. 13. The first figure in the first row shows the situation that there are exactly two convex sign-changes and no concave sign-change (all other cases are symmetric). Then the signs of all variables of the pseudotriangle, except the variable x_v corresponding to the central triangle of the triangle decomposition of the pseudotriangle, are determined by the equations of the system (see the second). Then the equation e_1 implies $x_v > 0$, in contradiction to equation e_2.

The first figure in the second row shows the situation that there are exactly three convex sign-changes and exactly one concave sign-change (all other cases are symmetric). Then the signs of most of the variables are determined by the equations of the system (see second figure). In this situation equation e_1 implies $x_v > 0$ (see third figure). Then the signs of the remaining variables are determined by the equations and lead to a contradiction in equation e_2 (see fourth figure). □

The fact, that the total number of sign-changes is even, leads to the following corollary.

Corollary 1. *In each pseudotriangle there are at least as many concave sign-changes as convex sign-changes.*

Definition 13. We call the following three types of oriented edges $e = (v, w)$ in $G^\star$ *sign-separating* edges:

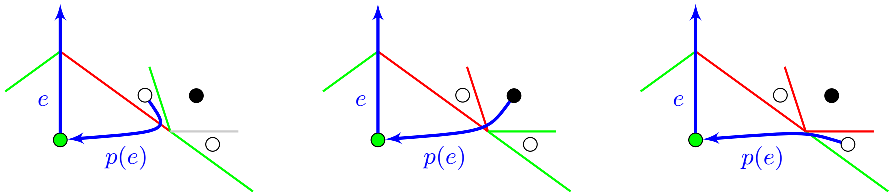

Figure 14: Construction of the predecessor $p(e)$ of a sign-separating edge e of type (A) or (B).

(A) v, w are normal vertices, the abstract K-gons of both vertices have a sign-change at the contact, and the two involved abstract pseudotriangles do not have a sign-change at the contact,

(B) v is a normal vertex, w is a stack vertex, and there is a sign-change at the corner corresponding to e,

(C) v is a stack vertex, w is a normal vertex, the abstract pseudotriangle corresponding to v has a sign-change in a convex corner, the abstract K-gon corresponding to w has a sign-change at the same point, but not a corner.

Notice that since sign-seperating edges are defined using an abstract notion of a K-gon contact structure or the equation system respectively, we might have different parallel edges connecting the same vertices.

Lemma 8. *If the solution of $A_{\mathcal{A}}x = \mathbf{e_1}$ is not nearly nonnegative, there exists a sign-separating edge.*

Proof. If there is a sign-change at a pseudotriangle, we know from Corollary 1 that there is a concave sign-change at this pseudotriangle. There is a sign-separating edge of type (B) at every concave sign-change. Therefore the statement is true in this case.

Assume there are no sign-changes at pseudotriangles. Then two pseudotriangles with different signs have to meet at some point. There is always a sign-separating edge of type (A) at such a point. Therefore the statement is true also in this case. $\square$

Lemma 9. *The multiset of sign-separating edges forms an Eulerian orientation.*

Proof. The strategy for the proof is the following: We assign to each sign-separating edge e a predecessor $p(e)$. The predecessor is a sign-separating edge whose endpoint is the starting point of e. Then we show that this assignment is injective. Since this a assignment is a function from the finite set of sign-separating edges to the same set, it has to be bijective. Finally, this implies the statement of the lemma.

Let $e = vw$ be a sign-separating edge of type (A) or of type (B). Then e corresponds to a corner of the K-gon corresponding to v with a sign-change. Assume the variable x_v of the K-gon is nonnegative. Then the segment starting at this corner with a negative edge has to have a sign-change at some point. Otherwise the sum of the edge-variables of this segment would be negative, in contradiction to $x_v \geq 0$. Let us look at the first sign-change of this segment and distinguish three cases according to the signs of the other two edges ending at this point. Figure 14 shows how to choose the predecessor of e in each of these cases.

Now let $e = (v, w)$ be a sign-separating edge of type (C). Since the pseudotriangle corresponding to v has at least as many concave sign-changes as convex sign-changes, and each concave sign-change corresponds to a sign-separating edge of type (B), we can find an injective assignment from the sign-separating edges of type (C) starting at v to the sign-separating edges of type (B) ending at v.

It remains to show that for sign-separating edges $e = (v, w)$ of types (A) and (C) there cannot be distinct sign-separating edges e_1, e_2 with $p(e_1) = p(e_2) = e$. In both cases e corresponds to a sign-change on a segment of the K-gon A corresponding to w. If $x_w \geq 0$, the sign-separating edges e_1 and e_2 have to correspond to the first corner of A we reach when going in the direction of the negative edge. Analogously, if $x_w < 0$, the sign-separating edges e_1 and e_2 have to correspond to the first corner of A we reach when going in the direction of the nonnegative edge. Thus in both cases $e_1 = e_2$. $\square$

Let E_{+-} be the set of sign-separating edges. For a normal vertex u and a stack vertex v it can happen that both of the edges (u, v) and (v, u) are sign-separating edges. Let w be the normal vertex corresponding to the (abstract) K-gon touching the (abstract) K-gon of v in the contact point where (u, v) and (v, u) have their assigned sign-changes. Then we change E_{+-} in the following way: We set $E_{+-} \leftarrow E_{+-} \setminus \{(v, u)\} \cup \{(v, w), (w, u)\}$. We call this a *repairing step.*

Lemma 10. *The edge (w, u) added to E_{+-} in a repairing step is no sign-separating edge and has not been added to E_{+-} in an earlier repairing step.*

Proof. The edge (w, u) is not a sign-separating edge of type (A) since the pseudotriangle corresponding to v has a sign-change at the contact point of the K-gons corresponding to w and u.

Assume $x_u \geq 0$ (the other case is symmetric). Then each sign-separating edge (v', u) of type (C) corresponds to a common negative edge of the K-gon of u and the pseudotriangle of v'. But the pseudotriangle touching the common point of the K-gons of u and w, which corresponds to $v'' \neq v$, has a nonnegative edge in common with the K-gon of u. Therefore, (v'', u) is no sign-separating edge and wu cannot have been added to E_{+-} in an earlier repairing step. $\square$

Note that the edge (w, u) was not in the original set E_{+-} since the pseudotriangle corresponding to v has a sign-change at the contact point of the K-gons corresponding to w and u and therefore the edge (w, u) is not sign-separating edge of type (A). Further the edge (w, u) cannot be added twice to E_{+-} by these operations.

Due to Lemma 10 the edges in E_{+-} form an Eulerian orientation after applying all possible repairing steps. Additionally, since every normal edge in E_{+-} is oriented the same way as in $\mathcal{A}$ and edges in E_{+-} oriented towards a stack vertex correspond to (abstract) concave corners in the linear equation system, the number of parallel such edges at $e \in E(G^\star)$ is bounded by $w(e)$. Hence, E_{+-} is flippable and changing it's edges in the K-contact-structure as described in Section 5 leads to a new K-contact-structure.

We cannot prove that iterating these modifications can guarantee any kind of progress. Therefore a proof is still missing that this heuristic always terminates with a solution. However, the heuristic has been subject to extensive experiments and these have always been successful (see Section 8).

8 Experiments

The heuristic introduced in Section 7 has been implemented in SageMath [1]. We ran a set of experiments in the following way: For each $n \in \{10, 20, \ldots, 100\}$ we generated ten triangulations (inner triangulations of a triangle) uniformly at random using the algorithm described in [14]. Then, for each $K \in \{5, 7, \ldots, 23\}$, we surrounded each triangulation T with a K-gon $a_1, \ldots, a_K$ and extended this graph to an inner triangulation of the K-gon $a_1, \ldots, a_K$ by connecting each of the outer vertices of T (nearly) to the same number of vertices a_i. This way we ended up with a total of 1000 graphs which we used as input for the heuristic. On a standard personal computer (AMD FX-6300 Six-Core Processor, 3.5 GHz, 8 GB RAM) the heuristic terminated for each graph after few seconds. Therefore we have the following conjecture.

Conjecture 1. *The heuristic described above terminates with a solution for all K, for every graph G which is an inner triangulation of a K-gon, and for every K-contact-structure of G to start the heuristic.*

Figures 15 and 16 show the numbers of iterations the heuristic needed to find a K-contact-structure for the graphs such that the solution of equation system is nonnegative. These experiments suggest that the number of iteration of the heuristic is $\mathcal{O}(K)$ for constant n and $o(n)$ for constant K. Since the size of the equation system is $\mathcal{O}(nK)$ and systems of linear equations can be solved exactly in polynomial time, this would imply that the heuristic has polynomial running time.

Figure 17 shows two examples of regular K-gon contact representations calculated with the heuristic. In Fig. 18 we can see a sequence of regular K-gon contact representations which has been constructed as in the experiments described above. It can be seen that the sequence converges to a circle contact representation.

- We believe that a proof of the conjecture will foster new interactions between discrete mathematics and geometry. In particular it may ultimately lead to a discrete proof of Schramm's Convex Packing Theorem.
- Even without proving the conjecture it may be possible to give a proof for the existence of equiangular K-gon contact representations which is based on the theory developed in this paper and the method from [18].

Acknowledgements. We want to thank Manfred Scheucher for supporting us with the implementation of the heuristic.

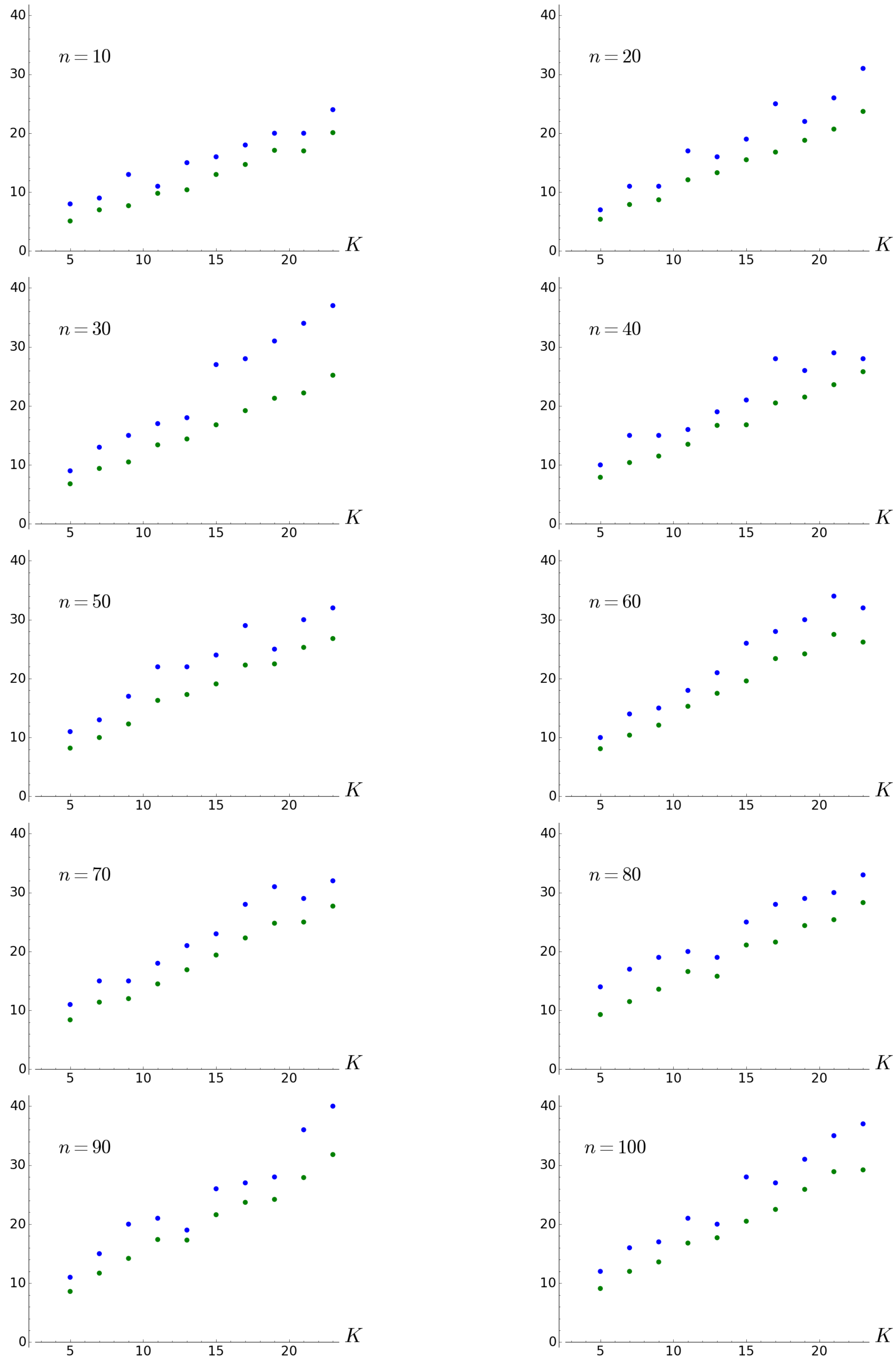

Figure 15: Plot of the average (green) and maximal (blue) number of iterations in dependence of K for fixed n.

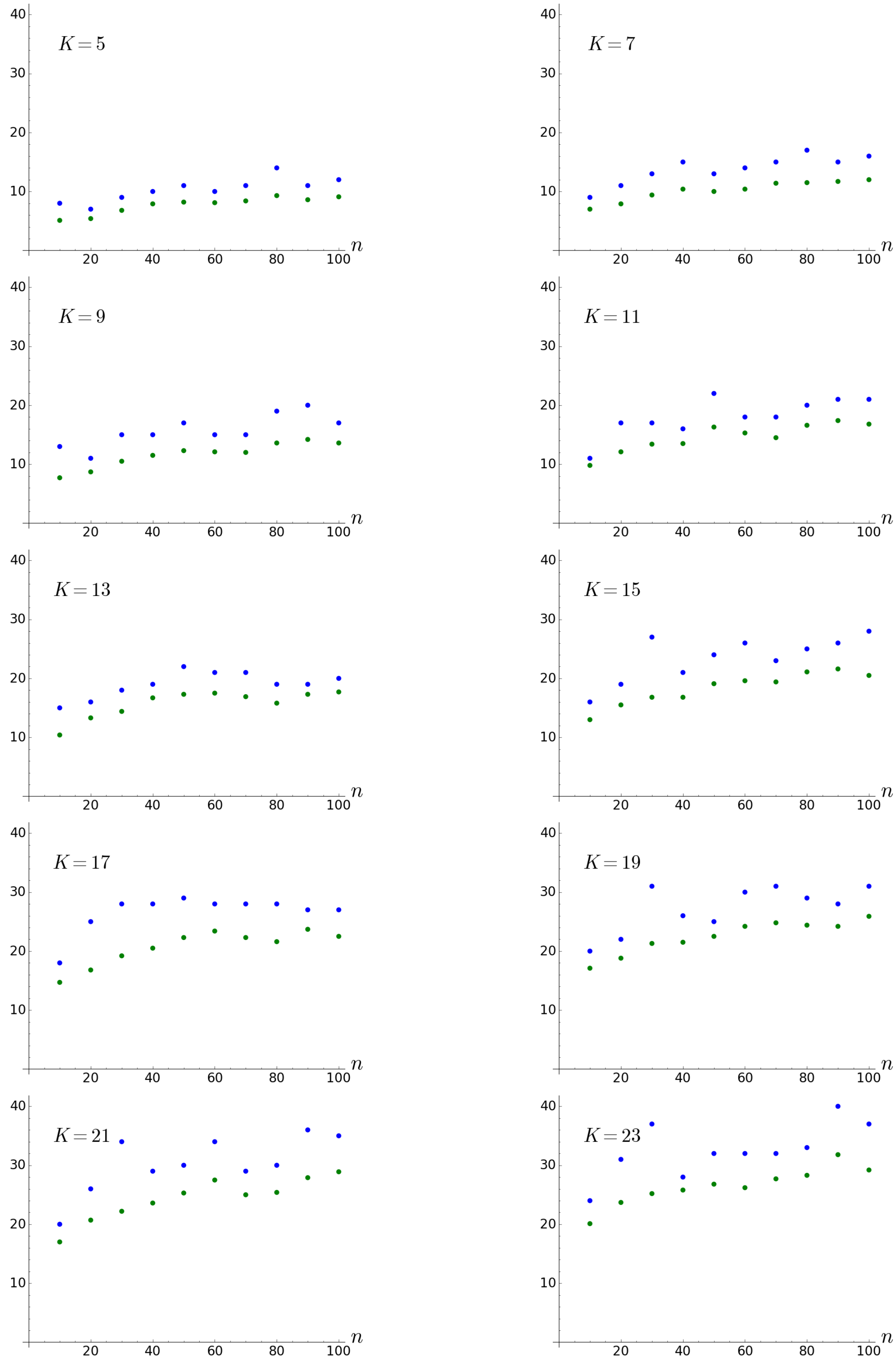

Figure 16: Plot of the average (green) and maximal (blue) number of iterations in dependence of n for fixed K.

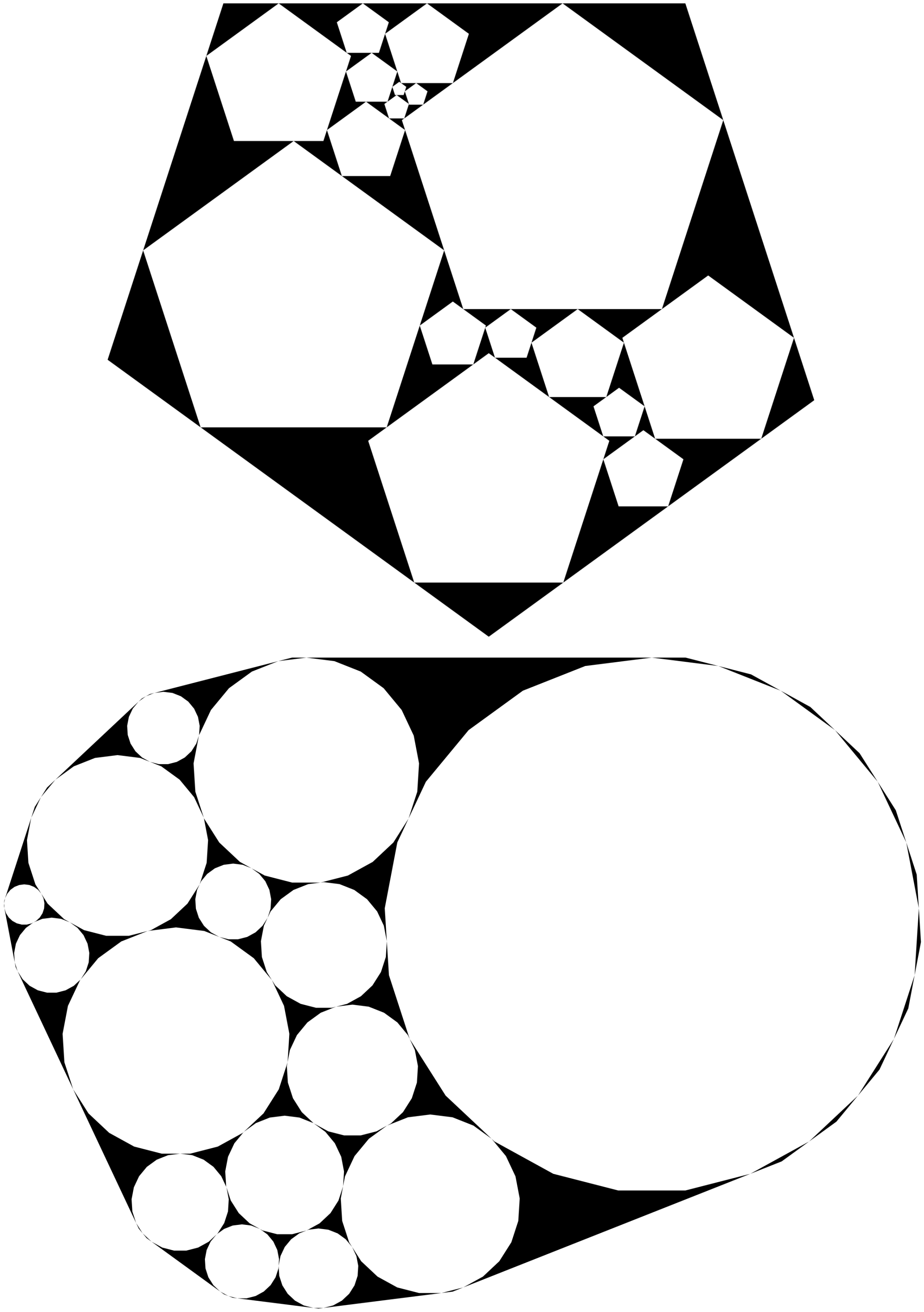

Figure 17: Two examples of regular K-gon contact representations computed with the heuristic, one example for $K = 5$ and one example for $K = 25$.

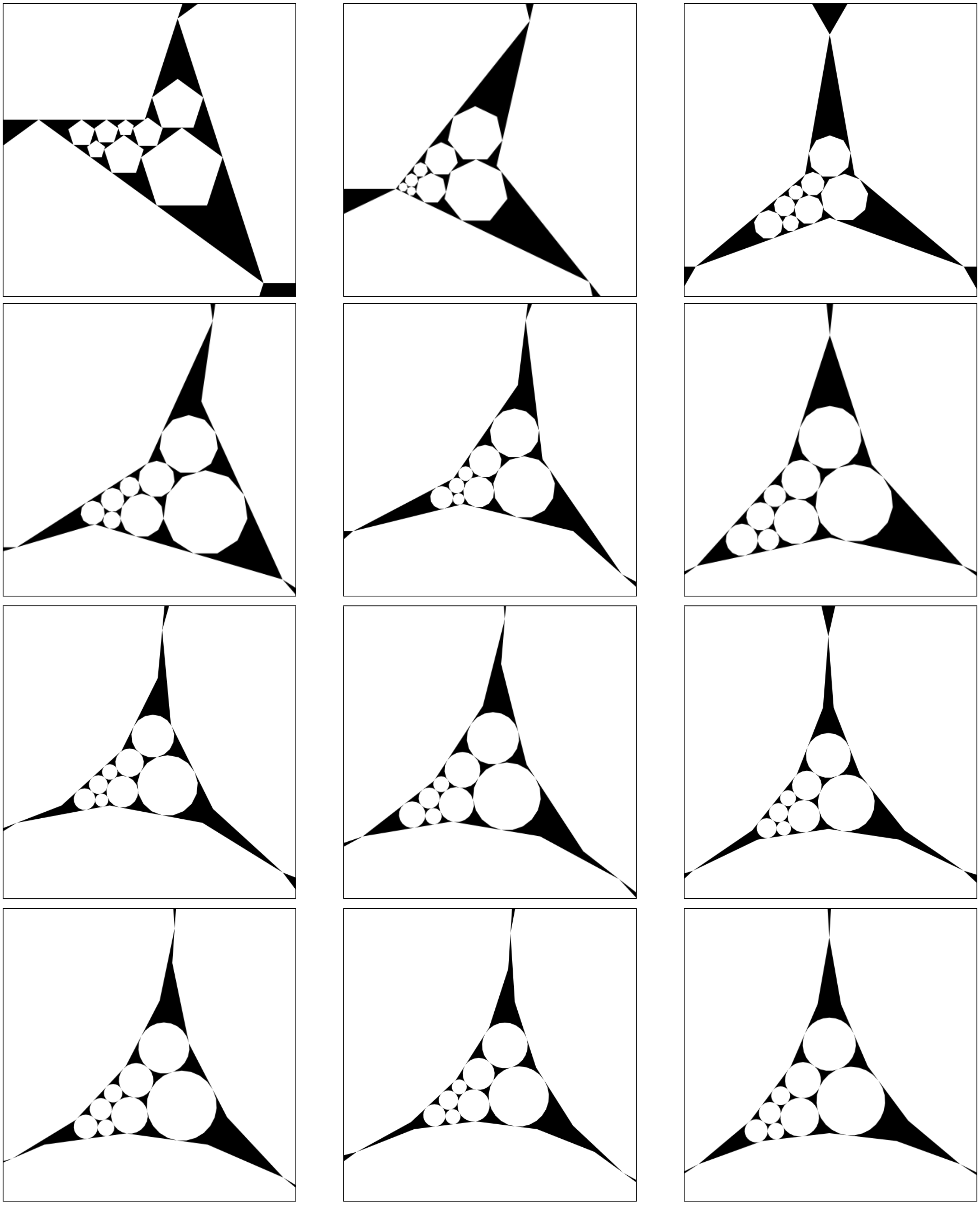

Figure 18: The regular K-gon contact representations of a triangulation embedded in surrounding K-gons for $K = 5, 7, \ldots, 27$. The K-gons corresponding to the three outer vertices of the triangulation are not drawn completely.

References

[1] *SageMath version 7.5.1*, 2017. URL: `http://doc.sagemath.org/index.html`.

[2] Philip L. Bowers. Circle packing: a personal reminiscence. In Mircea Pitici, editor, *The Best Writing on Mathematics 2010*, pages 330–345. Princeton University Press, 2010.

[3] R. Leonard Brooks, Cderic Smith, Arthur Harold Stone, and William T. Tutte. The dissection of rectangles into squares. *Duke Mathematical Journal*, 7(1):312–340, 1940.

[4] Adam L. Buchsbaum, Emden R. Gansner, Cecilia M. Procopiuc, and Suresh Venkatasubramanian. Rectangular layouts and contact graphs. *ACM Transactions on Algorithms*, 4:Art. 8, 28, 2008.

[5] Hubert de Fraysseix and Patrice Ossona de Mendez. On topological aspects of orientations. *Discrete Mathematics*, 229(1):57–72, 2001.

[6] Hubert de Fraysseix, Patrice Ossona de Mendez, and Pierre Rosenstiehl. On triangle contact graphs. *Combinatorics, Probability and Computing*, 3:233–246, 1994.

[7] Stefan Felsner. Lattice structures from planar graphs. *Electronic Journal of Combinatorics*, 11(1):R15, 2004.

[8] Stefan Felsner. Triangle contact representations. In *Midsummer Combinatorial Workshop, Praha*, 2009. URL: `page.math.tu-berlin.de/~felsner/Paper/prag-report.pdf`.

[9] Stefan Felsner. Rectangle and square representations of planar graphs. In *Thirty Essays on Geometric Graph Theory*, pages 213–248. Springer, 2013.

[10] Stefan Felsner and Kolja Knauer. Uld-lattices and δ-bonds. *Combinatorics, Probability and Computing*, 18(5):707–724, 2009.

[11] Stefan Felsner, Hendrik Schrezenmaier, and Raphael Steiner. Pentagon contact representations. In *Proc. Eurocomb*, pages 421–427, 2017.

[12] Daniel Gonçalves, Benjamin Lévêque, and Alexandre Pinlou. Triangle contact representations and duality. In *Proc. Graph Drawing*, pages 262–273, 2011.

[13] Thomas Picchetti. Finding a square dual of a graph. 2011. URL: `page.math.tu-berlin.de/~felsner/Diplomarbeiten/rapport_picchetti.pdf`.

[14] Dominique Poulalhon and Gilles Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3-4):505–527, 2006.

[15] Julia Rucker. Kontaktdarstellungen von planaren Graphen. Diplomarbeit, Technische Universität Berlin, 2011. URL: `page.math.tu-berlin.de/~felsner/Diplomarbeiten/dipl-Rucker.pdf`.

[16] Oded Schramm. Combinatorically prescribed packings and applications to conformal and quasiconformal maps. Modified version of PhD thesis from 1990. URL: `arXiv.org/0709.0710v1`.

[17] Oded Schramm. Square tilings with prescribed combinatorics. *Israel Journal of Mathematics*, 84(1-2):97–118, 1993.

[18] Hendrik Schrezenmaier. Zur Berechnung von Kontaktdarstellungen. Masterarbeit, Technische Universität Berlin, 2016. URL: `page.math.tu-berlin.de/~schrezen/Papers/Masterarbeit.pdf`.

[19] Hendrik Schrezenmaier. Homothetic triangle contact representations. In *Proc. WG*, pages 425–437, 2017.

[20] Raphael Steiner. Existenz und Konstruktion von Dreieckszerlegungen triangulierter Graphen und Schnyder woods. Bachelorarbeit, FernUniversität in Hagen, 2016. URL: `www.fernuni-hagen.de/mathematik/DMO/pubs/Bachelorarbeit_Raphael_Steiner.pdf`.

[21] Kenneth Stephenson. *Introduction to circle packing.* Cambridge University Press, 2005. The theory of discrete analytic functions.