THE CONVERGENCE OF AN INTERIOR POINT METHOD FOR AN ELLIPTIC CONTROL PROBLEM WITH MIXED CONTROL-STATE CONSTRAINTS.*

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Abstract. The paper addresses a primal interior point method for state-constrained PDE optimal control problems in function space. By a Lavrentiev regularization, the state constraint is transformed to a mixed control-state constraint with bounded Lagrange multiplier. Existence and convergence of the central path are established, and linear convergence of a short-step pathfollowing method is shown. The behaviour of the method is demonstrated by numerical examples.

Key words. interior point method, function space, optimal control, mixed control-state constraints, Lavrentiev regularization

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1. Introduction. The application of interior point methods to optimal control problems has received a good deal of interest in the past years. This parallels the fast development of numerical methods in large scale optimization where interior point methods play an important role. In the context of PDE control, their performance was carefully tested by Bergounioux et al. [4] for discretized versions of elliptic control problems. Similarly, Grund and Rösch [8] considered different codes of interior point methods for elliptic control problems with pointwise state-constraints.

Leibfritz and Sachs [9] applied an interior point method for solving the quadratic subproblems of a discretized version of an SQP method. Trust-region interior point techniques have been considered by M. Ulbrich, S. Ulbrich, and Heinkenschloss [17] for the optimal control of semilinear parabolic equations in a function space setting. Moreover, affine-scaling interior-point methods were presented for semilinear parabolic boundary control in [16].

In [18, 20] primal-dual interior point methods have been analysed for ODE problems in an infinite dimensional function space setting, and their computational realization by inexact pathfollowing methods has been suggested. In [19] this method has been enhanced on the control of elliptic PDE problems with control constraints.

A satisfactory convergence theory, however, had only been obtained for control constraints, whereas results for state constraints are scarce. The difficulty arises from the fact that Lagrange multipliers for state constraints are usually only measures, which hampers theoretical convergence analysis and affects the numerical solution.

Concerning the regularity of Lagrange multipliers, the situation changes for mixed control-state constraints such as constraints of bottleneck type. Under natural assumptions, their multipliers can be shown to be functions in certain L^p-spaces, we only mention [14, 5, 3]. In [11], the idea came up to add a tiny fraction of the control to the state constraint such that a mixed control-state constraint results. The Lagrange multiplier to this mixed constraint is a bounded and measurable function. This Lavrentiev regularization for state constraints has been analyzed in the context

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of primal-dual active set methods for elliptic control problems. Some results concerning the convergence of the solutions of the regularized problem to that of the original state constrained one can be found in [10, 11].

In the current paper, both ideas are combined. After a Lavrentiev type regularization, we are able to prove the convergence of a primal interior point method in function space. To our best knowledge, this has not jet been done for the control of PDEs with pointwise state constraints. We should underline that the regularization approach is crucial for our analysis.

The paper is organized as follows: We analyze the interior point method applied to the regularized state constrained optimal control problem defined in Section 2. We show existence and convergence of the central path defined by the interior point method in Section 3 and Section 4, respectively. In Section 5, we turn to the linear convergence of an implementable short-step pathfollowing method. The paper is concluded with a set of numerical examples in Section 6 and some remarks on the convergence for Lavrentiev parameter tending to zero in Section 7.

We confine ourselfes to a linear elliptic equation with quadratic objective. In this case, the optimal solution is unique. In case of nonlinar equations, the analysis would have to deal with local minima, second-order sufficient optimality conditions and, in many cases, with the known two-norm discrepancy. The analysis is presented for the case of state-constraints. However, the case of bound constraints on the control is covered by the theory as well.

2. Problem setting. In this paper, we consider the optimal control problem

$$\min\ J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{2.1}$$

subject to the elliptic boundary value problem

$$Ay = u \qquad \text{in } \Omega \tag{2.2}$$

$$\partial_n y + \alpha y = 0 \qquad \text{on } \Gamma \tag{2.3}$$

and to the pointwise mixed control-state constraints

$$y + \lambda u \geq y_c \qquad \text{a.e. in } \Omega. \tag{2.4}$$

In Section 7, we briefly discuss the pass to the limit $\lambda \downarrow 0$. In this setting, $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a bounded domain with $C^{0,1}$-boundary Γ, y_c, $y_d \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\Gamma)$ are fixed functions, and ν, $\lambda \in \mathbb{R}$, $\lambda > 0$, are given constants. By A we denote the differential operator

$$(Ay)(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} y(x)\right) + c_0(x)y(x)$$

with coefficients $a_{ij} \in C^{1,1}(\Omega)$, $c_0 \in L^\infty(\Omega)$ satisfying $a_{ij}(x) = a_{ji}(x)$ and the condition of uniform ellipticity

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N.$$

Moreover, we require $c_0(x) \geq 0, \alpha(x) \geq 0$ and assume that one of these two functions is not vanishing identically. Let us introduce the following

Notations. By $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $(\cdot,\cdot)$ we denote the natural norm and the associated inner product of $L^2(\Omega)$, respectively. We use $\|B\|_{L^p\to L^q}$ to denote the norm of a linear continuous operator $B:\ L^p(\Omega)\to L^q(\Omega)$. In the case $p=q=2$, this norm is just denoted by $\|B\|$. For $\|\cdot\|_{L^\infty}$ we write $\|\cdot\|_\infty$. Throughout the paper, c is a generic constant. Moreover we write L^p for $L^p(\Omega)$ to shorten the notation.
If $v\in L^2(\Omega)$ is a given function, then $v\le 0$ means $v(x)\le 0$ for a.a. $x\in\Omega$. In (2.3), ∂_n denotes the outward co-normal derivative at Γ.

The main scope of our paper is to discuss the convergence of the standard interior point method for the problem (2.1)–(2.4) in function space. The simplest and well known idea of introducing this method is the elimination of the mixed control-state constraint $y+\lambda u\ge y_c$ by a logarithmic barrier function. We substitute (2.1)–(2.4) by the problem

$$\min J_\mu(y,u) := \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 - \mu\int_\Omega \ln\left((y+\lambda u-y_c)(x)\right)dx \tag{2.5}$$

subject to $u\in L^2$ and

$$\begin{aligned} Ay &= u \quad \text{in } \Omega &(2.6)\\ \partial_n y+\alpha y &= 0 \quad \text{on } \Gamma. &(2.7)\end{aligned}$$

In our analysis, we shall transform the state-constrained problem (2.1)–(2.4) to the problem (3.4)–(3.5) with control constraints. We have two reasons for this transformation: The analysis of this transformed problem is simpler than that for (2.1)–(2.4), since we are able to prove the needed regularity of Lagrange multipliers. Moreover, it is easier to show the existence of the central path for the transformed problem.

3. Existence of the central path. In this section, we establish the existence of unique minima u_μ of (2.4)–(2.7) for all $\mu>0$. To do this, we show the existence of a unique solution v_μ of the transformed problem (P$_\mu$) below. We refer to the mappings $\mu\mapsto u_\mu$ and $\mu\mapsto v_\mu$ as the central path, even though continuity is proved only in Section 4. First we recall some known facts about the state-equation (2.2)–(2.3).

Theorem 3.1. *Under our assumptions, for all $u\in L^r(\Omega)$ with $r>\frac{N}{2}$, equation* (2.2) *has a unique solution $y\in H^1(\Omega)\cap C(\bar\Omega)$. There is a constant $c(\Omega,r)$ such that*

$$\|y\|_{H^1(\Omega)} + \|y\|_{C(\bar\Omega)} \le c\,\|u\|_{L^r(\Omega)}.$$

We refer to Casas [6] or Alibert and Raymond [1]. The reader might also consult the detailed presentation of these results in the monography [15]. The theorem ensures that, for $N\le 3$, the mapping $G: u\mapsto y$ is continuous from L^2 to $H^1(\Omega)\cap C(\bar\Omega)$. In particular, it is continuous in L^2. We denote the associated mapping by $S=EG$, where $E: H^1(\Omega)\to L^2$ is the embedding operator from $H^1\cap C(\bar\Omega)$ in L^2. Therefore, we have $S: L^2\to L^2$, continuously.

By S, problem (2.1)–(2.4) becomes equivalent to

$$\min \frac{1}{2}\|Su-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \tag{3.1}$$

subject to

$$\lambda u + Su - y_c \ge 0 \quad \text{a.e. in } \Omega. \tag{3.2}$$

Remark. S is known to be compact. By $\lambda > 0$, $-\lambda$ is not an eigenvalue of S. In fact, since $\lambda > 0$, we have $\lambda u + Su = 0 \Leftrightarrow \lambda u + y = 0 \Leftrightarrow u = -\frac{1}{\lambda}y$. This means $Ay = -\frac{1}{\lambda}y$, hence $Ay + \frac{1}{\lambda}y = 0$ and $\partial_n y + \alpha y = 0$. By coercivity, this equation has only the trivial solution.

To transform (3.1)–(3.2) into a control-constrained problem, we substitute

$$v := Su + \lambda u.$$

By our assumption,

$$D := (S + \lambda I)^{-1} \tag{3.3}$$

exists as a continuous linear operator in L^2. After this substitution, (3.1)–(3.2) is equivalent to

$$(\mathrm{P}) \qquad \min\, f(v) := \frac{1}{2}\|SDv - y_d\|^2 + \frac{\nu}{2}\|Dv\|^2 \tag{3.4}$$

subject to the constraints on the new control $v \in L^2$,

$$v - y_c \geq 0. \tag{3.5}$$

This simplification to a control-constrained problem can be made more explicit: By $v = Su + \lambda u = y + \lambda u$, we have $u = \lambda^{-1}(v - y)$. Inserting this in the state equation and in J, we see that (3.4)–(3.5) is equivalent to the elliptic control problem with control constraint,

$$\min \tilde{J}(y, v) = \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2\lambda^2}\|v - y\|^2 \tag{3.6}$$

subject to

$$\begin{aligned} Ay + \frac{1}{\lambda}y &= \frac{1}{\lambda}v \\ \partial_n y + \alpha y &= 0 \end{aligned} \tag{3.7}$$

and $v - y_c \geq 0$.

For the special choice $D = I$, our analysis of (3.4)–(3.5) covers problems with simple bounds on the control v. The interior point method for (3.4)–(3.5) (or (2.1)–(2.3), respectively) is equivalent to solving

$$\min f_\mu(v) := f(v) - \mu \int_\Omega \ln\left(v(x) - y_c(x)\right) dx. \tag{3.8}$$

Obviously, the quadratic functional f is continuously differentiable in L^2. Its derivative is given by

$$f'(v)\, h = (\tilde{p} + \nu D^* D v\,,\, h)$$

with adjoint state $\tilde{p} = D^* S^*(SDv - y_d)$. Here, S^*, $D^* : L^2 \to L^2$ are the Hilbert space adjoints to S, D, respectively. If $v_\varepsilon(x) - y_c(x) \geq \varepsilon > 0$ holds a.e. on Ω, then the functional

$$\phi(v) = \mu \int_\Omega \ln\left(v(x) - y_c(x)\right) dx$$

is directionally differentiable at v_ε in any direction $h \in L^\infty(\Omega)$, since $v_\varepsilon + t\,h - y_c \geq \varepsilon/2$ for sufficiently small t. In this case,

$$\phi'(v_\varepsilon)h = \int \frac{\mu}{v_\varepsilon(x) - y_c} h\,dx.$$

Suppose now that (3.8) admits a solution $v_\varepsilon = v_\varepsilon(\mu) \in L^2$ satisfying $v_\varepsilon(x) - y_c(x) \geq \varepsilon > 0$. Then we get from the differentiability properties mentioned above

$$f_\mu(v_\varepsilon) = f'(v_\varepsilon) - \phi'(v_\varepsilon) = 0, \tag{3.9}$$

since in this case $\tilde{v} + th, h \in L^\infty$, has distance $\varepsilon/2$ for all small t. Therefore, it holds

$$\tilde{p} + \nu D^* D v_\varepsilon - \frac{\mu}{v_\varepsilon - y_c} = 0 \quad \text{a.e. in } \Omega.$$

Define $\eta \in L^\infty(\Omega)$ by

$$\eta(x) := \frac{\mu}{v_\varepsilon(x) - y_c(x)}. \tag{3.10}$$

Then we have $\eta \geq 0$, $v_\varepsilon - y_c \geq 0$ and $\eta(v_\varepsilon - y_c) = \mu$ for almost all $x \in \Omega$. This function η will tend to a Lagrange multiplier for (2.1)–(2.4) as $\mu \downarrow 0$. However, we have to show that (3.8) is solvable, i.e. that the central path exists. Notice that, by $u = D\,v$, (3.8) and (2.4)–(2.7) are equivalent.

To verify this existence, we consider for fixed $\mu > 0$, $\varepsilon > 0$ the auxiliary problem

$$(\mathrm{P}_\mu^\varepsilon) \qquad \min_{v(x) - y_c(x) \geq \varepsilon} f_\mu(v),$$

where $v \in L^2$. We first prove that this problem is solvable. Next we show that the solution is not active for all sufficiently small $\varepsilon > 0$. In this way, finally a solution $u_\mu = (\lambda I + S)^{-1} v_\mu$ of (2.4)–(2.7) is found.

Lemma 3.2. *For all $\mu \geq 0$, it holds that $f_\mu(v) \to \infty$ if $\|v\| \to \infty$ and $v(x) \geq y_c(x) + \varepsilon$.*

Proof. Since $\|v\| = \|D^{-1} D v\| \leq \|S + \lambda I\|\,\|Dv\|$, we have

$$\begin{aligned} f_\mu(v) &= \frac{1}{2}\,\|SDv - y_d\|^2 + \frac{\nu}{2}\,\|Dv\|^2 - \mu \int_\Omega \ln(v - y_c)\,dx \\ &\geq \frac{\nu}{2}\,\|Dv\|^2 - \mu \int_\Omega (v - y_c)\,dx \qquad (3.11) \\ &\geq \frac{\nu\delta_0}{2}\,\|v\|^2 - \mu\,\|v - y_c\|_{L^1} \geq \frac{\nu\delta_0}{2}\,\|v\|^2 - \mu\,c\,\|v - y_c\|. \qquad (3.12) \end{aligned}$$

In (3.11), we have used $\ln(x) < x$ for all $x > 0$. With $\delta_0 = \|S + \lambda I\|^{-2} > 0$ we observe that $\|v\| \to \infty$ implies $f_\mu(v) \to \infty$. ∎

Theorem 3.3. *For all $\mu \geq 0$ and $0 < \varepsilon \leq 1$, problem $(\mathrm{P}_\mu^\varepsilon)$ has a unique solution $v_\varepsilon(\mu)$. There is a constant $c_v < \infty$ independent of μ and ε such that $\|v_\varepsilon(\mu)\| \leq c_v$.*

Proof. Obviously, f_μ is convex and continuous on the convex and closed subset $C_\varepsilon \subset L^2$ defined by

$$C_\varepsilon = \left\{ v \in L^2(\Omega) \,|\, v(x) - y_c(x) \geq \varepsilon > 0 \text{ for a.a. } x \in \Omega \right\}.$$

Therefore, f_μ is lower semicontinuous on C_ε. Take $\tilde v := y_c + 1$, then the logarithmic term vanishes and by Lemma 3.2, it holds

$$f_\mu(v) \geq f_\mu(y_c+1) = \frac{1}{2}\,\|SD\tilde v - y_d\|^2 + \frac{\nu}{2}\,\|D\tilde v\|^2$$

for all sufficiently large v, say $\|v\| \geq c_v$ with certain $c_v > 0$.

All $v \in C_\varepsilon$ with $\|v\| > c_v$ can be neglected for the search of the infinimum of f_μ. On $C_\varepsilon \cap \{v \in L^2 \,|\, \|v\| \leq c_v\}$, the functional f_μ is bounded, hence

$$j(\varepsilon) := \inf_{v \in C_\varepsilon} f_\mu(v)$$

is finite. Here and in what follows, we suppress for a while the dependence of the problem and its solutions on μ.

Let $v_n \in C_\varepsilon$, $\|v_n\| \leq c_v$, be an infimal sequence, i.e. $f_\mu(v_n) \to j(\varepsilon)$ for $n \to \infty$. We can assume w.l.o.g. weak convergence in L^2, $v_n \rightharpoonup v_\varepsilon \in C_\varepsilon$. By lower semicontinuity, a standard argument yields

$$f_\mu(v_\varepsilon) = j(\varepsilon),$$

hence v_ε is the solution $v_\varepsilon(\mu)$ of $(\mathrm{P}_\mu^\varepsilon)$. Uniqueness follows from the strict convexity of f_μ. $\square$

We recall problem $(\mathrm{P}_\mu^\varepsilon)$,

$$\min f_\mu(v) := \frac{1}{2}\|SDv - y_d\|^2 + \frac{\nu}{2}\|Dv\|^2 - \mu \int_\Omega \ln(v - y_c)\,dx$$
$$v(x) - y_c(x) \geq \varepsilon \quad \text{a.e. in } \Omega.$$

To shorten the notation, we continue to denote the optimal solution $v_\varepsilon(\mu)$ of $(\mathrm{P}_\mu^\varepsilon)$ by v_ε. Take any other $v \in C_\varepsilon$ and $t \in [0,1]$. Then $v_\varepsilon + t(v - v_\varepsilon) \in C_\varepsilon$, hence $f_\mu(v_\varepsilon + t(v - v_\varepsilon))$ is defined. Note that f_μ is not Gâteaux-differentiable in L^2, since $f_\mu(v_\varepsilon + ht)$ may be undefined for $h \in L^2$. However, it is directionally differentiable in the direction $v - v_\varepsilon$. From

$$0 \leq \frac{f_\mu(v_\varepsilon + t(v - v_\varepsilon)) - f_\mu(v_\varepsilon)}{t}$$

we find by $t \downarrow 0$ for the directional derivative

$$f'_\mu(v_\varepsilon)(v - v_\varepsilon) \geq 0 \quad \forall v \in C_\epsilon.$$

In terms of our transformation, this can be written as

$$\Big(D^*S^*(SDv_\varepsilon - y_d) + \nu D^*Dv_\varepsilon - \frac{\mu}{v_\varepsilon - y_c}\,,\, v - v_\varepsilon\Big) \geq 0 \quad \forall v \in C_\varepsilon. \tag{3.13}$$

Define $p_\varepsilon := D^*S^*(SDv_\varepsilon - y_d)$. Then we can re-write (3.13) as

$$\Big(p_\varepsilon + \nu D^*Dv_\varepsilon - \frac{\mu}{v_\varepsilon - y_c}\,,\, v - v_\varepsilon\Big) \geq 0 \quad \forall v \in C_\varepsilon. \tag{3.14}$$

We shall show that $\|p_\varepsilon\|_\infty$ is bounded, independently of ε:
The operator S is self-adjoint, $S = S^*$. Moreover, as $S = EG$, S is even linear and continuous from L^2 to L^∞. The same holds for S^*.

Let us discuss the form and the regularity properties of the operator D. We have $D = (S+\lambda I)^{-1}$. Put $w = Dz$. Then $z = Sw+\lambda Iw$. It follows $\lambda w = z-Sw = z-SDz$ and $w = \lambda^{-1}z - \lambda^{-1}SDz$. Therefore D admits the form

$$D = \lambda^{-1}(I - SD). \tag{3.15}$$

From this representation we easily get the additional regularity property $D : L^\infty \to L^\infty$, continuously.

This is also visible from (3.6)–(3.7), since $v \mapsto y$ is continuous from L^∞ to L^∞ and therefore also $v \mapsto u = \lambda^{-1}(v - y(v)) : L^\infty \to L^\infty$.

We know from Lemma 3.2 that $\|v_\varepsilon\|$ is bounded by a constant c_v that does not depend on ε. Now we estimate $\|p_\varepsilon\|_\infty$ by

$$\begin{aligned} \|p_\varepsilon\|_\infty &= \|D^*S^*(SDv_\varepsilon - y_d)\|_\infty \\ &\le \|D^*\|_{L^\infty}\, \|S^*\|_{L^2\to L^\infty}\, \|SDv_\varepsilon - y_d\| \le c_p, \end{aligned} \tag{3.16}$$

where c_p does not depend on ε, since $\|SDv_\varepsilon - y_d\| \le \|S\|_{L^2}\|D\|_{L^2}\|c_v\| + \|y_d\|$. Next we evaluate (3.13). Let us define the sets

$$\begin{aligned} M_+(\varepsilon) &:= \left\{ x \in \Omega \,\middle|\, p_\varepsilon(x) + \nu(D^*Dv_\varepsilon)(x) - \frac{\mu}{v_\varepsilon(x) - y_c(x)} > 0 \right\} \\ M_0(\varepsilon) &:= \left\{ x \in \Omega \,\middle|\, p_\varepsilon(x) + \nu(D^*Dv_\varepsilon)(x) - \frac{\mu}{v_\varepsilon(x) - y_c(x)} = 0 \right\}. \end{aligned}$$

Due to (3.13), $M_+(\varepsilon) \cup M_0(\varepsilon)$ covers Ω up to a set of measure zero. The variational inequality (3.13) implies $v_\varepsilon(x) - y_c = \varepsilon$ for almost all $x \in M_+(\varepsilon)$.

Theorem 3.4. *There exist constants $a, b > 0, \varepsilon_M$ such that the set $M_+(\varepsilon)$ has measure zero for all $0 < \varepsilon < \varepsilon_M$.*

Proof. For almost all $x \in M_+(\varepsilon)$, the constraint is active, i.e. $v_\varepsilon(x) - y_c(x) = \varepsilon$. Thus, by (3.16), we have for almost all $x \in M_+(\varepsilon)$

$$c_p + \nu\,(D^*Dv_\varepsilon)(x) - \frac{\mu}{\varepsilon} \ge p_\varepsilon(x) + \nu(D^*Dv_\varepsilon)(x) - \frac{\mu}{v_\varepsilon(x) - y_c(x)} > 0. \tag{3.17}$$

By (3.15),

$$D^*D = \lambda^{-2}(I - S^*D^*)(I - SD) = \lambda^{-2}I + K$$

with bounded $K : L^2 \to L^\infty$,

$$K = \lambda^{-2}\left\{S^*D^*DS - (S^*D^* + SD)\right\}.$$

Almost everywhere on $M_+(\varepsilon)$ it holds $v_\varepsilon(x) = y_c(x) + \varepsilon$, hence

$$c_p + \nu\,(D^*Dv_\varepsilon)(x) = c_p + \nu\,(\lambda^{-2}(y_c(x) + \varepsilon) + (K\,v_\varepsilon)(x)).$$

With the left-hand side of (3.17), Theorem 3.3 yields

$$c_p + \nu(\lambda^{-2}(\|y_c\|_\infty + \varepsilon) + c_v\,\|K\|_{L^2\to L^\infty}) > \frac{\mu}{\varepsilon}. \tag{3.18}$$

Clearly, the right hand side tends to infinity as $\varepsilon \downarrow 0$ while the left hand side remains bounded. Therefore, the inequality cannot be satisfied for small ε.

To quantify a and b, we set

$$\tilde{a} = \left(c_p \frac{\lambda^2}{\nu} + \|y_c\|_\infty + \|K\|_{L^2 \to L^\infty} c_v \lambda^2 \right), \tag{3.19}$$

and from (3.18) we get

$$\varepsilon^2 + \tilde{a}\varepsilon - \frac{\lambda^2}{\nu}\mu > 0\,.$$

Solving this quadratic inequality, we have

$$\varepsilon > -\frac{1}{2}\tilde{a} + \sqrt{\frac{1}{4}\tilde{a} + \frac{\lambda^2}{\nu}\mu} \; > \; 0.$$

With $a = \frac{1}{2}\tilde{a}$ and $b = \frac{\lambda^2}{\nu} a^{-2}$, we can write this as $\varepsilon > a(\sqrt{1 + b\mu} - 1)$, where $a, b > 0$. For smaller ε, $M_+(\varepsilon)$ must have measure zero. ∎

COROLLARY 3.5. *For all* $0 < \varepsilon < \varepsilon_M$*, the solution* $v_\varepsilon(\mu)$ *of* ($\mathrm{P}_\mu^\varepsilon$) *is the unique solution to* (P_μ).

Proof. For these ε, the set $M_+(\varepsilon)$ has measure zero. Therefore, it holds

$$p_\varepsilon(x) + \nu(D^* D v_\varepsilon)(x) - \frac{\mu}{v_\varepsilon(x) - y_c(x)} = 0$$

almost everywhere on Ω, hence $v_\varepsilon(\mu)$ satisfies the first-order necessary optimality conditions for the optimization problem (P_μ). This is a problem with convex objective functional; the necessary conditions are sufficient for optimality. Strong convexity yields uniqueness (notice that $\nu > 0$). Therefore, $v_\varepsilon(\mu)$ is the unique solution $v(\mu)$ of (P_μ). ∎

COROLLARY 3.6. *There exists a constant* $c_\mu > 0$ *such that for* $\mu \leq 1$ *the unique solution* v_μ *of* (3.8) *satisfies* $v_\mu \geq y_c + c_\mu \mu$ *a.e. on* Ω.

Proof. Let v_μ be the solution of (P_μ). Then an $\varepsilon > 0$ exists such that v_μ is a solution of ($\mathrm{P}_\mu^\varepsilon$) too. For that ε it holds $v_\mu \geq y_c + \varepsilon$. Choosing $c_\mu = \frac{\varepsilon}{\mu}$ yields $v_\mu \geq y_c + c_\mu \mu$. ∎

4. Convergence of the central path. Having established the existence of the central path $\mu \mapsto v_\mu$ for all $\mu > 0$, we can proceed with proving continuity of the path and convergence towards a solution of (P).

The unique minimizer of (3.8) can be characterized by (3.9) as

$$F(v_\mu; \mu) = (D^* S^* S D + \nu D^* D) v_\mu - D^* S^* y_d - \frac{\mu}{v_\mu - y_c} = 0 \quad \text{a.e. on } \Omega \tag{4.1}$$

Since $v_\mu - y_c \geq c_\mu \mu$ holds for $\mu \leq 1$ by Corollary 3.6, F is Fréchet differentiable in all directions $v \in L^\infty$. We denote the partial derivatives w.r.t. v and μ by $\partial_v F$ and $\partial_\mu F$, respectively. The derivative $\partial_v F$ is

$$\begin{aligned} \partial_v F(v; \mu) &= (D^* S^* S D + \nu D^* D) + \frac{\mu}{(v - y_c)^2} && (4.2)\\ &= (D^* S^* S D + \nu K) + \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v - y_c)^2} \right) \\ &= \bar{K} + \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v - y_c)^2} \right), && (4.3) \end{aligned}$$

where

$$\bar{K} = D^*S^*SD + \nu K$$

is a bounded operator from L^2 to L^∞. From (4.2) and (3.3) we see immediately that, for all $v \geq y_c + \varepsilon$, $\partial_v F(v;\mu) \in \mathcal{L}(L^2, L^2)$ is a symmetric positive definite operator with

$$\langle \xi, \partial_v F(v;\mu)\xi \rangle \geq \nu \langle D\xi, D\xi \rangle \geq \nu \|S + \lambda I\|^{-2} \|\xi\|^2.$$

The Lax-Milgram theorem guarantees the existence of a bounded inverse $\partial_v F(v;\mu)^{-1} : L^2 \to L^2$ with

$$\|\partial_v F(v;\mu)^{-1}\| \leq \frac{1}{\nu}(\|S\| + |\lambda|)^2. \tag{4.4}$$

In the next lemma we prove a further regularity property of $\partial_v F$. In all what follows, we write for short $v > y_c$ to express the existence of $\varepsilon \geq 0$ such that $v(x) > y_c(x) + \varepsilon$ for a.a. $x \in \Omega$.

Lemma 4.1. *Assume that $v > y_c$. Then the derivative $\partial_v F(v;\mu) : L^\infty \to L^\infty$ is a bijective operator with bounded inverse $\partial_v F(v;\mu)^{-1} : L^\infty \to L^\infty$, where $\|\partial_v F(v;\mu)^{-1}\|_{L^\infty \to L^\infty} \leq c_i$ is bounded independently of μ.*

Proof. Due to (4.4), for each $z \in L^\infty \subset L^2$ there is a solution $\xi \in L^2$ to $\partial_v F(v;\mu)\xi = z$ with

$$\|\xi\| \leq \frac{1}{\nu}(\|S\| + |\lambda|)^2 \|z\| \leq \frac{\sqrt{|\Omega|}}{\nu}(\|S\| + |\lambda|)^2 \|z\|_\infty. \tag{4.5}$$

Now we have by (4.3)

$$\left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v - y_c)^2}\right)\xi = z - \bar{K}\xi$$

and hence by (4.5)

$$\begin{aligned}
\|\xi\|_\infty &\leq \frac{\lambda^2}{\nu}\left(\|z\|_\infty + \|\bar{K}\|_{L^2 \to L^\infty}\|\xi\|\right) \\
&\leq \frac{\lambda^2}{\nu}\left(1 + \|\bar{K}\|_{L^2 \to L^\infty}\frac{\sqrt{|\Omega|}}{\nu}(\|S\| + |\lambda|)^2\right)\|z\|_\infty \\
&=: c_i \|z\|_\infty.
\end{aligned}$$

Thus, $\xi \in L^\infty$ holds, such that $\partial_v F(v;\mu) : L^\infty \to L^\infty$ is bijective and has a bounded inverse $\|\partial_v F(v;\mu)^{-1}\|_{L^\infty \to L^\infty} \leq c_i$. ☐

With the invertibility of $\partial_v F$ at hand we make use of the implicit function theorem in order to justify the notion of a central path. We obtain the

Corollary 4.2. *The mapping $\mu \mapsto v_\mu$ is continuously differentiable from $\mathbb{R}_+$ to L^∞.*

Now we turn to convergence of the central path towards a solution of (3.1).

Theorem 4.3. *For $\mu \to 0$, the central path converges towards a KKT point v_0 of* (3.1)*. There exists a constant $c_0 < \infty$ such that the following error estimate holds for all $\mu \leq 1$:*

$$\|v_0 - v_\mu\|_\infty \leq c_0\sqrt{\mu}. \tag{4.6}$$

Proof. First we will establish an L^2-bound on

$$v'_\mu = -\partial_v F(v_\mu;\mu)^{-1}\partial_\mu F(v_\mu;\mu) = \left(\bar{K} + \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}\right)\right)^{-1} \frac{1}{v_\mu - y_c} \tag{4.7}$$

and infer an L^∞-bound from that. From this we will determine the existence of and distance to the limit point v_0, and finally check the first order necessary conditions for v_0.
(i) L^2*-estimate.* We introduce the diagonal preconditioner

$$z_\mu = \sqrt{\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}} \tag{4.8}$$

and write (4.8) as

$$\begin{aligned} z_\mu v'_\mu &= \left(z_\mu^{-1}\left(\bar{K} + \frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}\right) z_\mu^{-1}\right)^{-1} z_\mu^{-1} \frac{1}{v_\mu - y_c} \\ &= \left(z_\mu^{-1}\bar{K} z_\mu^{-1} + I\right)^{-1} \left(\frac{\nu(v_\mu - y_c)^2}{\lambda^2} + \mu\right)^{-1/2}. \end{aligned}$$

Since $z_\mu^{-1}\bar{K}z_\mu^{-1}$ is positive semidefinite, we may conclude that $\|z_\mu v'_\mu\| \le \sqrt{|\Omega|/\mu}$. From $z_\mu \ge \sqrt{\nu}/\lambda$ a.e. we finally obtain

$$\|v'_\mu\| \le \sqrt{\frac{\nu|\Omega|}{\mu\lambda}}. \tag{4.9}$$

(ii) L^∞*-estimates.* Using the splitting (4.3) to move the coupling term $\bar{K}v'_\mu$ in $\partial_v F(v_\mu;\mu)v'_\mu = -\partial_\mu F(v_\mu;\mu)$ to the right hand side, and the fact that

$$ax + \frac{b}{x} \ge 2\sqrt{ab}$$

holds for arbitrary $a, b, x > 0$, we obtain

$$\begin{aligned} \|v'_\mu\|_\infty &\le \left\| \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}\right)^{-1} \frac{1}{v_\mu - y_c}\right\|_\infty + \left\| \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}\right)^{-1} \bar{K}v'_\mu\right\|_\infty \\ &\le \left\| \left(2\sqrt{\frac{\nu\mu}{\lambda^2}}\right)^{-1}\right\|_\infty + \left\| \left(\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}\right)^{-1}\right\|_\infty \|\bar{K}\|_{L^2\to L^\infty}\|v'_\mu\| \end{aligned}$$

and infer from (4.9)

$$\|v'_\mu\|_\infty \le \left\|\frac{\lambda}{2\sqrt{\nu\mu}}\right\|_\infty + \frac{\lambda^2}{\nu}\|\bar{K}\|_{L^2\to L^\infty}\sqrt{\frac{\nu|\Omega|}{\mu\lambda}} \le \frac{c_0}{\sqrt{\mu}}$$

for some $c_0 < \infty$.
(iii) *Distance to the limit point.* The distance between two points on the central path is therefore bounded by

$$\|v_{\mu_1} - v_{\mu_2}\|_\infty \le \int_{\mu_1}^{\mu_2} \|v'_\mu\|_\infty \, d\mu \le \frac{c_0}{2}(\sqrt{\mu_2} - \sqrt{\mu_1}). \tag{4.10}$$

Since for any sequence (μ_k) with $\mu_k \to 0$ the corresponding sequence (v_{μ_k}) of central path points forms a Cauchy sequence, the path converges towards some limit point v_0. Passing to the limit $\mu_1 \to 0$ verifies the error bound (4.6).
(iv) *First order necessary conditions.* Recalling the Lagrange multiplier approximations η_μ from (3.10) we write (3.9) as $f'(v_\mu) = \eta_\mu$. Due to the continuity of f' and the convergence of $v_\mu \to v_0$ in L^2, the multiplier approximations converge towards $\eta_0 = f'(v_0)$ in L^2. Since $\eta_\mu \geq 0$ and $\eta_\mu(v_\mu - y_c) = \mu$ for almost all $x \in \Omega$ and therefore $(\eta_\mu, v_\mu - y_c) = \mu|\Omega|$, the same holds by continuity for η_0, i.e. $\eta_0 \geq 0$ and $(\eta_0, v_0 - y_c) = 0$. Since the first order necessary conditions are satisfied, v_0 is a KKT point for (3.4). ∎

5. Convergence of a short step pathfollowing method. For the analysis of interior point methods, local norms are an invaluable tool. Here we use the scaled norm

$$\|v\|_\mu = \|z_\mu v\|_\infty$$

with the scaling z_μ defined in (4.8), which is closely connected to the energy norms used in the theory of self-concordant barrier functionals [12, 13].

We consider a short-step pathfollowing method with classical predictor. Since we are interested in actually implementable algorithms, we have to use an inexact Newton corrector that accounts for the error due to discretisation. We consider this error by an inner residual r^k and an inexact Newton correction Δv_h^k and replace the infinite dimensional Newton equation

$$\partial_v F(v^k; \mu^{k+1}) \Delta v^k = -F(v^k; \mu^{k+1})$$

for the exact correction Δv^k by

$$\partial_v F(v^k; \mu^{k+1}) \Delta v_h^k = -F(v^k; \mu^{k+1}) + r^k.$$

The iteration index is denoted by a superscript. Another source of inexactness is e.g. the iterative solution of the state equation. The algorithm reads as follows.

ALGORITHM 5.1.
Choose $0 < \sigma < 1$, $\delta > 0$, $\mu^0 > 0$, *and* $v^0 > y_c$.
For $k = 0, 1, \ldots$
  $\mu^{k+1} = \sigma \mu^k$,
  solve $\partial_v F(v^k; \mu^{k+1}) \Delta v_h^k = -F(v^k; \mu^{k+1})$
    up to a relative accuracy of $\|\Delta v_h^k - \Delta v^k\|_{\mu^{k+1}} \leq \delta \|\Delta v^k\|_{\mu^{k+1}}$,
  $v^{k+1} = v^k + \Delta v_h^k$.

Note that the accuracy matching in Algorithm 5.1 will require mesh refinement as $\mu \to 0$. Alternatively, on a fixed discretization the algorithm can be performed only up to some $\mu_{\min} > 0$ while still meeting the accuracy requirement.

The remainder of the section is devoted to proving that for suitable choices of σ, δ, μ^0, and v^0, all iterates of this algorithm are well defined and converge towards the solution point v_0. First we formulate the main result, the proof of which is postponed to the end of this section.

THEOREM 5.2. *Let a tolerance* $\vartheta < 1/(18c_z)$, $\mu^0 > 0$, *and an initial iterate* v^0 *with* $\|v^0 - v_{\mu^0}\|_{\mu^0} \leq 2\vartheta\sqrt{\mu^0}$ *be given. Choose* $\delta \leq 1/4$ *and a reduction factor* σ *satisfying*

$$1 - \sigma \leq \frac{\vartheta}{3\vartheta(c_z + 1/2) + c_z}.$$

Then the iterates v^k defined by Algorithm 5.1 are all well defined and converge linearly towards the limit point v_0. More precisely,

$$\left\|v^k - v_{\mu^k}\right\|_{\mu^k} \le 2\vartheta\sqrt{\mu^k} \quad \text{and} \quad \left\|v^k - v_0\right\|_{\mu^k} \le (c_0 + 2\vartheta)\sigma^{k/2}\sqrt{\mu^0}.$$

We stress that this conceptual algorithm is deliberately designed to be simplistic in order to facilitate convergence analysis. We do not recommend to use it for actual computation. First, the admissible choice of parameters σ and v_0 depends on the problem specific constant c_z, which will usually be unavailable in actual computation. Second, the bounds given for ϑ and σ are global worst-case bounds that will be unnecessarily restrictive locally. Adaptive stepsize and accuracy selection will result in a far more efficient algorithm.

The proof of Theorem 5.2 will require the usual ingredients for Newton convergence theorems which we will provide now: boundedness of $\partial_v F^{-1}$, Lipschitz continuity of the local norms, and Lipschitz continuity of $\partial_v F$. First we turn to $\partial_v F^{-1}$ and derive the analogue of Lemma 4.1 for the scaled norm.

LEMMA 5.3. *There is some constant $1 \le c_z < \infty$ independent of μ, such that*

$$\|\partial_v F(v;\mu)^{-1}\zeta\|_\mu \le c_z \|z_\mu^{-1}\zeta\|_\infty \tag{5.1}$$

for all $v \in B_\mu(v_\mu;\vartheta\sqrt{\mu}) = \{v \in L^\infty : \|v - v_\mu\|_\mu \le \vartheta\sqrt{\mu}\}$ with $\vartheta < 1$.

Proof. From $z_\mu \ge \sqrt{\mu}/(v_\mu - y_c)$ we see that

$$\left\|\frac{v - v_\mu}{v_\mu - y_c}\right\|_\infty = \left\|\frac{v - v_\mu}{\sqrt{\mu}}\sqrt{\frac{\mu}{(v_\mu - y_c)^2}}\right\|_\infty \le \left\|z_\mu \frac{v - v_\mu}{\sqrt{\mu}}\right\|_\infty = \frac{\|v - v_\mu\|_\mu}{\sqrt{\mu}} \le \vartheta$$

for $v \in B_\mu(v_\mu;\vartheta\sqrt{\mu})$. For almost all $x \in \Omega$, we therefore have $-(v - v_\mu) \le \vartheta(v_\mu - y_c)$, which implies

$$v \ge (1-\vartheta)v_\mu + \vartheta y_c = (1-\vartheta)(v_\mu - y_c) + y_c \ge (1-\vartheta)c_\mu\mu + y_c > y_c \tag{5.2}$$

due to Corollary 3.6. Lemma 4.1 now provides the invertibility of $\partial_v F(v;\mu)$. As in the proof of Theorem 4.3, we have $z_\mu \partial_v F(v;\mu)^{-1} z_\mu = (z_\mu^{-1}\bar{K}z_\mu^{-1} + I)^{-1}$ and hence the L^2-estimate

$$\|z_\mu \partial_v F(v;\mu)^{-1}\zeta\| = \|(z_\mu^{-1}\bar{K}z_\mu^{-1} + I)^{-1} z_\mu^{-1}\zeta\| \le \|z_\mu^{-1}\zeta\|. \tag{5.3}$$

Defining $\phi = \partial_v F(v;\mu)^{-1}\zeta$ we have $(\bar{K} + z_\mu^2)\phi = \zeta$ and

$$\begin{aligned}\|z_\mu \partial_v F(v;\mu)^{-1}\zeta\|_\infty &= \|z_\mu\phi\|_\infty = \|z_\mu^{-1}(\zeta - \bar{K}\phi)\|_\infty \\ &\le \|z_\mu^{-1}\zeta\|_\infty + \frac{\lambda}{\sqrt{\nu}}\|\bar{K}\|_{L^2\to L^\infty}\|\phi\| \le \|z_\mu^{-1}\zeta\|_\infty + \frac{\lambda^2}{\nu}\|\bar{K}\|_{L^2\to L^\infty}\|z_\mu\phi\|.\end{aligned}$$

Using (5.3) yields

$$\|z_\mu \partial_v F(v;\mu)^{-1}\zeta\|_\infty \le \left(1 + \frac{\lambda^2\|\bar{K}\|_{L^2\to L^\infty}}{\nu\sqrt{|\Omega|}}\right)\|z_\mu^{-1}\zeta\|_\infty$$

and establishes the constant c_z. □

Next we prove Lipschitz continuity of the scaled norms.

LEMMA 5.4. *For all $v \in L^\infty$ and $0 < \sigma \le 1$,*

$$\|v\|_{\sigma\mu} \le \sigma^{-c_z} \|v\|_\mu \tag{5.4}$$

holds. Moreover, the derivative of the central path is bounded by $\|v'_\mu\|_\mu \le c_z/\sqrt{\mu}$.

Proof. We begin with estimating the derivative of the central path in the scaled norm. Lemma 5.3 applied to $v'_\mu = -\partial_v F(v_\mu;\mu)^{-1}\partial_\mu F(v_\mu;\mu)$ results in

$$\|v'_\mu\|_\mu \le c_z \|z_\mu^{-1}\partial_\mu F(v_\mu;\mu)\|_\infty = c_z\|z_\mu^{-1}(v_\mu - y_c)^{-1}\|_\infty \le \frac{c_z}{\sqrt{\mu}}. \tag{5.5}$$

Notice that $v_{\sigma\mu} - y_c > 0$. We proceed with bounding the expression

$$\phi(\sigma) = \frac{v_\mu - y_c}{v_{\sigma\mu} - y_c}.$$

Note that since $\mu \mapsto v_\mu$ is a differentiable mapping from $\mathbb{R}_+$ to L^∞, ϕ maps $]0,1]$ differentiably into L_∞. Using $z_{\tau\mu} \ge \sqrt{\tau\mu}/(v_{\tau\mu} - y_c)$ and (5.5), we start with

$$\begin{aligned}
\|\phi'(\tau)\|_{L_\infty} &= \left\| \frac{v_\mu - y_c}{(v_{\tau\mu} - y_c)^2} v'_{\tau\mu}\mu \right\|_\infty \le \left\| \frac{v_\mu - y_c}{v_{\tau\mu} - y_c}\right\|_\infty \left\| \frac{\sqrt{\tau\mu}}{v_{\tau\mu} - y_c} v'_{\tau\mu}\right\|_\infty \frac{\mu}{\sqrt{\tau\mu}} \\
&\le \|\phi(\tau)\|_{L_\infty} \|v'_{\tau\mu}\|_{\tau\mu} \frac{\mu}{\sqrt{\tau\mu}} \le \|\phi(\tau)\|_{L_\infty} \frac{c_z}{\sqrt{\tau\mu}}\frac{\mu}{\sqrt{\tau\mu}} \le \|\phi(\tau)\|_{L_\infty}\frac{c_z}{\tau}
\end{aligned}$$

for $\sigma \le \tau \le 1$. From this we infer

$$\begin{aligned}
\|\phi(\sigma)\|_{L_\infty} &= \left\| \phi(1) + \int_\sigma^1 \phi'(\tau)\,d\tau \right\|_\infty \\
&\le \|\phi(1)\|_{L_\infty} + \int_\sigma^1 \|\phi'(\tau)\|_{L_\infty}\,d\tau \le 1 + \int_\sigma^1 \|\phi(\tau)\|_{L_\infty}\frac{c_z}{\tau}\,d\tau.
\end{aligned}$$

The Bellmann-Gronwall lemma now yields $\|\phi(\sigma)\|_{L_\infty} \le \sigma^{-c_z}$ for $0 < \sigma \le 1$, hence

$$\left\| \frac{v_\mu - y_c}{v_{\sigma\mu} - y_c}\right\|_\infty \le \sigma^{-c_z}. \tag{5.6}$$

Next we estimate

$$\left\|\frac{z_{\sigma\mu}}{z_\mu}\right\|_\infty = \left\| \frac{\frac{\nu}{\lambda^2} + \frac{\sigma\mu}{(v_{\sigma\mu}-y_c)^2}}{\frac{\nu}{\lambda^2} + \frac{\mu}{(v_\mu - y_c)^2}}\right\|_\infty^{1/2} \le \max\left(1, \sqrt{\sigma}\left\|\frac{v_\mu - y_c}{v_{\sigma\mu} - y_c}\right\|_\infty\right), \tag{5.7}$$

the case depending on whether

$$\frac{\sigma\mu}{(v_{\sigma\mu} - y_c)^2} \le \frac{\mu}{(v_\mu - y_c)^2}$$

holds. Dropping the factor $\sqrt{\sigma} \le 1$ for simplicity, we obtain from (5.7) and (5.6)

$$\|v\|_{\sigma\mu} = \left\| z_\mu \frac{z_{\sigma\mu}}{z_\mu} v\right\|_\infty \le \max\left(1, \sigma^{-c_z}\right)\|v\|_\mu$$

that proves the claim. $\square$

Finally, we prove Lipschitz continuity of $\partial_v F$, weighted with $\partial_v F^{-1}$.

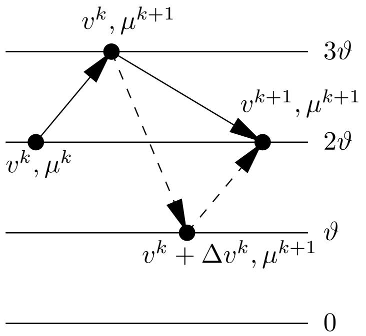


FIG. 5.1. *Errors of the iterates during the proof of Theorem 5.2.*

LEMMA 5.5. *For all $0 < \vartheta < 1$, and all v, $\hat{v} \in B_\mu(v_\mu, \vartheta\sqrt{\mu})$, the following Lipschitz condition holds:*

$$\left\|\partial_v F(v;\mu)^{-1}(\partial_v F(v;\mu) - \partial_v F(\hat{v};\mu))(v-\hat{v})\right\|_\mu \leq \frac{2c_z}{(1-\vartheta)^3\sqrt{\mu}} \|v-\hat{v}\|_\mu^2 . \tag{5.8}$$

Proof. Using Lemma 5.3 and in view of the representation (4.2) of $\partial_v F$, we have

$$\begin{aligned}
&\left\|\partial_v F(v;\mu)^{-1}(\partial_v F(v;\mu) - \partial_v F(\hat{v};\mu))(v-\hat{v})\right\|_\mu \\
&\qquad \leq c_z \left\|z_\mu^{-1}(\partial_v F(v;\mu) - \partial_v F(\hat{v};\mu))(v-\hat{v})\right\|_\infty \\
&\qquad = c_z \left\| z_\mu^{-1} \left(\frac{\mu}{(v-y_c)^2} - \frac{\mu}{(\hat{v}-y_c)^2} \right)(v-\hat{v}) \right\|_\infty .
\end{aligned}$$

Using the fact that the Lipschitz constant of x^{-2} for $x \geq a > 0$ is given by $2a^{-3}$, and that $v - y_c \geq (1-\vartheta)(v_\mu - y_c)$ for $v \in B_\mu(v_\mu, \vartheta\sqrt{\mu})$ due to (5.2), we can proceed with

$$\begin{aligned}
&\left\|\partial_v F(v;\mu)^{-1}(\partial_v F(v;\mu) - \partial_v F(\hat{v};\mu))(v-\hat{v})\right\|_\mu \\
&\qquad \leq c_z \left\| z_\mu^{-1} \mu \frac{2(v-\hat{v})}{((1-\vartheta)(v_\mu - y_c))^3}(v-\hat{v}) \right\|_\infty \\
&\qquad = \frac{2c_z}{(1-\vartheta)^3} \left\| \frac{\mu}{z_\mu^3 (v_\mu - y_c)^3}\, z_\mu^2 (v-\hat{v})^2 \right\|_\infty \\
&\qquad \leq \frac{2c_z}{(1-\vartheta)^3} \left\| \frac{\mu}{z_\mu^3 (v_\mu - y_c)^3} \right\|_\infty \|v-\hat{v}\|_\mu^2 \\
&\qquad \leq \frac{2c_z}{\sqrt{\mu}(1-\vartheta)^3} \|v-\hat{v}\|_\mu^2,
\end{aligned}$$

where the last inequality is a direct consequence of (4.8). ∎

Now we are prepared to prove Theorem 5.2.

Proof. First we give an outline of the proof (see Fig. 5.1). We use induction and assume that $\|v^k - v_{\mu^k}\|_{\mu^k} \leq 2\vartheta\sqrt{\mu^k}$. Decreasing the homotopy parameter μ^k by a factor of σ will lead to an error bound $\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \leq 3\vartheta\sqrt{\mu^{k+1}}$. Then the inexact Newton corrector reduces the error again to $\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \leq 2\vartheta\sqrt{\mu^{k+1}}$. We show this by deriving an error bound $\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \leq \vartheta\sqrt{\mu^{k+1}}$ for the exact Newton corrector and adding a residual bounded by $\vartheta\sqrt{\mu^{k+1}}$.

(i) To begin with, we split the error as follows:

$$
\begin{aligned}
\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} &\le \|v^k - v_{\mu^k}\|_{\mu^{k+1}} + \|v_{\mu^k} - v_{\mu^{k+1}}\|_{\mu^{k+1}} \\
&\le \|v^k - v_{\mu^k}\|_{\mu^{k+1}} + \int_{\mu^{k+1}}^{\mu^k} \|v'_\tau\|_{\mu^{k+1}}\, d\tau.
\end{aligned}
$$

In view of $\mu^{k+1} = \sigma\mu^k$, application of Lemma 5.4 and the induction assumption, and setting $\mu_{k+1} = \tilde{\sigma}\tau$ with $\tilde{\sigma} = \mu_{k+1}/\tau$ leads this to

$$
\begin{aligned}
\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} &\le \sigma^{-c_z}\|v^k - v_{\mu^k}\|_{\mu^k} + \int_{\mu^{k+1}}^{\mu^k} (\mu^{k+1}/\tau)^{-c_z}\|v'_\tau\|_\tau\, d\tau \\
&\le \sigma^{-c_z} 2\vartheta\sqrt{\mu^k} + \int_{\mu^{k+1}}^{\mu^k} (\mu^{k+1}/\tau)^{-c_z} \frac{c_z}{\sqrt{\tau}}\, d\tau. \qquad (5.9)
\end{aligned}
$$

The integral evaluates to

$$
\begin{aligned}
\frac{c_z}{(\mu^{k+1})^{c_z}} \int_{\sigma\mu^k}^{\mu^k} \tau^{c_z - 1/2}\, d\tau &= \frac{c_z}{(\sigma\mu^k)^{c_z}} \frac{(\mu^k)^{c_z+1/2} - (\sigma\mu^k)^{c_z+1/2}}{c_z + 1/2} \\
&= \frac{c_z}{c_z + 1/2}\sigma^{-c_z}(1 - \sigma^{c_z+1/2})\sqrt{\mu^k} \\
&\le c_z\sigma^{-c_z}(1-\sigma)\sqrt{\mu^k}. \qquad (5.10)
\end{aligned}
$$

For the last estimate, we applied the first-order Taylor expansion of $\sigma^{c_z+1/2}$ at $\sigma = 1$. Inserting (5.10) into (5.9) leads to the estimate

$$
\begin{aligned}
\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} &\le \sigma^{-c_z}(2\vartheta + c_z(1-\sigma))\sqrt{\mu^k} \\
&= \sigma^{-(c_z+1/2)}(2\vartheta + c_z(1-\sigma))\sqrt{\mu^{k+1}}. \qquad (5.11)
\end{aligned}
$$

Next we rewrite the assumption on σ:

$$
\begin{aligned}
1 - \sigma \le \frac{\vartheta}{3\vartheta(c_z + 1/2) + c_z} &\Leftrightarrow (3\vartheta(c_z + 1/2) + c_z)(1-\sigma) \le \vartheta \\
&\Leftrightarrow c_z(1-\sigma) \le \vartheta - 3\vartheta(c + 1/2)(1-\sigma).
\end{aligned}
$$

Adding 2ϑ on both sides leads to

$$
2\vartheta + c_z(1-\sigma) \le 3\vartheta\big(1 + (c_z + 1/2)(\sigma - 1)\big) \le 3\vartheta\sigma^{c_z+1/2}. \qquad (5.12)
$$

Combining (5.11) and (5.12) yields

$$
\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \le 3\vartheta\sqrt{\mu^{k+1}}. \qquad (5.13)
$$

(ii) Since $c_z \ge 1$, the assumption $\vartheta \le 1/(18c_z)$ implies

$$
\frac{3\vartheta}{(1-3\vartheta)^3} \le 6\vartheta \le \frac{1}{3c_z}
$$

and thus

$$
\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \le \frac{2}{3}\,\frac{(1-3\vartheta)^3}{2c_z}\sqrt{\mu^{k+1}}.
$$

This is two third of the Lipschitz constant provided by Lemma 5.5 for the ball $B_{\mu^{k+1}}(v_{\mu^{k+1}}, 3\vartheta\sqrt{\mu^{k+1}})$. Thus, the conditions for local convergence of the *exact* Newton corrector as considered in the refined Newton-Mysovskii theorem [7] are satisfied. For an iteration sequence x^k that was started close enough to the solution x^*, it states

$$\|x^{k+1} - x^k\| \le \frac{\omega}{2}\|x^k - x^*\|^2.$$

In our case, the Lipschitz constant ω is given by Lemma 5.5 as

$$\omega = \frac{2c_z}{(1-3\vartheta)^3\sqrt{\mu^{k+1}}}.$$

Since $\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}}$ is sufficiently small as verified above. Therefore, the error of the next iterate $v^k + \Delta v^k$ is bounded by

$$\begin{aligned}\|v^k + \Delta v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} &\le \frac{1}{2}\,\frac{2c_z}{(1-3\vartheta)^3\sqrt{\mu^{k+1}}}\|v^k - v_{\mu^{k+1}}\|^2_{\mu^{k+1}} \\ &\le \frac{1}{3}\|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \le \vartheta\sqrt{\mu^{k+1}}\end{aligned}$$

in view of (5.13).
(iii) The length of the Newton step Δv^k can be estimated by

$$\|\Delta v^k\|_{\mu^{k+1}} \le \|v^k + \Delta v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} + \|v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} \le 4\vartheta\sqrt{\mu^{k+1}},$$

and the error of the *inexact* iterate v^{k+1} is bounded by

$$\begin{aligned}\|v^{k+1} - v_{\mu^{k+1}}\|_{\mu^{k+1}} &\le \|v^k + \Delta v^k - v_{\mu^{k+1}}\|_{\mu^{k+1}} + \delta\|\Delta v^k\|_{\mu^{k+1}} \\ &\le (1+4\delta)\vartheta\sqrt{\mu^{k+1}}.\end{aligned}$$

With the accuracy matching $\delta \le 1/4$ we obtain $\|v^{k+1} - v_{\mu^{k+1}}\|_{\mu^{k+1}} \le 2\vartheta\sqrt{\mu^{k+1}}$, which completes the induction step.
(iv) Moreover, together with Theorem 4.3, we obtain

$$\begin{aligned}\|v_0 - v^k\|_\infty &\le \|v_0 - v_{\mu^k}\|_\infty + \frac{\lambda}{\sqrt{\nu}}\|v_{\mu^k} - v^k\|_{\mu^k} \\ &\le c_0\sqrt{\mu^k} + 2\vartheta\sqrt{\mu^k} \le (c_0 + 2\vartheta)\sigma^{k/2}\sqrt{\mu^0},\end{aligned}$$

which proves r-linear convergence of v^k to the KKT point v_0. ◻

6. Numerical tests. In Section 5, we have formulated our algorithm in a fairly abstract way. Now we reformulate it in terms of PDEs. To implement the method, we have to solve a discretized version of equation (4.1), i.e. of

$$F(v_\mu;\mu) = (D^*S^*SD + \nu D^*D)v_\mu - D^*S^*y_d - \frac{\mu}{v_\mu - y_c} = 0 \quad \text{a.e. on } \Omega\,. \tag{6.1}$$

Let us transform this equation to a standard optimality sytem in terms of PDEs:

First, we mention that $v = \lambda u + y$, $u = Dv$, and $y = SDv$ holds. Therefore, the equation above is equivalent to

$$D^*(S^*(y - y_d) + \nu u) - \frac{\mu}{y + \lambda u - y_c} = 0\,. \tag{6.2}$$

Next, we define $p_1 := S^*(y - y_d) = S(y - y_d)$. By definition of S, p_1 solves the PDE $Ap_1 = y - y_d$ subject to homogeneous boundary conditions. Inserting this in (6.2), we obtain after multiplying (6.2) by $D^{*-1} = S^* + \lambda I = S + \lambda I$,

$$p_1 + \nu u - \frac{\lambda\mu}{y + \lambda u - y_c} = S^* \frac{\mu}{y + \lambda u - y_c} =: -p_2. \tag{6.3}$$

The function p_2 solves the equation $Ap_2 = -\frac{\mu}{y+\lambda u - y_c}$ with homogeneous boundary conditions. Setting $p = p_1 + p_2$, we arrive at the *adjoint equation*

$$\begin{aligned} Ap &= y - y_d - \tfrac{\mu}{y+\lambda u - y_c} && \text{in } \Omega \\ \partial_n p + p &= 0 && \text{on } \Gamma. \end{aligned}$$

In view of this, equation (6.3) is nothing more then the *gradient equation*

$$p + \nu u - \frac{\lambda\mu}{y + \lambda u - y_c} = 0 \quad \text{in } \Omega. \tag{6.4}$$

Together with the state equation, the nonlinear optimality system

$$\begin{aligned} Ay &= u \quad \text{in } \Omega & \qquad Ap &= y - y_d - \frac{\mu}{y + \lambda u - y_c} && \text{in } \Omega \\ \partial_n y + \alpha y &= 0 \quad \text{on } \Gamma & \qquad \partial_n p + \alpha p &= 0 && \text{on } \Gamma \end{aligned}$$

$$p + \nu u - \frac{\lambda\mu}{y + \lambda u - y_c} = 0 \quad \text{a.e. in } \Omega \tag{6.5}$$

must be solved. Notice that (6.5) can be directly obtained as the first-order optimality system for the problem (2.5)–(2.7) by using the standard formal Lagrange technique. Our abstract approach that was used to simplify the analysis is consistent with this.

For the computations, (6.5) is discretized by a finite element method in the space $H^1(\Omega) \supset V_h = \text{span}\{\phi_1, \cdots, \phi_n\}$ with standard piecewise linear elements. The functions y, p, and u are chosen as

$$y = \sum_1^n y_i \phi_i, \qquad p = \sum_1^n p_i \phi_i, \qquad u = \sum_1^n u_i \phi_i.$$

First we consider the numerical approximation of our state equation (2.2–2.3). We will restrict the examples to

$$-\Delta y + y = u \quad \text{in } \Omega \tag{6.6}$$

$$\partial_n y = 0 \quad \text{on } \Gamma. \tag{6.7}$$

Let $\mathbf{K}$ and $\mathbf{M}$ be the stiffness- and mass matrices associated with V_h. Note, that $\phi_i(x_j) = \delta_{ij}$. Then the functions y_h and p_h and u_h can be identified by their coefficient vectors $\mathbf{y} = (y_1, y_2, ..., y_n)^T$, $\mathbf{u} = (u_1, u - 2, ..., u_n)^T$, and $\mathbf{y} = (y_1, y_2, ..., y_n)^T$, respectively. The finite element approximation of (6.6)–(6.7) is

$$(\mathbf{K} + \mathbf{M})\mathbf{y} = \mathbf{M}\mathbf{u}. \tag{6.8}$$

To simplify the construction of test examples we introduce a desired control u_d, (cf. (6.10) below) which does not change the validity of our theorems. Further, we drop the transformation $D^{-1} = S + \lambda I$, which we established to abbreviate the formulation of our problem. The discretization of u_d and y_d is taken as above.

Using the discretization (6.8), in view of (6.5), and multiplying the gradient equation (6.3) with $-\mathbf{M}$, we arrive at

$$\tilde{F}_h(\mathbf{y},\mathbf{u},\mathbf{p};\mu) = \begin{pmatrix} -\mathbf{M} & 0 & \mathbf{K}+\mathbf{M} \\ 0 & -\nu\mathbf{M} & -\mathbf{M} \\ \mathbf{K}+\mathbf{M} & -\mathbf{M} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbf{M}\mathbf{y}_d \\ \nu\mathbf{M}\mathbf{u}_d \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{M}\left(\dfrac{\mu}{\mathbf{y}+\lambda\mathbf{u}-\mathbf{y_c}}\right) \\ \mathbf{M}\dfrac{\lambda\mu}{\mathbf{y}+\lambda\mathbf{u}-\mathbf{y_c}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.9}$$

This is the discrete optimality system we use for our computations. In (6.9), the vector $\dfrac{\mu}{\mathbf{y}+\lambda\mathbf{u}-\mathbf{y}_c}$ is defined by

$$\left(\frac{\mu}{\mathbf{y}+\lambda\mathbf{u}-\mathbf{y}_c}\right)_i = \frac{\mu}{\mathbf{y}_i+\lambda\mathbf{u}_i-(\mathbf{y}_c)_i}.$$

Note that, due to the linearity of the state equation, the computational all-at-once approach used here is indeed an implementation of the inexact Newton method described in Section 5.

We have tested our method using the example

$$\text{(PT)} \qquad \min J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u-u_d\|^2_{L^2(\Omega)} \tag{6.10}$$

subject to

$$-\Delta y + y = u \qquad \text{in}\,\Omega \tag{6.11}$$
$$\partial_n y = 0 \qquad \text{on}\,\Gamma \tag{6.12}$$

and to the pointwise mixed control-state constraints

$$y + \lambda u \geq y_c \qquad \text{a.e. in}\,\Omega \tag{6.13}$$

with $\Omega = (0,1)\times(0,1)$.

It is easy to verify that (PT) fits into the setting of (P). For all $\lambda > 0$, the Lagrange multiplier η associated with (6.13) belongs to $L^2(\Omega)$.

After having solved (6.9) by our primal algorithm, we have calculated the Lagrange multiplier η by the relation

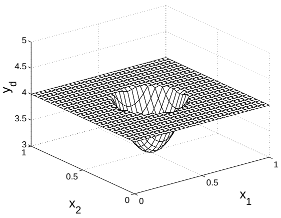
$$\eta = \frac{\mu}{y+\lambda u - y_c}.$$

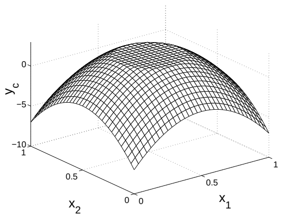
The method was implemented using Matlab and its PDE-toolbox for mesh generation and matrix-assembling. The stopping criterion for the outer iteration was $\mu \leq \epsilon = 10^{-12}$. A regular cross (Friedrichs-Keller) triangulation has been used with fixed mesh size $h = 0.025$. In the following, the numerical solutions are denoted by $(\cdot)_\mu$, the exact optimal control, optimal state, and the optimal adjoint state are $\bar{u}$, $\bar{y}$, and $\bar{p}$ respectively. For fixed mesh size, the numerical solutions converged to the projection of the exact solution onto the finite element space. All computations were performed on a Pentium IV/2.8GHz machine with 1GB RAM running under Linux.

6.1. Example 1. This example is a slight update of Example 1 in [11]. We choose $\bar{u} = 2$, $\bar{p} = -2$ and $\bar{y} = 2$. The desired state y_d, the bound y_c, and the Lagrange multiplier η are given by

$$\begin{aligned} y_d(x_1, x_2) &= 4 - \max\left\{-20\left((x_1 - 0.5)^2 - (x_2 - 0.5)^2\right) + 1 - 2\lambda, 0\right\}, \\ y_c(x_1, x_2) &= \min\left\{-20\left((x_1 - 0.5)^2 - (x_2 - 0.5)^2\right) + 3, 2 + 2\lambda\right\}, \\ \eta(x_1, x_2) &= \max\left\{-20\left((x_1 - 0.5)^2 - (x_2 - 0.5)^2\right) + 1 - 2\lambda, 0\right\}. \end{aligned}$$

Moreover, we have chosen $u_d = -\lambda\eta(x_1, x_2)$. In (6.10) we take $\nu = 1$ and $\lambda = 10^{-3}$. The Algorithm 5.1 was applied with $\sigma = 0.75$ and $\mu^0 = 1$.
The following figures show the functions y_d, y_c and the Lagrange multiplier η.





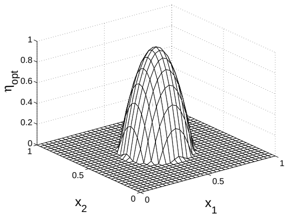


FIG. 6.1. *Desired state y_d* FIG. 6.2. *State contraint y_c* FIG. 6.3. *Multiplier η*

The figures 6.4–6.7 show the numerical solutions y_h, u_h, p_h, and η_h with $\lambda = 10^{-3}$.

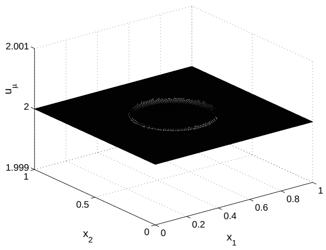


FIG. 6.4. *Control u_h*

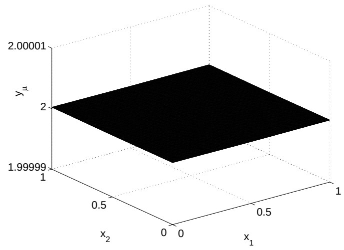


FIG. 6.5. *State y_h*

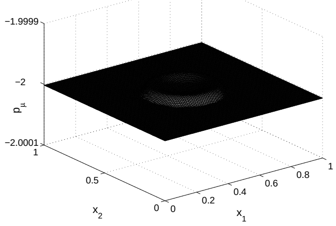


FIG. 6.6. *Adjoint state p_h*

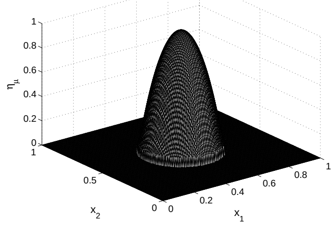


FIG. 6.7. *Lagrange multiplier η_h*

Tabular 6.1 contains the relative errors $err(u) = \frac{\|u_\mu - \bar{u}\|_{l^2}}{\|\bar{u}\|_{l^2}}$, $err(y) = \frac{\|y_\mu - \bar{y}\|_{l^2}}{\|\bar{y}\|_{l^2}}$, $err(p) = \frac{\|p_\mu - \bar{p}\|_{l^2}}{\|\bar{p}\|_{l^2}}$, and $err(\eta) = \frac{\|\eta_\mu - \bar{\eta}\|_{l^2}}{\|\bar{\eta}\|_{l^2}}$ for the problem regularized with $\lambda = 10^{-3}$

depending on μ. It shows the linear convergence in u, y, and p. This is also reflected by the figures 6.8–6.11. For a comparison with results computed by a primal-dual active set strategy we refer to [11]. For $\mu < 10^{-10}$, the discretization error dominates the values of the error function $err(\eta)$.

μ	$err(u)$	$err(y)$	$err(p)$	$err(\eta)$
1.0775e-02	8.9940e-03	8.9802e-03	8.9740e-03	5.9016e-01
1.0611e-03	1.1585e-03	1.1293e-03	1.1472e-03	4.9188e-01
1.0450e-04	3.0274e-04	1.2221e-04	2.6728e-04	4.6082e-01
1.0290e-05	1.7956e-04	1.7204e-05	1.5513e-04	3.0395e-01
1.0134e-06	5.2173e-05	3.0048e-06	4.2306e-05	1.4558e-01
1.1088e-07	1.4471e-05	5.6443e-07	9.8070e-06	6.4546e-02
1.0919e-08	3.5160e-06	9.4222e-08	1.8734e-06	2.2455e-02
1.0753e-09	7.3268e-07	1.4464e-08	3.1009e-07	1.0979e-02
1.0589e-10	1.2286e-07	1.9308e-09	4.2994e-08	1.0093e-02
1.0428e-11	1.5516e-08	2.1608e-10	4.8831e-09	1.0096e-02
1.0269e-12	1.6071e-09	2.1836e-11	4.9450e-10	1.0103e-02

TABLE 6.1
Relative errors for Example 1.

The figures 6.8–6.11 show the differences between the numerical solutions u_μ,y_μ p_μ and η_μ and the exact solutions $\bar{u}$, $\bar{y}$, $\bar{p}$ and $\bar{\eta}$, measured in the L^2-norm at a regularization parameter $\lambda = 10^{-3}$ depending on the path-parameter μ. Both axes are scaled logarithmically. The behavior of the Lagrange multiplier for $\mu \to 0$ is remarkable: It converges very slowly up to $\mu \approx 10^{-5}$, between $\mu \approx 10^{-5}$ and $\mu = 10^{-9}$ it converges linearly, and for $\mu < 10^{-9}$ we see a saturation caused by numerical errors. Compare also the figures 6.12–6.15 below.

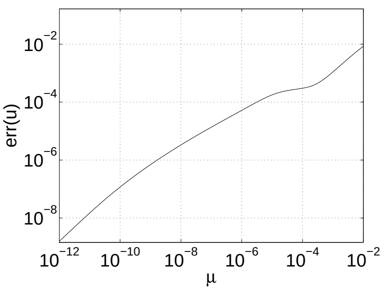


FIG. 6.8. $err(u)$

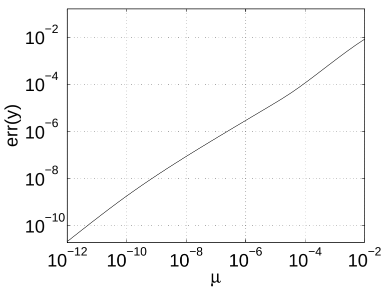


FIG. 6.9. $err(y)$

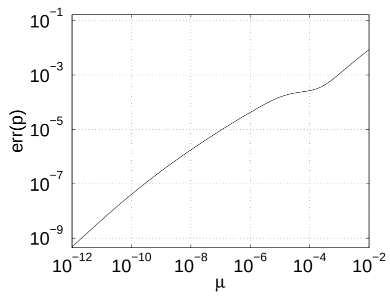


FIG. 6.10. $err(p)$

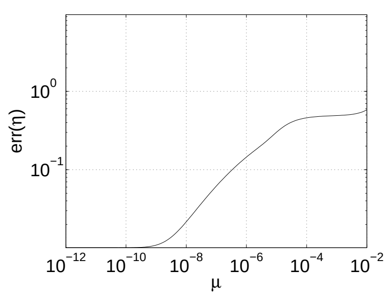


FIG. 6.11. $err(\eta)$

The next figures show the evolution of the multiplier η_μ along the central path.

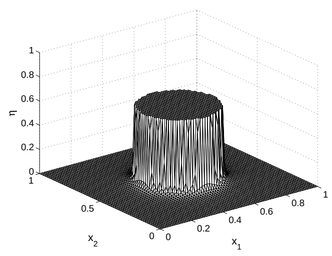


FIG. 6.12. *Multiplier* η_h *at* $\mu = 10^{-2}$

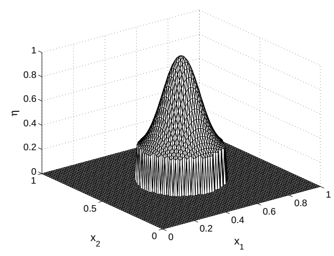


FIG. 6.13. *Multiplier* η_h *at* $\mu = 10^{-6}$

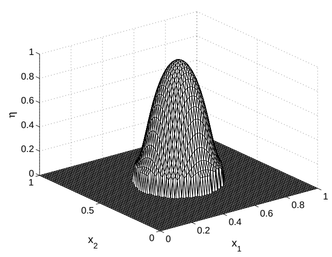


FIG. 6.14. *Multiplier* η_h *at* $\mu = 10^{-7}$

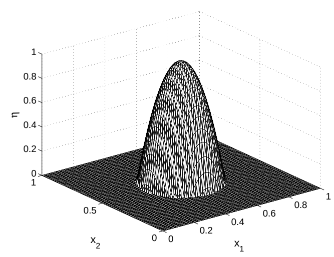


FIG. 6.15. *Multiplier* η_h *at* $\mu = 10^{-10}$

6.2. Example 2. This example is constructed such that $\bar{y}$, $\bar{u}$ and $\bar{p}$ are trigonometric functions of the form $\varphi(x_1, x_2) = c\cos(\pi x_1)\cos(2\pi x_2)$. We choose $c = 1$ for $\bar{y}$ and $c = (-5\nu\pi^2)$ for $\bar{p}$. From the state equation and the optimality condition we get $\bar{u} = -\Delta\bar{y} + \bar{y} = \left(5\pi^2 + 1\right)\bar{y}$, and $u_d = \bar{u} + \frac{1}{\nu}\bar{p} - \frac{\lambda}{\nu}\eta = \bar{y} - \frac{\lambda}{\nu}\eta$, respectively.
By $\hat{y} = 2\sin(2\pi x_1) - 1.5$, $\bar{\eta} = \max\{\hat{y} - \bar{y}, 0\}$, and $y_c = \min\{\hat{y}, \bar{y}\} - \lambda\bar{u}$, the complementary slackness condition is fullfilled. All these functions are continuous. Therefore, the adjoint equation does not contain measures as data. From the adjoint equation we get $y_d = \Delta\bar{p} - \bar{p} + \bar{y} - \bar{\eta} = \left(\left(5\nu\pi^2\right)\left(5\pi^2 + 1\right) + 1\right)\bar{y} - \bar{\eta}$. Figures 6.16–6.18 present the functions y_d y_c and η.

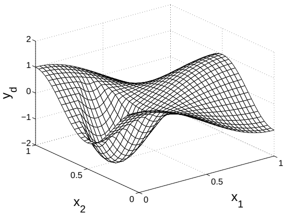


FIG. 6.16. *Desired state y_d*

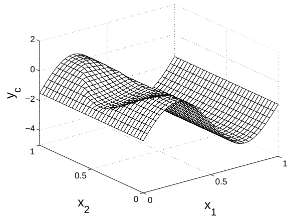


FIG. 6.17. *State constraints y_c*

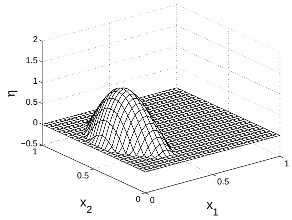


FIG. 6.18. *Multiplier $\bar{\eta}$*

The following figures show the numerical solutions for $\nu = 10^{-3}$ and $\lambda = 10^{-3}$.

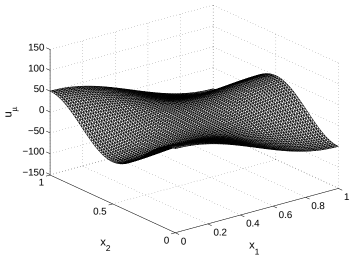


FIG. 6.19. *Control u_μ*

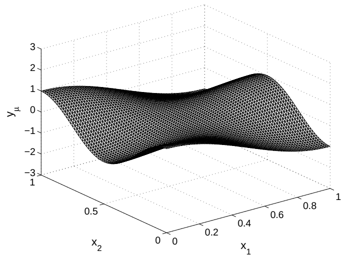


FIG. 6.20. *State y_μ*

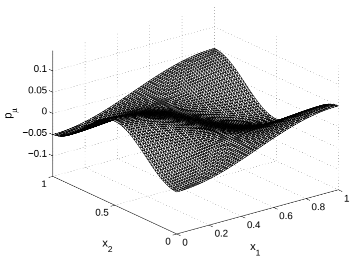


FIG. 6.21. *Adjoint state p_μ*

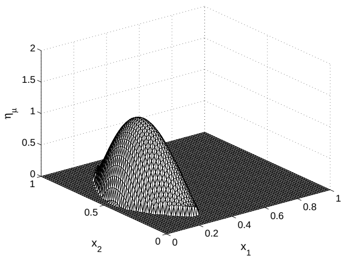


FIG. 6.22. *Multiplier η_μ*

Table 6.2 displays the relative error $err(\cdot)$ for u, y, p, and η. Here, the discretization error dominates $err(\cdot)$ for all values $\mu < 10^{-6}$.

μ	$err(u)$	$err(y)$	$err(p)$	$err(\eta)$
1.0775e-02	1.7515e-02	4.2897e-02	1.4888e-02	3.2848e-01
1.0611e-03	5.0088e-03	6.7794e-03	3.8653e-03	1.4802e-01
1.0450e-04	1.3658e-03	1.1136e-03	9.0134e-04	5.9756e-02
1.0290e-05	4.4222e-04	3.9979e-04	3.0695e-04	2.1376e-02
1.0134e-06	2.8956e-04	3.7943e-04	2.4390e-04	1.1738e-02
1.1088e-07	2.7751e-04	3.8078e-04	2.3967e-04	1.0866e-02
1.0919e-08	2.7666e-04	3.8100e-04	2.3932e-04	1.0830e-02
1.0753e-09	2.7662e-04	3.8102e-04	2.3930e-04	1.0833e-02

TABLE 6.2
Relative errors for Example 2.

Figures 6.23–6.26 present the differences between the numerical solutions and the optimal solutions at $\nu = 10^{-3}$ and $\lambda = 10^{-3}$.

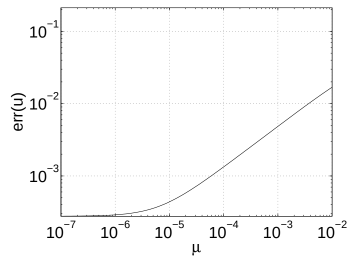


FIG. 6.23. $err(u)$

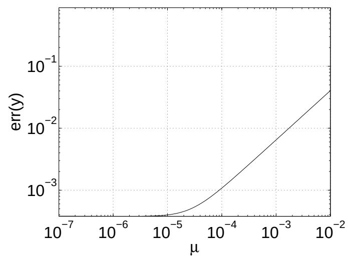


FIG. 6.24. $err(y)$

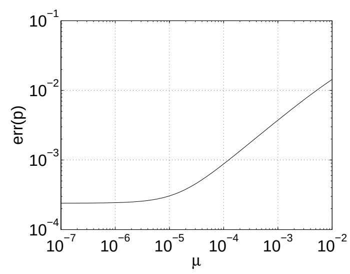


FIG. 6.25. $err(p)$

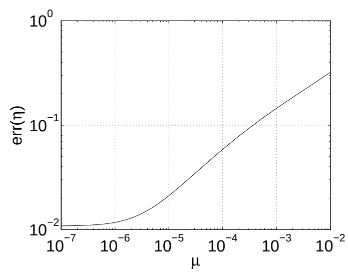


FIG. 6.26. $err(\eta)$

The figures illustrate the evolution of the Lagrange multiplier η_μ along the central path.

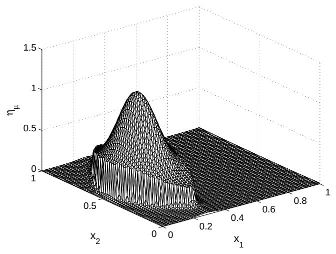


FIG. 6.27. *Multiplier η_{mu} at $\mu = 0.001$*

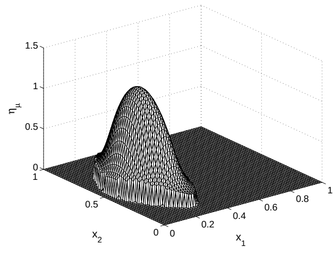


FIG. 6.28. *Multiplier η_μ at $\mu = 10^{-4}$*

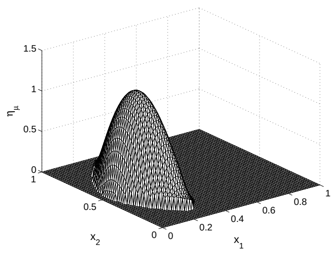


FIG. 6.29. *Multiplier η_μ at $\mu = 10^{-5}$*

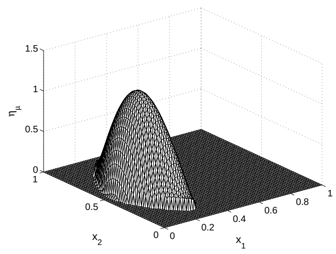


FIG. 6.30. *Multiplier η_μ at $\mu = 10^{-6}$*

7. Pass to the limit $\lambda \downarrow 0$. We have considered our method for fixed $\lambda > 0$. Nevertheless, we briefly mention for convenience how close the solution $\bar{u}$ of the problem with pure state constraint $y \geq 0$ is approximated by $\bar{u}_\lambda$, the one associated to the Lavrentiev regularized constraint $\lambda u + y \geq 0$. It is known from [10] that $\|\bar{u} - \bar{u}_\lambda\| \to 0$ as $\lambda \downarrow 0$. To get a convergence rate, we require the following two assumptions:

Uniform boundedness: *There is $M > 0$ such that*

$$\|\bar{u}_\lambda\|_\infty \leq M \quad \forall\, \lambda > 0. \tag{7.1}$$

Notice that (7.1), together with $\bar{u}_\lambda \to \bar{u}$ in L^2, implies that $\|\bar{u}\|_\infty \leq M$.

Slater condition: *There is some $u_0 \in L^\infty$ and an $\varepsilon > 0$ such that*

$$y_0(x) \geq y_c(x) + \varepsilon \quad \forall x \in \bar{\Omega} \tag{7.2}$$

holds for the associated state $y_0 = S\,u_0$.

Under these assumptions, the first of them being quite strong, but often satisfied in concrete examples, the estimate

$$\|\bar{u} - \bar{u}_\lambda\| \leq C\,\sqrt{\lambda} \tag{7.3}$$

is obtained by a fairly standard technique, cf. for instance Alt [2]. We briefly sketch the main steps: In view of the assumptions (7.1) and (7.2), there exist positive constants c_0 and λ_0, and controls $\tilde{u}(\lambda)$, $\hat{u}(\lambda)$ with associated states $\tilde{y}(\lambda)$, $\hat{y}(\lambda)$ having the following properties: For all $\lambda \in (0, \lambda_0]$, it holds

$$\lambda\,\tilde{u}(\lambda) + \tilde{y}(\lambda) \geq y_c, \qquad \|\bar{u} - \tilde{u}(\lambda)\| \;\leq c_0\,\lambda \tag{7.4}$$

$$\hat{y}(\lambda) \geq y_c, \qquad \|\bar{u}_\lambda - \hat{u}(\lambda)\| \leq c_0\,\lambda. \tag{7.5}$$

The upper estimate is obtained as follows: We define $\tilde{u}(\lambda) = (1-\rho)\bar{u} + \rho u_0$ with suitable $\rho > 0$. Then by $\bar{y} \geq y_c$ and (7.2),

$$\begin{aligned}
\lambda\,\tilde{u}(\lambda) + \tilde{y}(\lambda) &= \lambda\,(1-\rho)\bar{u} + \lambda\rho u_0 + (1-\rho)\bar{y} + \rho y_0 \\
&\geq \lambda\,(1-\rho)\bar{u} + \lambda\rho u_0 + (1-\rho)y_c + \rho(y_c + \varepsilon) \\
&= \lambda\,(1-\rho)\bar{u} + \lambda\rho u_0 + \rho\varepsilon + y_c.
\end{aligned}$$

For $0 < \rho \le 1$, we get $\|\lambda(1-\rho)\bar{u} + \lambda\rho u_0\|_\infty \le \lambda\,(M + \|u_0\|_\infty)$. Take

$$\rho = \frac{\lambda}{\varepsilon}(M + \|u_0\|_\infty) \tag{7.6}$$

and assume that λ is so small, say $\lambda \le \lambda_0$, such that $\rho \le 1$ holds. Then

$$\lambda(1-\rho)\bar{u} + \lambda\rho u_0 + \rho\varepsilon + y_c \ge -\lambda\,(M + \|u_0\|_\infty) + \rho\varepsilon + y_c \ge y_c$$

so that $\lambda\,\tilde{u}(\lambda) + \tilde{y}(\lambda) \ge y_c$. Moreover,

$$\|\bar{u} - \tilde{u}(\lambda)\| = \|\bar{u} - (1-\rho)\bar{u} - \rho u_0\| \le \rho(M + \|u_0\|_\infty) \le c_0\lambda$$

because of (7.6).

In the same way, the relations (7.5) are shown by the ansatz $\hat{u}(\lambda) = (1-\rho)\bar{u}_\lambda + \rho u_0$ with certain $\rho \in (0,1)$. We exploit $\lambda\bar{u}_\lambda + \bar{y}_\lambda \ge y_c$, hence $\bar{y}_\lambda \ge y_c - \lambda M$. The term $-\lambda M$ can be compensated by adding a small multiple of y_0.

Invoking (7.4), (7.5), the estimate (7.3) is now obtained immediately: The functional $f(u)$ is uniformly Lipschitz with constant L on the set of all u with $\|u\|_\infty \le M$. We find by Taylor expansion

$$\begin{aligned} f(\bar{u}_\lambda) - f(\bar{u}) &\ge f'(\bar{u})(\bar{u}_\lambda - \bar{u}) + \frac{\kappa}{2}\|\bar{u}_\lambda - \bar{u}\|^2 \\ &= f'(\bar{u})(\hat{u}(\lambda) - \bar{u}) + \frac{\kappa}{2}\|\bar{u}_\lambda - \bar{u}\|^2 + f'(\bar{u})(\bar{u}_\lambda - \hat{u}(\lambda)) \\ &\ge \frac{\kappa}{2}\|\bar{u}_\lambda - \bar{u}\|^2 - c_1\,L\lambda \end{aligned}$$

since $\hat{u}(\lambda)$ satisfies the constraints of (P), and hence the variational inequality is fulfilled. Moreover, (7.5) was used. On the other hand,

$$f(\bar{u}_\lambda) - f(\bar{u}) = f(\bar{u}_\lambda) - f(\tilde{u}(\lambda)) + f(\tilde{u}(\lambda)) - f(\bar{u}) \le 0 + c_2 L\lambda$$

is found. Altogether, $\kappa\|\bar{u}_\lambda - \bar{u}\|^2 \le 2\,(c_1 + c_2)L\lambda$ follows from the last two inequalities, implying the estimate (7.3).

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