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Optimal Control of Low-Frequency Electromagnetic Fields in Multiply Connected Conductors

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A class of optimal control problems for electromagnetic fields is considered. Special emphasis is laid on a non-standard H-based formulation of the equations of low-frequency electromagnetism in multiply connected conductors. By this technique, the low-frequency Maxwell equations can be solved with reduced computational complexity. While the magnetic field H in the conductor is obtained from an elliptic equation with the $\operatorname{curl} \sigma^{-1} \operatorname{curl}$ operator, an elliptic equation with the $\operatorname{div} \mu \nabla$ operator is set up for a potential ψ in the isolator. Both equations are coupled by appropriate interface conditions. In all problems, the electrical current is controlled in the conducting domain. We discuss two optimal control problems with distributed control. A standard quadratic tracking type objective functional is minimized in the first problem, while a convex nondifferentiable functional with L^1-sparsity term is considered in the second. For all problems, the associated sensitivity analysis is performed.

Keywords: Electromagnetic fields; Maxwell equations; eddy current equations; H-based approximation; low frequency approximation; optimal control; sparse optimal control

AMS Subject Classification: 49K20, 35Q60, 35J25

1. Introduction

Our paper is a contribution to the fast developing numerical analysis of optimal control of electromagnetic fields. In associated mathematical models, often a vector potential ansatz is used for the magnetic induction B, namely $B = \operatorname{curl} A$. In this case, the associated Maxwell equations have to be solved for a 3D vector formulation in the whole computational domain. This domain should be taken sufficiently large, so that the choice of standard boundary conditions will guarantee a sufficiently precise solution.

The vector potential ansatz is quite popular since it is simple and can be used also if the conducting domain has a complicated geometrical shape. However, it suffers from a large computational complexity, and this can be a severe drawback for control problems.

Another way of modeling is the H-based eddy current formulation, in which a scalar magnetic potential is introduced in the non-conducting domain, thus leading

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there to a standard scalar elliptic equation; the vector unknown H is only kept in the conductor. This approach is theoretically slightly more complicated, since some additional conditions must be imposed, if the conductor is not simply connected. On the other hand, the computational savings can be considerable: in fact, if the domain Ω that contains the whole setting has to be chosen large (for instance, this is the case if the conductor is a torus of moderate thickness with very big radius), tackling the problem by means of a scalar unknown in the exterior of the conductor reduces dramatically the global number of degrees of freedom. We refer to [2] for a more detailed numerical analysis of E- or H-formulation for eddy current equations.

The main aim of our paper is to study such an H-based formulation in the context of optimal control of electric and magnetic fields. In this way, our problems will be close to the setting in [25, 26], but the associated mathematical analysis is significantly different. The main novelty of our paper is as follows: In contrast to the papers mentioned above, we consider a time-harmonic problem in a complex setting by a model of Maxwell equations that was not yet used in the optimal control theory. Moreover, we also investigate a problem of sparse control. To our best knowledge, sparse controls have not yet been studied in the optimal control of Maxwell equations. Due to the lack of boundedness of the solutions to the Maxwell system, this leads to slightly new aspects.

Optimal control of electromagnetic fields is a quite active subject, important for various applications. We mention the control of induction heating as in [17, 18, 32], heat sources such as in [31], the optimal control of MHD processes as in [4, 11, 13, 14, 16, 19, 20, 27], optimal control problems for time-harmonic eddy current problems as in [22, 23], inverse problems for electromagnetic fields as in [3], or the control of magnetic fields in flow measurement as in [25, 26].

2. Models of electromagnetism

2.1. *Time-harmonic Maxwell and eddy current equations*

For establishing our eddy current formulation, we follow [2] and begin with the well known Maxwell system

$$\frac{\partial D}{\partial t} + J_T = \operatorname{curl} H \tag{2.1}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0 \tag{2.2}$$

$$\operatorname{div} D = \rho \tag{2.3}$$

$$\operatorname{div} B = 0, \tag{2.4}$$

where B, H, D, and E denote the magnetic induction, the magnetic field, the electric induction, and the electric field, respectively.

These fields are related through some constitutive equations. A linear dependence of the form $D = \varepsilon E$, $B = \mu H$ is usually assumed, where the symmetric and (uniformly) positive definite matrices ε and μ are called electric permittivity and magnetic permeability, respectively. We assume that the entries of ε and μ are bounded and measurable real functions on Ω.

The (total) current J_T is the sum of the generated current and an imposed current J_e. By the generalized Ohm's law, we have

$$J_T = \sigma E + J_e, \tag{2.5}$$

where σ is the electrical conductivity, that is assumed to be a symmetric and (uniformly) positive definite matrix in the conducting region and to vanish in the non-conducting region. Again, we assume that the entries of σ are bounded and measurable real functions on Ω_C.

In time-harmonic models, it is assumed that the impressed current J_e is an alternating current of the form

$$J_e(\mathbf{x}, t) = J(\mathbf{x}) \cos(\omega t + \phi),$$

where J is a real vector function that accounts for direction and strength of the current, ω is the angular frequency and ϕ is the phase angle. Expressing these quantities in a complex setting, we have

$$J_e(\mathbf{x}, t) = \mathrm{Re}\,[J(\mathbf{x}) e^{i\,\omega t + i\,\phi}] = \mathrm{Re}\,[\mathbf{J}_e(\mathbf{x}) e^{i\,\omega t}].$$

The complex vector function $\mathbf{J}_e = J\, e^{i\,\phi}$ will be our control; we assume that it is supported in the conducting region, namely, it is vanishing inside the non-conducting region.

This time-periodic impressed current J_e generates associated time-periodic solutions in the form

$$E(\mathbf{x}, t) = \mathrm{Re}\,[\mathbf{E}(\mathbf{x}) e^{i\,\omega t}], \qquad H(\mathbf{x}, t) = \mathrm{Re}\,[\mathbf{H}(\mathbf{x}) e^{i\,\omega t}].$$

Inserting these quantities in the Maxwell equations, and using $D = \varepsilon E$ and $B = \mu H$, we finally arrive in a standard way at the equations of the *time-harmonic Maxwell system*

$$\begin{aligned} \operatorname{curl} \mathbf{H} - (i\omega\varepsilon + \sigma)\mathbf{E} &= \mathbf{J}_e \\ \operatorname{curl} \mathbf{E} + i\omega\mu \mathbf{H} &= \mathbf{0}. \end{aligned}$$

We shall assume that the term $i\omega\varepsilon\mathbf{E}$ can be neglected (this is often the case for low-frequency problems). Thus we end up with the *time-harmonic eddy current system*

$$\begin{aligned} \operatorname{curl} \mathbf{H} - \sigma \mathbf{E} &= \mathbf{J}_e \\ \operatorname{curl} \mathbf{E} + i\omega\mu \mathbf{H} &= \mathbf{0} \end{aligned} \tag{2.6}$$

that holds in the whole space $\mathbb{R}^3$.

2.2. *Eddy current formulation in weak and strong form*

The function spaces used in our paper will include complex functions. For instance, on a bounded measurable set $\mathcal{D} \subset \mathbb{R}^3$, $L^p(\mathcal{D})$, $1 \le p < \infty$, is defined as the space of all complex valued functions $v : \mathcal{D} \to \mathbb{C}$ such that $|v|^p$ is integrable on $\mathcal{D}$. To distinguish this space from the one with real-valued functions, we introduce

$$L^p_{\mathbb{R}}(\mathcal{D}) = \{v : \mathcal{D} \to \mathbb{R},\ |v|^p \text{ is integrable}\}.$$

The spaces $L^\infty(\mathcal{D})$ (complex) and $L^\infty_{\mathbb{R}}(\mathcal{D})$ (real) are defined accordingly.

Concerning the geometry of our domains, we assume that:

Assumption 2.1 (Geometry) *$\Omega \subset \mathbb{R}^3$ is a bounded and simply connected Lipschitz domain with connected boundary Γ; Ω is the "holdall" computational domain containing all conductors. The subdomain $\Omega_C \subset \Omega$ that denotes the conductor is a bounded Lipschitz set. We require that Ω_C is the union of finitely many disjoint open and connected sets $(\Omega_C)_l$, $l \in \{1, \dots, k\}$, the so-called (connected) components of Ω_C. Assume further that $\operatorname{cl}\Omega_C \cap \partial\Omega = \emptyset$. The set $\Omega_I := \Omega \setminus \operatorname{cl}\Omega_C$ stands for the non-conducting domain. For simplicity, it is assumed to be connected.*

The following definition is also useful:

Definition 1 Let $g \in \mathbb{N} \cup \{0\}$ be the number of all "handles" of Ω_I (precisely, the rank of the first homology group of $\operatorname{cl}\Omega_I$, or, equivalently, the first Betti number of Ω_I). Due to our assumption on Ω, it is also the number of "handles" of Ω_C. If all the components $(\Omega_C)_l$ are simply connected, we have $g = 0$.

Loosely speaking, a "handle" is a part of the domain that contains closed curves that are not the boundary of any surface. Therefore the Stokes theorem does not apply, and the line integral of a curl-free vector field along them can be different from 0: the consequence is that, in a domain with "handles", there exist curl-free vector fields that are not gradients. In our situation, the magnetic field $\mathbf{H}$ is curl-free in the non-conducting region Ω_I, but, if Ω_I has "handles", it is not equal there to the gradient of a scalar potential.

This geometrical assumption allows fairly general forms of conductors (see Figure 1). For instance, the conducting domain can include finitely many tori which might form together more complicated geometrical figures like the Borromean rings. Also any knot (for example, a trefoil knot) is allowed as a conducting domain. We quote these geometrical examples since their shape requires additional mathematical conditions for the well-posedness of our equations. Let us also mention that a very complete presentation of the topological structure of three-dimensional domains and how the geometrical shape has an influence on the representation of vector fields can be found in [5].

Definition 2 We denote by $\boldsymbol{\rho}_j$, $j \in \{1, \dots, g\}$, a basis of the space of μ-harmonic fields

$$\mathcal{H}^\mu_I = \{\mathbf{v} : \Omega_I \to \mathbb{R}^3 \,:\, \operatorname{curl}\mathbf{v} = \mathbf{0} \text{ in } \Omega_I, \operatorname{div}(\mu\mathbf{v}) = 0 \text{ in } \Omega_I, \mu\mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_I\}, \tag{2.7}$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega_I$. Classical result of algebraic topology assure that the dimension of $\mathcal{H}^\mu_I$ is indeed equal to the first Betti number of Ω_I (for a more detailed presentation of this aspect, see [2]).

The functions $\boldsymbol{\rho}_j$ are determined as follows. First, one constructs a basis $\mathbf{T}^0_j$, $j \in \{1, \dots, g\}$, of the first de Rham cohomology group as done in [1]. Then one projects them on the subspace μ-orthogonal to the gradients, namely, one considers

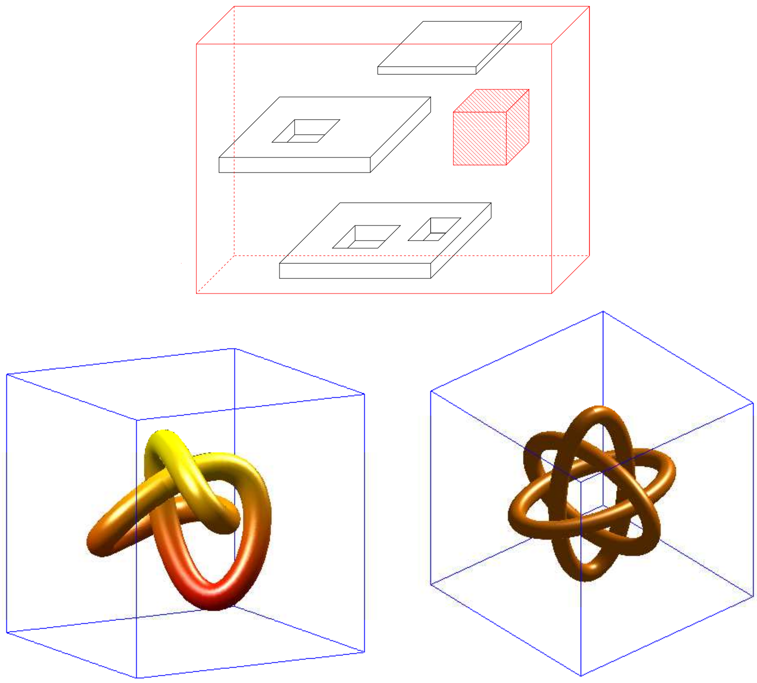

Figure 1. Geometrical configurations (courtesy of Ana Alonso Rodríguez). Top: three internal conductors of different topological shape are drawn, while the red domain is not a part of Ω (the first Betti number of Ω_I is $g = 3$). Bottom: the conductor Ω_C is a trefoil knot (left; the first Betti number of Ω_I is $g = 1$) or the union of the three Borromean rings (right; the first Betti number of Ω_I is $g = 3$).

$\boldsymbol{\rho}_j = \mathbf{T}_j^0 - \operatorname{grad} \eta_j$, where η_j is the solution to

$$\eta_j \in H^1(\Omega_I)/\mathbb{C} \ : \ \int_{\Omega_I} \mu \operatorname{grad} \eta_j \cdot \operatorname{grad} \xi = \int_{\Omega_I} \mu \mathbf{T}_j^0 \cdot \operatorname{grad} \xi \quad \forall \xi \in H^1(\Omega_I)/\mathbb{C} \, .$$

It is easily checked that $\boldsymbol{\rho}_j \in \mathcal{H}_I^\mu$; moreover, recalling that the loop fields $\mathbf{T}_j^0$ satisfy

$$\oint_{\sigma_n} \mathbf{T}_j^0 \cdot d\mathbf{s} = k_{nj} \, ,$$

where the cycles $\{\sigma_n\}$, $n \in \{1, \dots, g\}$, are a basis of the first homology group of $\operatorname{cl} \Omega_I$ and $K = (k_{nj})$ is a non-singular matrix, it is easy to see that the fields $\boldsymbol{\rho}_j$ thus defined are linearly independent (just compute the line integral of a linear combination of them on each cycle σ_n).

Remark 1 The functions $\boldsymbol{\rho}_j$ can be computed once "offline" before the numerical solution of the optimal control problem is started. They are only needed when at least one of the conducting subdomains $(\Omega_C)_l$ is not simply connected (such as a torus). Instead, when all the components of Ω_C are simply connected (e.g., balls, cubes, balls with holes) these functions $\boldsymbol{\rho}_j$ are not necessary (in fact, they are vanishing). However, we recall that Ω_I is assumed to be connected and this excludes that the components of Ω_C are tori with interior holes or balls with interior holes.

From the Ampére equation $(2.6)_1$ we see that the magnetic field satisfies $\operatorname{curl} \mathbf{H} = \mathbf{0}$ in Ω_I (remember that σ and $\mathbf{J}_e$ are vanishing in Ω_I). Therefore, if the non-conducting domain Ω_I has "handles", $\mathbf{H}_{|\Omega_I}$ is not equal to a gradient in Ω_I, but however it can be written as $\nabla \psi + \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j$ (see, e.g., [2, Appen. A.3]). We are thus led to write the weak formulation of our eddy current system in the state space

$$\mathbf{V}_0 = \{(\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V} \text{ that satisfy the interface conditions (2.8) below}\},$$

where

$$\mathbf{V} = H(\text{curl}; \Omega_C) \times H^1(\Omega_I)/\mathbb{C} \times \mathbb{C}^g$$

and

$$\mathbf{H} \times \mathbf{n} - \nabla\psi \times \mathbf{n} - \sum_{j=1}^{g} \alpha_j \boldsymbol{\rho}_j \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma. \tag{2.8}$$

Both spaces $\mathbf{V}$ and $\mathbf{V}_0$ are equipped with the norm

$$\|(\mathbf{H}, \Psi, \boldsymbol{\alpha})\|_{\mathbf{V}} = \left(\|\mathbf{H}\|^2_{H(\text{curl};\Omega_C)} + \|\psi\|^2_{H^1(\Omega_I)/\mathbb{C}} + |\boldsymbol{\alpha}|^2\right)^{1/2},$$

where

$$\|\mathbf{H}\|_{H(\text{curl};\Omega_C)} = \left(\int_{\Omega_C} (\text{curl}\,\mathbf{H} \cdot \text{curl}\,\overline{\mathbf{H}} + \mathbf{H} \cdot \overline{\mathbf{H}})\right)^{1/2}$$

and

$$\|\psi\|_{H^1(\Omega_I)/\mathbb{C}} = \left(\int_{\Omega_I} \nabla\psi \cdot \nabla\overline{\psi}\right)^{1/2}.$$

In $H^1(\Omega_I)/\mathbb{C}$, this H^1-seminorm is equivalent to the standard norm of $H^1(\Omega_I)$ (see, e.g., [10, Chap. IV, Sect. 7.2]). The space $\mathbf{V}_0$ defined above is a (complex) Hilbert space, because it is closed in $\mathbf{V}$. Notice that the trace mappings $\mathbf{H} \mapsto \mathbf{H} \times \mathbf{n}$ and $\psi \mapsto \nabla\psi \times \mathbf{n}$ are continuous from $H(\text{curl}; \Omega_C)$ to $H^{-1/2}(\text{div}_\tau; \Gamma)$ and from $H^1(\Omega_I)$ to $H^{-1/2}(\text{div}_\tau; \Gamma)$, respectively, where, for a smooth surface Γ, the trace space $H^{-1/2}(\text{div}_\tau; \Gamma)$ is defined as

$$H^{-1/2}(\text{div}_\tau; \Gamma) := \{\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)^3 \;:\; \boldsymbol{\lambda} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \text{div}_\tau\,\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\}$$

(see, e.g., [2, Appen. A.1], where also a more general characterization is discussed, when Γ is a Lipschitz closed surface).

We also define the norms

$$\|\mathbf{Q}\|_{\Omega_C} := \left(\int_{\Omega_C} |\mathbf{Q}(\mathbf{x})|^2\right)^{\frac{1}{2}}, \qquad \|\mathbf{Q}\|_{\mu,\Omega_C} := \left(\int_{\Omega_C} \mu(\mathbf{x})\mathbf{Q}(\mathbf{x}) \cdot \overline{\mathbf{Q}(\mathbf{x})}\right)^{\frac{1}{2}},$$

and, analogously, the norms $\|\mathbf{Q}\|_{\sigma,\Omega_C}$ and $\|\mathbf{Q}\|_{\mu,\Omega_I}$.

Let us introduce now the symmetric and positive definite matrix M by setting

$$M_{nj} = \int_{\Omega_I} \mu\boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_j\,;$$

we will also use the vector norm $|\mathbf{q}|_M = (M\mathbf{q}\cdot\overline{\mathbf{q}})^{\frac{1}{2}}$, where $\mathbf{q}\in\mathbb{C}^g$.

Finally, we define a sesquilinear form $a : \mathbf{V}\times\mathbf{V}\to\mathbb{C}$ by

$$a[\mathbf{u},\mathbf{v}] = \int_{\Omega_C} \sigma^{-1}\operatorname{curl}\mathbf{H}\cdot\operatorname{curl}\overline{\mathbf{W}} + \int_{\Omega_C} i\omega\mu\,\mathbf{H}\cdot\overline{\mathbf{W}} + \int_{\Omega_I} i\omega\mu\,\nabla\psi\cdot\nabla\overline{\eta} + i\omega M\boldsymbol{\alpha}\cdot\overline{\boldsymbol{\beta}},$$

where $\mathbf{u} = (\mathbf{H},\psi,\boldsymbol{\alpha})$ and $\mathbf{v} = (\mathbf{W},\eta,\boldsymbol{\beta})$. The form $a[\cdot,\cdot]$ is obviously continuous on $\mathbf{V}\times\mathbf{V}$ and it is also coercive (see, e.g., [2, p. 37]). The sesquilinear form a is chosen to set up the weak formulation of the system (2.11) below. Notice that the electrical field $\mathbf{E}$ was eliminated by the equation (2.6).

Definition 3 A triplet $\mathbf{u} = (\mathbf{H},\psi,\boldsymbol{\alpha})\in\mathbf{V}_0$ is said to be a weak solution of the eddy current model associated with $\mathbf{J}_e\in L^2(\Omega_C)^3$ if

$$a[\mathbf{u},\mathbf{v}] = \int_{\Omega_C}\sigma^{-1}\mathbf{J}_e\cdot\operatorname{curl}\overline{\mathbf{W}} \qquad \forall\,\mathbf{v} := (\mathbf{W},\eta,\boldsymbol{\beta})\in\mathbf{V}_0. \tag{2.9}$$

Note that the interpretation of this weak problem is given by the system (2.11) reported here below.

Lemma 2.2 (Well posedness) *For all $\mathbf{J}_e\in L^2(\Omega_C)^3$, the equation (2.11) has a unique weak solution $(\mathbf{H},\psi,\boldsymbol{\alpha})$. Moreover, there is a constant $c>0$ not depending on $\mathbf{J}_e$ such that*

$$\|(\mathbf{H},\psi,\boldsymbol{\alpha})\|_{\mathbf{V}} \le c\,\|\mathbf{J}_e\|_{\Omega_C}. \tag{2.10}$$

Proof. The mapping $\Theta : H(\operatorname{curl};\Omega_C)\to\mathbb{C}$ defined by

$$\Theta(\mathbf{W},\eta,\boldsymbol{\beta}) := \int_{\Omega_C}\sigma^{-1}\overline{\mathbf{J}_e}\cdot\operatorname{curl}\mathbf{W}$$

(i.e., the conjugate complex value of the right hand side of (2.9)) is continuous and linear on $\mathbf{V}$, hence it belongs in particular to $(\mathbf{V}_0)'$. Moreover, the sesquilinear form a is coercive on $\mathbf{V}_0$, hence the Lemma of Lax and Milgram ensures the existence of a unique solution of the variational equation (2.9) and of a constant $c_0>0$ such that

$$\|(\mathbf{H},\psi,\boldsymbol{\alpha})\|_{\mathbf{V}} \le c_0\,\|\Theta\|_{(\mathbf{V}_0)'} \le c\,\|\mathbf{J}_e\|_{\Omega_C}$$

holds. □

Let us denote by $\mathbf{n}_\Omega$ the unit outward normal vector on $\partial\Omega$. It is not difficult to show that the variational equation (2.9) is the weak formulation of the following strong form of the eddy current problem:

Theorem 2.3 (Strong eddy current problem) *If the solution $(\mathbf{H},\psi,\boldsymbol{\alpha})\in\mathbf{V}_0$ to the variational problem (2.9) is sufficiently smooth, then it satisfies the strong eddy* *current equations*

$$\begin{aligned}
\operatorname{curl}(\sigma^{-1}\operatorname{curl}\mathbf{H}) + i\omega\mu\,\mathbf{H} &= \operatorname{curl}(\sigma^{-1}\mathbf{J}_e) && \textit{in } \Omega_C \\
\mathbf{H}\times\mathbf{n} &= \nabla\psi\times\mathbf{n} + \textstyle\sum_{j=1}^{g}\alpha_j\boldsymbol{\rho}_j\times\mathbf{n} && \textit{on } \Gamma \\
\mu\mathbf{H}\cdot\mathbf{n} &= \mu\nabla\psi\cdot\mathbf{n} && \textit{on } \Gamma \\
-\operatorname{div}(\mu\,\nabla\psi) &= 0 && \textit{in } \Omega_I \\
\mu\nabla\psi\cdot\mathbf{n}_\Omega &= 0 && \textit{on } \partial\Omega
\end{aligned} \tag{2.11}$$

and the geometrical conditions

$$(M\boldsymbol{\alpha})_j = -(i\omega)^{-1}\int_\Gamma \sigma^{-1}(\operatorname{curl}\mathbf{H} - \mathbf{J}_e)\cdot(\boldsymbol{\rho}_j\times\mathbf{n}) \quad \forall\, j\in\{1,\dots,g\}. \tag{2.12}$$

We do not give the complete proof here, as it is similar to that presented in [2, p. 42–43]; moreover, in Section 3.3 we will give all the details of the procedure that shows how to derive the strong form of the variational formulation of the adjoint problem.

3. The optimal control problem

3.1. *The optimal current problem and its well-posedness*

Let us discuss now the following steady state optimal control problem of elliptic type, where the impressed current $\mathbf{J}_e$ is the control function.

As fixed data, vector functions $\mathbf{H}_d \in L^2(\Omega)^3$, $\mathbf{E}_d \in L^2(\Omega_C)^3$ and constants $\nu_C \geq 0$, $\nu_A \geq 0$, $\nu_B \geq 0$, $\nu_E \geq 0$, $\nu \geq 0$ with $\nu_C + \nu_A + \nu_B + \nu_E + \nu > 0$ are given. In Ω_I the reference magnetic field $\mathbf{H}_d$ is split as $\nabla\psi_d + \sum_{j=1}^{g}\alpha_{d,j}\boldsymbol{\rho}_j$. Moreover, a set of admissible controls $\mathbf{J}_{ad} \subset L^2(\Omega_C)^3$ is given and is assumed to be nonempty, bounded, convex and closed. Several types of controls and admissible sets that are useful in realistic applications have been presented in [30].

Thanks to Lemma 2.2, for each control $\mathbf{J}_e \in \mathbf{J}_{ad}$ there exists a unique weak solution of (2.11). To indicate the correspondence of this solution to the given control $\mathbf{J}_e$, we denote this solution by $(\mathbf{H}_{\mathbf{J}_e}, \psi_{\mathbf{J}_e}, \boldsymbol{\alpha}_{\mathbf{J}_e})$. In what follows, we will skip the subscript e from the controls and denote them just by $\mathbf{J}$.

As optimization criterion, we use the following (reduced) objective functional F,

$$\begin{aligned}
F(\mathbf{J}) := \frac{\nu_C}{2}\,\|\mathbf{H}_\mathbf{J} - \mathbf{H}_d\|^2_{\mu,\Omega_C} + \frac{\nu_A}{2}\,\|\nabla\psi_\mathbf{J} - \nabla\psi_d\|^2_{\mu,\Omega_I} + \frac{\nu_B}{2}\,|\boldsymbol{\alpha}_\mathbf{J} - \boldsymbol{\alpha}_d|^2_M \\
+ \frac{\nu_E}{2}\,\|\mathbf{E}_\mathbf{J} - \mathbf{E}_d\|^2_{\sigma,\Omega_C} + \frac{\nu}{2}\,\|\mathbf{J}\|^2_{\Omega_C},
\end{aligned} \tag{3.1}$$

where $\mathbf{E}_\mathbf{J}$ denotes the electric field associated with $\mathbf{J}$. These weighted norms are more natural than the standard L^2-norms, as in the terms of F the magnetic energy and the electric energy (per unit time) of $\mathbf{H}$ and $\mathbf{E}$, respectively, appear; moreover, this choice will later lead to some simplifications in the adjoint equation. For the L^2-norm, the theory is similar and can be covered by setting μ and σ to one in all the terms that are associated with the objective functional.

The electric field $\mathbf{E}$ is equal to $\mathbf{E} = \sigma^{-1}(\operatorname{curl}\mathbf{H} - \mathbf{J})$, hence

$$\begin{aligned} F(\mathbf{J}) = \frac{\nu_C}{2}\,\|\mathbf{H}_\mathbf{J} - \mathbf{H}_d\|^2_{\mu,\Omega_C} + \frac{\nu_A}{2}\,\|\nabla\psi_\mathbf{J} - \nabla\psi_d\|^2_{\mu,\Omega_I} + \frac{\nu_B}{2}\,|\boldsymbol{\alpha}_\mathbf{J} - \boldsymbol{\alpha}_d|^2_M \\ + \frac{\nu_E}{2}\,\|\sigma^{-1}(\operatorname{curl}\mathbf{H}_\mathbf{J} - \mathbf{J}) - \mathbf{E}_d\|^2_{\sigma,\Omega_C} + \frac{\nu}{2}\,\|\mathbf{J}\|^2_{\Omega_C}. \end{aligned} \tag{3.2}$$

This objective functional F aims at minimizing the weighted distance to desired (or measured) magnetic and electric fields, while the norm of the control function $\mathbf{J}$ is included as a Tikhonov regularization term weighted by ν.

The optimal control problem, written in short form, is

$$\min_{\mathbf{J}\in\mathbf{J}_{ad}} F(\mathbf{J}). \tag{3.3}$$

As usual, a control $\mathbf{J}^* \in \mathbf{J}_{ad}$ is said to be optimal if $F(\mathbf{J}^*) \le F(\mathbf{J})$ for all $\mathbf{J} \in \mathbf{J}_{ad}$, namely, $F(\mathbf{J}^*) = \min_{\mathbf{J}\in\mathbf{J}_{ad}} F(\mathbf{J})$.

Theorem 3.1 *The optimal control problem* (3.3) *admits at least one optimal control denoted by* $\mathbf{J}^*$. *The optimal control is unique, if* $\nu_E + \nu > 0$.

Proof. Thanks to Lemma 2.2, the mappings $\mathbf{J} \mapsto \mathbf{H}_\mathbf{J}$, $\mathbf{J} \mapsto \psi_\mathbf{J}$ and $\mathbf{J} \mapsto \boldsymbol{\alpha}_\mathbf{J}$ are well defined, linear and continuous from $L^2(\Omega_C)^3$ to $H(\operatorname{curl};\Omega_C)$, $H^1(\Omega_I)/\mathbb{C}$ and $\mathbb{C}^g$, respectively. Therefore, the reduced objective functional F is continuous and convex, hence also weakly lower semicontinuous. Moreover, the set $\mathbf{J}_{ad}$ of admissible controls is weakly sequentially compact in $L^2(\Omega_C)^3$ so that the existence of an optimal control $\mathbf{J}^* \in \mathbf{J}_{ad}$ with

$$F(\mathbf{J}^*) = \min_{\mathbf{J}\in\mathbf{J}_{ad}} F(\mathbf{J})$$

is an immediate consequence. Notice that F is bounded from below by zero so that the existence of a non-negative infimum is guaranteed. If $\nu_E + \nu > 0$, then the functional F is strictly convex and that implies the uniqueness of the optimal control. □

3.2. *The adjoint equation and the necessary optimality conditions*

The next step of our analysis is the derivation of first-order necessary optimality conditions for an optimal control $\mathbf{J}^*$. By convexity of F and $\mathbf{J}_{ad}$, they are also sufficient for optimality.

Prior to this, let us mention the following simple calculation concerning the directional derivative of the complex but real valued function $g : z \mapsto |z|^2$. For any fixed $z \in \mathbb{C}$ and varying $h \in \mathbb{C}$, we have

$$|z+h|^2 = |z|^2 + z\,\overline{h} + \overline{z}\,h + |h|^2 = |z|^2 + 2\,\mathrm{Re}\,[z\,\overline{h}] + |h|^2\,.$$

Therefore, the complex function g has the directional derivative

$$g'(z,h) := \lim_{t\to 0} \frac{|z + t\,h|^2 - |z|^2}{t} = 2\,\mathrm{Re}\,[z\,\overline{h}] = 2\,\mathrm{Re}\,[\overline{z}\,h]$$

(here, $t \in \mathbb{R}$). Notice that the mapping $h \mapsto 2\,\mathrm{Re}\,[\overline{z}\,h]$ is not complex linear. However, it is real linear, because $\mathrm{Re}\,[\overline{z}\,\alpha h] = \alpha\,\mathrm{Re}\,[\overline{z}\,h]$ for all real α. The function g is not holomorphic, i.e., not differentiable in the sense of complex analysis.

As a simple consequence, we are in the situation that the control-to-state mapping is linear and bounded, but the quadratic tracking type functional F is not differentiable. However, it is directionally differentiable. The directional derivative $F'(\widehat{\mathbf{J}}, \mathbf{J})$ of the objective functional F at $\widehat{\mathbf{J}} \in \mathbb{C}^3$ in the direction $\mathbf{J} \in \mathbb{C}^3$ can be determined in the same way as for g above. Let us explain this for the quadratic functional $F_{\mathbf{H}}$ defined by $F_{\mathbf{H}}(\mathbf{J}) := \|\mathbf{H}_{\mathbf{J}} - \mathbf{H}_d\|^2_{\mu,\Omega_C}$. Denote by $\mathbf{G} : L^2(\Omega_C)^3 \to L^2(\Omega_C)^3$ the linear and continuous mapping $\mathbf{J} \to \mathbf{H}_{\mathbf{J}}$. We have for $t \in \mathbb{R}$

$$
\begin{aligned}
F_{\mathbf{H}}(\widehat{\mathbf{J}} + t\mathbf{J}) - F_{\mathbf{H}}(\widehat{\mathbf{J}}) &= t \int_{\Omega_C} \mu\, \mathbf{G}(\mathbf{J}) \cdot \overline{(\mathbf{G}(\widehat{\mathbf{J}}) - \mathbf{H}_d)} + t \int_{\Omega_C} \mu\, (\mathbf{G}(\widehat{\mathbf{J}}) - \mathbf{H}_d) \cdot \overline{\mathbf{G}(\mathbf{J})} \\
&\quad + t^2 \int_{\Omega_C} \mu\, \mathbf{G}(\mathbf{J}) \cdot \overline{\mathbf{G}(\mathbf{J})} \\
&= 2\,t\ \mathrm{Re}\left[\int_{\Omega_C} \mu\, (\mathbf{G}(\widehat{\mathbf{J}}) - \mathbf{H}_d) \cdot \overline{\mathbf{G}(\mathbf{J})}\right] + t^2 \|\mathbf{G}(\mathbf{J})\|^2_{\mu,\Omega_C} \\
&= 2\,t\ \mathrm{Re}\left[\int_{\Omega_C} \mu\, (\mathbf{H}_{\widehat{\mathbf{J}}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}}}\right] + t^2 \|\mathbf{H}_{\mathbf{J}}\|^2_{\mu,\Omega_C}.
\end{aligned}
$$

Now, it follows immediately

$$
F'_{\mathbf{H}}(\widehat{\mathbf{J}}, \mathbf{J}) = \lim_{t \downarrow 0} \frac{1}{t}(F_{\mathbf{H}}(\widehat{\mathbf{J}} + t\,\mathbf{J}) - F_{\mathbf{H}}(\widehat{\mathbf{J}})) = 2 \int_{\Omega_C} \mathrm{Re}\ \left[\mu\, (\mathbf{H}_{\widehat{\mathbf{J}}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}}}\,\right].
$$

The other terms of F are treated analogously. Therefore, the directional derivative of F in the direction $\mathbf{J}$ at an arbitrary fixed (not necessarily optimal or admissible) control $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$ with associated solution $\widehat{\mathbf{H}} := \mathbf{H}_{\widehat{\mathbf{J}}}$, $\widehat{\psi} := \psi_{\widehat{\mathbf{J}}}$ and $\widehat{\boldsymbol{\alpha}} := \boldsymbol{\alpha}_{\widehat{\mathbf{J}}}$ is given by

$$
\begin{aligned}
F'(\widehat{\mathbf{J}}, \mathbf{J}) = \nu_C &\int_{\Omega_C} \mathrm{Re}\,[\mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}}}] \\
&+ \nu_A \int_{\Omega_I} \mathrm{Re}\,[\mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi_{\mathbf{J}}}] + \nu_B\, \mathrm{Re}\,[M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_{\mathbf{J}}}] \\
&+ \nu_E \int_{\Omega_C} \mathrm{Re}\,[\big(\sigma^{-1}(\mathrm{curl}\,\widehat{\mathbf{H}} - \widehat{\mathbf{J}}) - \mathbf{E}_d\big) \cdot (\mathrm{curl}\,\overline{\mathbf{H}_{\mathbf{J}}} - \overline{\mathbf{J}})] \\
&+ \nu \int_{\Omega_C} \mathrm{Re}\,[\widehat{\mathbf{J}} \cdot \overline{\mathbf{J}}].
\end{aligned}
$$

Notice that $\sigma^{-1}(\operatorname{curl}\widehat{\mathbf{H}} - \widehat{\mathbf{J}}) = \widehat{\mathbf{E}} := \mathbf{E}_{\widehat{\mathbf{J}}}$. Re-arranging the terms , we see

$$
\begin{aligned}
F'(\widehat{\mathbf{J}}, \mathbf{J}) = \operatorname{Re} \Big\{ \int_{\Omega_C} & \nu_C\, \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}}} \\
& + \int_{\Omega_I} \nu_A\, \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi_{\mathbf{J}}} + \nu_B\, M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_{\mathbf{J}}} \\
& + \int_{\Omega_C} \nu_E\big(\widehat{\mathbf{E}} - \mathbf{E}_d\big) \cdot \operatorname{curl}\overline{\mathbf{H}_{\mathbf{J}}} \\
& - \int_{\Omega_C} \nu_E\big(\widehat{\mathbf{E}} - \mathbf{E}_d\big) \cdot \overline{\mathbf{J}} + \nu \int_{\Omega_C} \widehat{\mathbf{J}} \cdot \overline{\mathbf{J}} \Big\} .
\end{aligned} \tag{3.4}
$$

Note that taking the real part of a complex number is an additive (but only real linear) operation. In (3.4), the variable direction $\mathbf{J}$ is appearing explicitly in the last two integrals, while it occurs implicitly in the first four terms through the mappings $\mathbf{J} \mapsto \mathbf{H}_{\mathbf{J}}$, $\mathbf{J} \mapsto \psi_{\mathbf{J}}$ and $\mathbf{J} \mapsto \boldsymbol{\alpha}_{\mathbf{J}}$. By introducing an adjoint state, this implicit dependence on $\mathbf{J}$ can be transformed in a standard way to an explicit dependence.

Definition 4 (Adjoint equation) Let $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$ be a given control with states $\widehat{\mathbf{H}} := \mathbf{H}_{\widehat{\mathbf{J}}}$, $\widehat{\mathbf{E}} := \mathbf{E}_{\widehat{\mathbf{J}}}$, $\widehat{\psi} := \psi_{\widehat{\mathbf{J}}}$, $\widehat{\boldsymbol{\alpha}} := \boldsymbol{\alpha}_{\widehat{\mathbf{J}}}$, and let $\mathbf{H}_d \in L^2(\Omega_C)^3$, $\psi_d \in H^1(\Omega_I)/\mathbb{C}$, $\boldsymbol{\alpha}_d \in \mathbb{C}^g$, $\mathbf{E}_d \in L^2(\Omega_C)^3$ be given as above. The equation for $(\mathbf{W}, \eta, \boldsymbol{\beta})$,

$$
\begin{aligned}
\int_{\Omega_C} & \sigma^{-1} \operatorname{curl}\mathbf{W} \cdot \operatorname{curl}\overline{\mathbf{H}} - i\omega \int_{\Omega_C} \mu\mathbf{W} \cdot \overline{\mathbf{H}} - i\omega \int_{\Omega_I} \mu\nabla\eta \cdot \nabla\overline{\psi} - i\omega M\boldsymbol{\beta} \cdot \overline{\boldsymbol{\alpha}} \\
&= \int_{\Omega_C} \nu_C\, \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}} \\
&\quad + \int_{\Omega_I} \nu_A\, \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi} + \nu_B\, M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}} \\
&\quad + \int_{\Omega_C} \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \operatorname{curl}\overline{\mathbf{H}} \qquad \forall\, (\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V}_0
\end{aligned} \tag{3.5}
$$

is called the *adjoint equation* of equation (2.9). The solution $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}}) \in \mathbf{V}_0$ is called the *adjoint state associated with* $\widehat{\mathbf{J}}$.

Corollary 3.2 *For all given* $\mathbf{H}_d \in L^2(\Omega_C)^3$, $\psi_d \in H^1(\Omega_I)/\mathbb{C}$, $\boldsymbol{\alpha}_d \in \mathbb{C}^g$, $\mathbf{E}_d \in L^2(\Omega_C)^3$, $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$, *the adjoint equation* (3.5) *has a unique solution* $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}})$.

This result follows as Lemma 2.2 by the Lemma of Lax and Milgram. Notice that the mapping

$$
\begin{aligned}
(\mathbf{H}, \psi, \boldsymbol{\alpha}) \mapsto \int_{\Omega_C} & \nu_C\, \mu\overline{(\widehat{\mathbf{H}} - \mathbf{H}_d)} \cdot \mathbf{H} \\
& + \int_{\Omega_I} \nu_A\, \mu\overline{(\nabla\widehat{\psi} - \nabla\psi_d)} \cdot \nabla\psi + \nu_B\, M\overline{(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)} \cdot \boldsymbol{\alpha} \\
& + \int_{\Omega_C} \nu_E\overline{(\widehat{\mathbf{E}} - \mathbf{E}_d)} \cdot \operatorname{curl}\mathbf{H}
\end{aligned}
$$

(i.e., the conjugate complex value of the right hand side of (3.5)) is linear and continuous from $\mathbf{V}_0$ to $\mathbb{C}$, hence it belongs to $(\mathbf{V}_0)'$.

To see that the adjoint state transforms the implicit appearance of the control $\mathbf{J}$ in (3.4) to an explicit one, we prove the following auxiliary result:

LEMMA 3.3 *We have that*

$$\begin{aligned}\mathrm{Re}\Big[\nu_C \int_{\Omega_C} \mu(\widehat{\mathbf{H}} - \mathbf{H}_d)\cdot\overline{\mathbf{H}_\mathbf{J}} + \nu_A \int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d)\cdot\nabla\overline{\psi_\mathbf{J}} \\ + \nu_B\, M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}_\mathbf{J}} + \nu_E \int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d)\cdot\operatorname{curl}\overline{\mathbf{H}_\mathbf{J}}\Big] \\ = \mathrm{Re}\int_{\Omega_C} \sigma^{-1}\operatorname{curl}\mathbf{W}_{\widehat{\mathbf{J}}}\cdot\overline{\mathbf{J}},\end{aligned} \tag{3.6}$$

where the function $\mathbf{W}_{\widehat{\mathbf{J}}}$ is the first component of the adjoint state $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}})$ associated with $\widehat{\mathbf{J}}$.

Proof. We write down the variational equation defining the weak solution $\mathbf{H}_\mathbf{J}$, $\psi_\mathbf{J}$ and $\boldsymbol{\alpha}_\mathbf{J}$, and insert the solution $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}})$ of the adjoint equation as test function; we obtain

$$\begin{aligned}\int_{\Omega_C} \sigma^{-1}\operatorname{curl}\mathbf{H}_\mathbf{J}\cdot\operatorname{curl}\overline{\mathbf{W}_{\widehat{\mathbf{J}}}} + i\omega\int_{\Omega_C} \mu\mathbf{H}_\mathbf{J}\cdot\overline{\mathbf{W}_{\widehat{\mathbf{J}}}} \\ + i\omega\int_{\Omega_I} \mu\nabla\psi_\mathbf{J}\cdot\nabla\overline{\eta_{\widehat{\mathbf{J}}}} + i\omega M\boldsymbol{\alpha}_\mathbf{J}\cdot\overline{\boldsymbol{\beta}_{\widehat{\mathbf{J}}}} \\ = \int_{\Omega_C} \sigma^{-1}\mathbf{J}\cdot\operatorname{curl}\overline{\mathbf{W}_{\widehat{\mathbf{J}}}}.\end{aligned} \tag{3.7}$$

On the other hand, inserting $(\mathbf{H}_\mathbf{J}, \psi_\mathbf{J}, \boldsymbol{\alpha}_\mathbf{J})$ as test function in the adjoint equation (3.5), we find

$$\begin{aligned}\int_{\Omega_C} \sigma^{-1}\operatorname{curl}\overline{\mathbf{H}_\mathbf{J}}\cdot\operatorname{curl}\mathbf{W}_{\widehat{\mathbf{J}}} - i\omega\int_{\Omega_C} \mu\overline{\mathbf{H}_\mathbf{J}}\cdot\mathbf{W}_{\widehat{\mathbf{J}}} \\ - i\omega\int_{\Omega_I} \mu\nabla\overline{\psi_\mathbf{J}}\cdot\nabla\eta_{\widehat{\mathbf{J}}} - i\omega M\overline{\boldsymbol{\alpha}_\mathbf{J}}\cdot\boldsymbol{\beta}_{\widehat{\mathbf{J}}} \\ = \nu_C\int_{\Omega_C} \mu(\widehat{\mathbf{H}} - \mathbf{H}_d)\cdot\overline{\mathbf{H}_\mathbf{J}} \\ + \nu_A\int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d)\cdot\nabla\overline{\psi_\mathbf{J}}] + \nu_B\, M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}_\mathbf{J}} \\ + \nu_E\int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d)\cdot\operatorname{curl}\overline{\mathbf{H}_\mathbf{J}}\,.\end{aligned} \tag{3.8}$$

We see that the left hand side of (3.7) is the complex conjugate of the left-hand side of (3.8). Therefore, the conjugate complex value of the right-hand side of (3.7) is equal to the right-hand side of (3.8), i.e.,

$$
\begin{aligned}
\int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{W}_{\widehat{\mathbf{J}}} \cdot \overline{\mathbf{J}} = \nu_C \int_{\Omega_C} & \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}}} \\
& + \nu_A \int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi_{\mathbf{J}}}] + \nu_B \, M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_{\mathbf{J}}} \\
& + \nu_E \int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}_{\mathbf{J}}}.
\end{aligned}
$$

The claim of the theorem follows by taking the real part of each side above. □

Theorem 3.4 (Necessary optimality conditions) *Let $\mathbf{J}^*$ be an optimal control of problem 3.3 and let $\mathbf{H}_{\mathbf{J}^*}$ and $\mathbf{E}_{\mathbf{J}^*}$ be the associated optimal magnetic and electric fields, respectively. Then there exists a unique solution $(\mathbf{W}_{\mathbf{J}^*}, \eta_{\mathbf{J}^*}, \boldsymbol{\beta}_{\mathbf{J}^*})$ of the adjoint equation* (3.5) *such that the variational inequality*

$$
\operatorname{Re} \int_{\Omega_C} \Big(\sigma^{-1} \operatorname{curl} \mathbf{W}_{\mathbf{J}^*} - \nu_E \, (\mathbf{E}_{\mathbf{J}^*} - \mathbf{E}_d) + \nu \, \mathbf{J}^*\Big) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \geq 0 \quad \forall \, \mathbf{J} \in \mathbf{J}_{ad} \tag{3.9}
$$

is satisfied.

Proof. The optimal control $\mathbf{J}^*$ must obey the standard variational inequality

$$
F'(\mathbf{J}^*, \mathbf{J} - \mathbf{J}^*) \geq 0 \quad \forall \, \mathbf{J} \in \mathbf{J}_{ad}. \tag{3.10}
$$

We show that this is equivalent to the variational inequality (3.9). We first consider the expression (3.4) for $F'(\widehat{\mathbf{J}}, \mathbf{J})$ with the particular choice $\widehat{\mathbf{J}} := \mathbf{J}^*$ and have

$$
\begin{aligned}
& F'(\mathbf{J}^*, \mathbf{J} - \mathbf{J}^*) \\
& \quad = \operatorname{Re} \Bigg[\nu_C \int_{\Omega_C} \mu(\mathbf{H}_{\mathbf{J}^*} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{\mathbf{J}-\mathbf{J}^*}} \\
& \qquad + \nu_A \int_{\Omega_I} \mu(\nabla\psi_{\mathbf{J}^*} - \nabla\psi_d) \cdot \nabla\overline{\psi_{\mathbf{J}-\mathbf{J}^*}} + \nu_B \, M(\boldsymbol{\alpha}_{\mathbf{J}^*} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_{\mathbf{J}-\mathbf{J}^*}} \\
& \qquad + \nu_E \int_{\Omega_C} (\mathbf{E}_{\mathbf{J}^*} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}_{\mathbf{J}-\mathbf{J}^*}} - \nu_E \int_{\Omega_C} (\mathbf{E}_{\mathbf{J}^*} - \mathbf{E}_d) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \\
& \qquad + \nu \int_{\Omega_C} \mathbf{J}^* \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \Bigg].
\end{aligned}
$$

Thanks to Lemma 3.3, we obtain

$$
\begin{aligned}
& F'(\mathbf{J}^*, \mathbf{J} - \mathbf{J}^*) \\
& \quad = \operatorname{Re} \Bigg[\int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{W}_{\mathbf{J}^*} \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \\
& \qquad - \int_{\Omega_C} \nu_E \, (\mathbf{E}_{\mathbf{J}^*} - \mathbf{E}_d) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) + \int_{\Omega_C} \nu \, \mathbf{J}^* \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \Bigg] \\
& \quad = \operatorname{Re} \int_{\Omega_C} \Big(\sigma^{-1} \operatorname{curl} \mathbf{W}_{\mathbf{J}^*} - \nu_E \, (\mathbf{E}_{\mathbf{J}^*} - \mathbf{E}_d) + \nu \, \mathbf{J}^*\Big) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}),
\end{aligned} \tag{3.11}
$$

where $\mathbf{W}_{\mathbf{J}^*}$ is the first component of the adjoint state associated with $\mathbf{J}^*$. □

Definition 5 For convenience, for each given control $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$ we define

$$\mathbf{D}_{\widehat{\mathbf{J}}} := \sigma^{-1} \operatorname{curl} \mathbf{W}_{\widehat{\mathbf{J}}} - \nu_E \left(\mathbf{E}_{\widehat{\mathbf{J}}} - \mathbf{E}_d\right). \tag{3.12}$$

By this definition, the variational inequality (3.9) simplifies to

$$\operatorname{Re} \int_{\Omega_C} \left(\mathbf{D}_{\mathbf{J}^*} + \nu\, \mathbf{J}^*\right) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \geq 0 \quad \forall\, \mathbf{J} \in \mathbf{J}_{ad}. \tag{3.13}$$

The next result is easy to obtain, but it is important for the choice of the descent direction in gradient-type methods for the numerical solution of our optimal control problem.

Corollary 3.5 *At an arbitrarily given control $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$, the maximum*

$$\max_{\|\mathbf{J}\|_{\Omega_C}=1} F'(\widehat{\mathbf{J}}, \mathbf{J}),$$

i.e., the direction of steepest ascent, is attained by

$$\mathbf{J}^\sharp = \frac{\mathbf{D}_{\widehat{\mathbf{J}}} + \nu\, \widehat{\mathbf{J}}}{\|\mathbf{D}_{\widehat{\mathbf{J}}} + \nu\, \widehat{\mathbf{J}}\|_{\Omega_C}}. \tag{3.14}$$

Proof. The integral in (3.11) can be written in terms of the inner product $(\cdot\,,\,\cdot)_{\Omega_C}$ of the space $L^2(\Omega_C)^3$ by

$$F'(\widehat{\mathbf{J}}, \mathbf{J}) = \operatorname{Re} \left(\mathbf{D}_{\widehat{\mathbf{J}}} + \nu\, \widehat{\mathbf{J}}\,,\, \mathbf{J}\right)_{\Omega_C}.$$

Invoking the Cauchy-Schwarz inequality, we estimate

$$F'(\widehat{\mathbf{J}}, \mathbf{J}) \leq \left| \operatorname{Re} \left(\mathbf{D}_{\widehat{\mathbf{J}}} + \nu\, \widehat{\mathbf{J}}\,,\, \mathbf{J}\right)_{\Omega_C} \right| \leq \left\|\mathbf{D}_{\widehat{\mathbf{J}}} + \nu\, \widehat{\mathbf{J}}\right\|_{\Omega_C}$$

if $\|\mathbf{J}\|_{\Omega_C} = 1$. This maximal value at the end of this inequality is attained by the function $\mathbf{J}^\sharp$ defined in (3.14). □

3.3. *The strong form of the adjoint equation*

We present here the strong formulation of the adjoint equation. As before, we write $\widehat{\mathbf{E}} = \sigma^{-1}(\operatorname{curl} \widehat{\mathbf{H}} - \widehat{\mathbf{J}})$.

Theorem 3.6 (Strong adjoint equation) *Let $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$ be a given control with associated states $\widehat{\mathbf{H}} := \mathbf{H}_{\widehat{\mathbf{J}}}$, $\widehat{\mathbf{E}} := \mathbf{E}_{\widehat{\mathbf{J}}}$, $\widehat{\psi} := \psi_{\widehat{\mathbf{J}}}$, $\widehat{\boldsymbol{\alpha}} := \boldsymbol{\alpha}_{\widehat{\mathbf{J}}}$, and let $\mathbf{H}_d \in L^2(\Omega_C)^3$, $\psi_d \in H^1(\Omega_I)/\mathbb{C}$, $\boldsymbol{\alpha}_d \in \mathbb{C}^g$, $\mathbf{E}_d \in L^2(\Omega_C)^3$ be given data. If the adjoint* *state* $(\mathbf{W}, \eta, \boldsymbol{\beta})$ *is sufficiently smooth, then it satisfies the system*

$$
\begin{aligned}
\operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{W}) - i\omega\mu\, \mathbf{W} &= \nu_C \mu\, (\widehat{\mathbf{H}} - \mathbf{H}_d) \\
&\quad + \nu_E \operatorname{curl}\left(\widehat{\mathbf{E}} - \mathbf{E}_d\right) && \text{in } \Omega_C \\
\mathbf{W} \times \mathbf{n} &= \nabla\eta \times \mathbf{n} + \textstyle\sum_{j=1}^{g} \beta_j \boldsymbol{\rho}_j \times \mathbf{n} && \text{on } \Gamma \\
\mu\, \mathbf{W} \cdot \mathbf{n} - \mu \nabla\eta \cdot \mathbf{n} &= -(i\omega)^{-1} \nu_C \mu\, (\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \mathbf{n} \\
&\quad + (i\omega)^{-1} \nu_A \mu\, (\nabla\widehat{\psi} - \nabla\psi_d) \cdot \mathbf{n} && \text{on } \Gamma \\
-\operatorname{div}(\mu \nabla\eta) &= (i\omega)^{-1} \nu_A \operatorname{div}\left(\mu\, (\nabla\widehat{\psi} - \nabla\psi_d)\right) && \text{in } \Omega_I \\
\mu \nabla\eta \cdot \mathbf{n}_\Omega &= -(i\omega)^{-1} \nu_A \mu\, (\nabla\widehat{\psi} - \nabla\psi_d) \cdot \mathbf{n}_\Omega && \text{on } \partial\Omega\,,
\end{aligned}
\tag{3.15}
$$

with the geometrical conditions

$$
\begin{aligned}
(M\boldsymbol{\beta})_j = (i\omega)^{-1} \textstyle\int_\Gamma \left(\sigma^{-1} \operatorname{curl} \mathbf{W} - \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d)\right) \cdot (\boldsymbol{\rho}_j \times \mathbf{n}) \\
-(i\omega)^{-1} \nu_B [M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)]_j \quad \forall\, j \in \{1, \ldots, g\}.
\end{aligned}
\tag{3.16}
$$

Proof. (i) First, we assume that in (3.5) $\psi = 0$ and $\boldsymbol{\alpha} = \mathbf{0}$ hold. By the interface conditions (2.8), which are to be understood as equations in $H^{-1/2}(\operatorname{div}_\tau; \Gamma)$, we have then $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on Γ and also $\overline{\mathbf{H}} \times \mathbf{n} = \mathbf{0}$ on Γ, since $\mathbf{n}$ is real. Integrating by parts in the weak formulation (3.5), we obtain

$$
\begin{aligned}
&\int_{\Omega_C} \operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{W}) \cdot \overline{\mathbf{H}} - i\omega \int_{\Omega_C} \mu \mathbf{W} \cdot \overline{\mathbf{H}} \\
&\qquad = \int_{\Omega_C} \nu_C \mu\, (\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}} + \int_{\Omega_C} \nu_E \operatorname{curl}(\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \overline{\mathbf{H}}
\end{aligned}
\tag{3.17}
$$

for all $\mathbf{H} \in H(\operatorname{curl}; \Omega_C)$ with $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on Γ. This implies that the first equation of (3.15) holds in the sense of distributions, because $\mathbf{H}$ can be chosen arbitrarily out of $C_0^\infty(\Omega_C)$.

(ii) Next, we allow ψ to vary while still $\boldsymbol{\alpha} = \mathbf{0}$ is required. Then, by the condition (2.8) that holds in $\mathbf{V}_0$, in particular we have

$$
\mathbf{H} \times \mathbf{n} = \nabla\psi \times \mathbf{n}. \tag{3.18}
$$

Note also that (3.5) holds not only for each $\psi \in H^1(\Omega_I)/\mathbb{C}$ but also for each $\psi \in H^1(\Omega_I)$. Performing an integration by parts in both Ω_C and Ω_I in (3.5) and using (3.18) and the first equation in (3.15), we find (remember that $\mathbf{n}$ is the unit outward normal vector to Ω_I)

$$
\begin{aligned}
&\int_\Gamma \sigma^{-1} \operatorname{curl} \mathbf{W} \cdot (\nabla\overline{\psi} \times \mathbf{n}) - \int_\Gamma \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot (\nabla\overline{\psi} \times \mathbf{n}) \\
&\quad = \int_\Gamma i\omega\, \overline{\psi}\, \mu \nabla\eta \cdot \mathbf{n} + \int_{\partial\Omega} i\omega\, \overline{\psi}\, \mu \nabla\eta \cdot \mathbf{n}_\Omega - \int_{\Omega_I} i\omega\, \overline{\psi} \operatorname{div}(\mu\nabla\eta) \\
&\quad\quad + \int_\Gamma \nu_A\, \overline{\psi} \mu (\nabla\widehat{\psi} - \nabla\psi_d) \cdot \mathbf{n} + \int_{\partial\Omega} \nu_A\, \overline{\psi} \mu (\nabla\widehat{\psi} - \nabla\psi_d) \cdot \mathbf{n}_\Omega \\
&\quad\quad - \int_{\Omega_I} \nu_A \overline{\psi} \operatorname{div}\left(\mu (\nabla\widehat{\psi} - \nabla\psi_d)\right).
\end{aligned}
\tag{3.19}
$$

Here, and in the following, the integrals on the interface Γ at the left hand side are defined in the duality sense on $H^{-1/2}(\mathrm{div}_\tau;\Gamma)$; those on Γ or $\partial\Omega$ at the right hand side are defined in the duality sense on $H^{-1/2}(\Gamma)$ or $H^{-1/2}(\partial\Omega)$, respectively. Selecting arbitrary $\psi \in C_0^\infty(\Omega_I)$, we deduce

$$-\operatorname{div}(\mu\nabla\eta) = (i\omega)^{-1}\nu_A \operatorname{div}\big(\mu(\nabla\widehat{\psi} - \nabla\psi_d)\big)$$

in the sense of distributions in Ω_I. This yields the fourth equation of (3.15), so that in particular the integral on Ω_I in (3.19) vanishes. Next, we vary ψ freely on $\partial\Omega$ subject to $\psi = 0$ in a neighborhood of Γ (and therefore, by (3.18), $\mathbf{H}\times\mathbf{n} = \mathbf{0}$ on Γ). Then we also find

$$\mu\nabla\psi\cdot\mathbf{n}_\Omega = -(i\omega)^{-1}\nu_A\mu(\nabla\widehat{\psi} - \nabla\psi_d)\cdot\mathbf{n}_\Omega$$

on $\partial\Omega$, i.e., the last equation of (3.15).

(iii) In the next step we verify the two interface conditions on Γ in (3.15). The first one is included in the definition of the space $\mathbf{V}_0$ that underlies the definition of a weak solution. For the second one, still assuming that $\boldsymbol{\alpha} = \mathbf{0}$, let us start from equation (3.19). In view of what we have proved in (ii) and taking into account (3.18), equation (3.19) can be re-written as

$$\begin{aligned}\int_\Gamma \sigma^{-1}\operatorname{curl}\mathbf{W}\cdot(\nabla\overline{\psi}\times\mathbf{n}) - \int_\Gamma \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\cdot(\nabla\overline{\psi}\times\mathbf{n})& \\ = \int_\Gamma i\omega\,\overline{\psi}\,\mu\nabla\eta\cdot\mathbf{n} + \int_\Gamma \nu_A\,\overline{\psi}\mu(\nabla\widehat{\psi} - \nabla\psi_d)\cdot\mathbf{n}&\end{aligned} \tag{3.20}$$

for each $\psi \in H^1(\Omega_I)$. We transform the first two terms of this equation as follows:

$$\begin{aligned}&\int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\big)\cdot(\nabla\overline{\psi}\times\mathbf{n}) \\ &\quad = -\int_\Gamma \nabla\overline{\psi}\cdot\big[\big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\big)\times\mathbf{n}\big] \\ &\quad = \int_\Gamma \mathrm{div}_\tau\big[\big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\big)\times\mathbf{n}\big]\,\overline{\psi} \\ &\quad = \int_\Gamma \operatorname{curl}\big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\cdot\mathbf{n}\,\overline{\psi} \\ &\quad = \int_\Gamma \big(i\omega\,\mu\mathbf{W} + \nu_C\mu(\widehat{\mathbf{H}} - \mathbf{H}_d)\big)\cdot\mathbf{n}\,\overline{\psi},\end{aligned} \tag{3.21}$$

where in the third line we have introduced the surface divergence div_τ, that, in particular, for each $\mathbf{a} \in H(\mathrm{curl};\Omega_C)$ satisfies

$$-\int_\Gamma \nabla\overline{\psi}\cdot(\mathbf{a}\times\mathbf{n}) = \int_\Gamma \mathrm{div}_\tau(\mathbf{a}\times\mathbf{n})\,\overline{\psi}$$

(see [24, p. 49]). Moreover, in the fourth line above we have used the identity

$$\mathrm{div}_\tau(\mathbf{a}\times\mathbf{n}) = \operatorname{curl}\mathbf{a}\cdot\mathbf{n}$$

(see, e.g., [2, Appen. A.1])), while in the fifth line we have invoked the first equation of (3.15) already obtained before. Now, we insert (3.21) in (3.20) and arrive at

$$\int_\Gamma \big(i\omega\,\mu\mathbf{W} - i\omega\,\mu\nabla\eta\big)\cdot\mathbf{n}\,\overline{\psi} = \int_\Gamma \big(-\nu_C\mu(\widehat{\mathbf{H}}-\mathbf{H}_d) + \nu_A\,\mu(\nabla\widehat{\psi}-\nabla\psi_d)\big)\cdot\mathbf{n}\,\overline{\psi}$$

for each $\psi \in H^1(\Omega_I)$, hence for each $\psi_{|\Gamma} \in H^{1/2}(\Gamma)$. Dividing by $i\omega$ yields the second interface condition of (3.15) that holds in $H^{-1/2}(\Gamma)$.

(iv) To verify the geometrical condition, we finally let also $\boldsymbol{\alpha}$ vary in $\mathbb{C}^g$. Then, in view of (2.8), $\mathbf{H}$, ψ, $\boldsymbol{\alpha}$ obey

$$\mathbf{H}\times\mathbf{n} = \nabla\psi\times\mathbf{n} + \sum_{j=1}^{g}\alpha_j\boldsymbol{\rho}_j\times\mathbf{n} \quad \text{on } \Gamma. \tag{3.22}$$

Moreover, we have to take into account the terms depending on $\boldsymbol{\alpha}$ in the variational equation. We return to (3.20), add $-i\omega M\boldsymbol{\beta}\cdot\overline{\boldsymbol{\alpha}}-\nu_B\,M(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}}$, and insert (3.22) instead of (3.18) for $\mathbf{H}\times\mathbf{n}$. This yields

$$\begin{aligned}
0 &= \int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}}-\mathbf{E}_d)\big)\cdot\big((\nabla\overline{\psi}+\sum_{j=1}^{g}\overline{\alpha_j}\boldsymbol{\rho}_j)\times\mathbf{n}\big) - i\omega M\boldsymbol{\beta}\cdot\overline{\boldsymbol{\alpha}}\\
&\quad - \int_\Gamma i\omega\,\overline{\psi}\,\mu\nabla\eta\cdot\mathbf{n} - \int_\Gamma \nu_A\,\overline{\psi}\mu(\nabla\widehat{\psi}-\nabla\psi_d)\cdot\mathbf{n} - \nu_B\,M(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}}\\
&= \int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}}-\mathbf{E}_d)\big)\cdot(\nabla\overline{\psi}\times\mathbf{n})\\
&\quad - \int_\Gamma i\omega\,\overline{\psi}\,\mu\nabla\eta\cdot\mathbf{n} - \int_\Gamma \nu_A\,\overline{\psi}\mu(\nabla\widehat{\psi}-\nabla\psi_d)\cdot\mathbf{n}\\
&\quad + \int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}}-\mathbf{E}_d)\big)\cdot(\sum_{j=1}^{g}\overline{\alpha_j}\boldsymbol{\rho}_j\times\mathbf{n})\\
&\quad - i\omega M\boldsymbol{\beta}\cdot\overline{\boldsymbol{\alpha}} - \nu_B\,M(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}}\\
&= \int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}}-\mathbf{E}_d)\big)\cdot(\sum_{j=1}^{g}\overline{\alpha_j}\boldsymbol{\rho}_j\times\mathbf{n})\\
&\quad - i\omega M\boldsymbol{\beta}\cdot\overline{\boldsymbol{\alpha}} - \nu_B\,M(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_d)\cdot\overline{\boldsymbol{\alpha}}\,,
\end{aligned} \tag{3.23}$$

in view of (3.21) and the second interface condition in (3.15). Since this must hold for arbitrary $\boldsymbol{\alpha}\in\mathbb{C}^g$, the last equation amounts to

$$i\omega(M\boldsymbol{\beta})_j = \int_\Gamma \big(\sigma^{-1}\operatorname{curl}\mathbf{W} - \nu_E(\widehat{\mathbf{E}}-\mathbf{E}_d)\big)\cdot(\boldsymbol{\rho}_j\times\mathbf{n}) - \nu_B[M(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_d)]_j,$$

for all $j\in\{1,\dots,g\}$, i.e., (3.16) is verified. □

At first view, the adjoint system (3.15) exhibits a different structure than the state equation. In particular, the vector field $\mu\nabla\eta$ is not divergence free. However, we can cover both equations by the following unified form:

$$
\begin{aligned}
\operatorname{curl}(\sigma^{-1}\operatorname{curl}\mathbf{Q}) \pm i\omega\mu\,\mathbf{Q} &= \mathbf{f}_C + \operatorname{curl}\mathbf{F}_C && \text{in } \Omega_C\\
\mathbf{Q}\times\mathbf{n} - \nabla\chi\times\mathbf{n} - \textstyle\sum_{j=1}^{g}\zeta_j\boldsymbol{\rho}_j\times\mathbf{n} &= \mathbf{0} && \text{on } \Gamma\\
\mu\mathbf{Q}\cdot\mathbf{n} - \mu\nabla\chi\cdot\mathbf{n} &= \pm(i\omega)^{-1}\mathbf{f}_C\cdot\mathbf{n} - \mathbf{G}_I\cdot\mathbf{n} && \text{on } \Gamma\\
-\operatorname{div}(\mu\,\nabla\chi) &= -\operatorname{div}\mathbf{G}_I && \text{in } \Omega_I\\
\mu\nabla\chi\cdot\mathbf{n}_\Omega &= \mathbf{G}_I\cdot\mathbf{n}_\Omega && \text{on } \partial\Omega\\
\textstyle\int_\Gamma \sigma^{-1}\operatorname{curl}\mathbf{Q}\cdot(\boldsymbol{\rho}_j\times\mathbf{n}) \pm i\omega(M\boldsymbol{\zeta})_j &= r_j + \textstyle\int_\Gamma \mathbf{F}_C\cdot(\boldsymbol{\rho}_j\times\mathbf{n})\\
&\quad \forall\, j\in\{1,\ldots,g\}.
\end{aligned}
\tag{3.24}
$$

having plus sign for the state problem and minus sign for the adjoint problem.

Precisely, for the state equation we have

$$
\mathbf{f}_C = \mathbf{0}\ ,\ \mathbf{F}_C = \sigma^{-1}\mathbf{J}_e\ ,\ \mathbf{G}_I = \mathbf{0}\ ,\ r_j = 0\,,
$$

whereas for the adjoint equation we have

$$
\begin{aligned}
&\mathbf{f}_C = \nu_C\mu(\widehat{\mathbf{H}} - \mathbf{H}_d)\ ,\ \mathbf{F}_C = \nu_E(\widehat{\mathbf{E}} - \mathbf{E}_d)\ ,\\
&\mathbf{G}_I = -(i\omega)^{-1}\nu_A\mu(\nabla\widehat{\psi} - \nabla\psi_d)\ ,\ r_j = \nu_B[M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)]_j\,.
\end{aligned}
$$

In particular, this says that, for the solution of the adjoint equation, the quantity $\mu\mathbf{W}\cdot\mathbf{n} - \mu\nabla\eta\cdot\mathbf{n}$ has a jump on Γ; hence the overall field

$$
\mathbf{K}_\Omega = \begin{cases} \mu\mathbf{W} & \text{in } \Omega_C\\ \mu\nabla\eta + \sum_{j=1}^{g}\beta_j\mu\boldsymbol{\rho}_j & \text{in } \Omega_I \end{cases}
$$

does not have a square-summable divergence, even if the desired fields $\mu\mathbf{H}_d$ and $\mu\nabla\psi_d$ were divergence free in Ω_C and Ω_I, respectively (this property is true for the state variables, that satisfy $\operatorname{div}(\mu\widehat{\mathbf{H}}) = 0$ in Ω_C and $\operatorname{div}(\mu\nabla\widehat{\psi}) = 0$ in Ω_I). In contrast to this, the solution of the state equation is the magnetic field

$$
\mathbf{H}_\Omega = \begin{cases} \mathbf{H} & \text{in } \Omega_C\\ \nabla\psi + \sum_{j=1}^{g}\alpha_j\boldsymbol{\rho}_j & \text{in } \Omega_I\,, \end{cases}
$$

whose associated magnetic induction $\mathbf{B}_\Omega = \mu\mathbf{H}_\Omega$ is divergence free, as the magnetic Gauss law requires.

EXAMPLE 3.7 *Let us consider the particular choice $\nu_A = \nu_C$ and assume in addition that the desired fields $\mathbf{H}_d$ and ψ_d are compatible on the interface, i.e.,*

$$
\mu\,\mathbf{H}_d\cdot\mathbf{n} = \mu\nabla\psi_d\cdot\mathbf{n} \qquad \text{on } \Gamma. \tag{3.25}
$$

Since also $\mu\widehat{\mathbf{H}}\cdot\mathbf{n} = \mu\nabla\widehat{\psi}\cdot\mathbf{n}$ holds on Γ, the second interface condition of the adjoint system then simplifies to

$$
\mu\mathbf{W}\cdot\mathbf{n} = \mu\nabla\eta\cdot\mathbf{n} \qquad \text{on } \Gamma.
$$

Therefore, the jump between $\mu\mathbf{W}\cdot\mathbf{n}$ and $\mu\nabla\eta\cdot\mathbf{n}$ disappears and the field $\mathbf{K}_\Omega$

defined here above is divergence free in Ω*, provided that the desired field* $\mathbf{H}_d$ *and* $\nabla\psi_d$ *satisfy* $\operatorname{div}(\mu\mathbf{H}_d) = 0$ *in* Ω_C *and* $\operatorname{div}(\mu\nabla\psi_d) = 0$ *in* Ω_I.

Remark 2 For the sake of completeness, let us also point out that the weak form of (3.24) reads

$$\begin{aligned}
&\int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{Q} \cdot \operatorname{curl} \overline{\mathbf{p}} \pm i\omega \int_{\Omega_C} \mu \mathbf{Q} \cdot \overline{\mathbf{p}} \pm i\omega \int_{\Omega_I} \mu \nabla\chi \cdot \nabla\overline{\vartheta} \pm i\omega M \boldsymbol{\zeta} \cdot \overline{\boldsymbol{\pi}} \\
&\quad = \int_{\Omega_C} \mathbf{f}_C \cdot \overline{\mathbf{p}} + \int_{\Omega_C} \mathbf{F}_C \cdot \operatorname{curl} \overline{\mathbf{p}} \pm i\omega \int_{\Omega_I} \mathbf{G}_I \cdot \nabla\overline{\vartheta} + \mathbf{r} \cdot \overline{\boldsymbol{\pi}} \,,
\end{aligned} \tag{3.26}$$

for each $(\mathbf{p}, \vartheta, \boldsymbol{\pi}) \in \mathbf{V}_0$, with plus sign for the state problem and minus sign for the adjoint problem. The sesquilinear form at the left hand side is continuous and coercive in $V \times V$, for both choices of the sign. The right hand side is a sesquilinear functional on V, provided that $\mathbf{f}_C \in L^2(\Omega_C)^3$, $\mathbf{F}_C \in L^2(\Omega_C)^3$, $\mathbf{G}_I \in L^2(\Omega_I)^3$ and $\mathbf{r} \in \mathbb{C}^g$.

3.4. *An example: real current sources*

A realistic class of controls $\mathbf{J}$ has the particular form

$$\mathbf{J}(\mathbf{x}) = e^{i\phi} J(\mathbf{x}) \,, \tag{3.27}$$

where J is a real vector function and ϕ is fixed. Here, J varies in the admissible set

$$\begin{aligned} J_{ad} = \{ J \in L^2_{\mathbb{R}}(\Omega_C)^3 : \ & -j_{\max} \le J_\ell(\mathbf{x}) \le j_{\max} \\ & \text{for a.a. } \mathbf{x} \in \Omega_C, \ \text{all } \ell \in \{1,2,3\} \} \end{aligned} \tag{3.28}$$

with a given bound $j_{\max} > 0$. We consider this particular set J_{ad} as preparation for the discussion of sparse controls in Section 4.

To cover this ansatz by the optimal control problem (3.3), we define the functional

$$f(J) := F(e^{i\phi} J)$$

and consider the problem

$$\min_{J \in J_{ad}} f(J). \tag{3.29}$$

This is nothing more than a particular case of the optimal control problem (3.3) subject to the particular control set defined by (3.27) and (3.28).

The associated optimal control $\mathbf{J}^* = e^{i\phi} J^*$ has to obey the necessary optimality conditions of Theorem 3.4, in particular the variational inequality (3.9), or, using the notation (3.12),

$$\operatorname{Re} \int_{\Omega_C} (\mathbf{D}_{\mathbf{J}^*} + \nu \mathbf{J}^*) \cdot (\overline{\mathbf{J}} - \overline{\mathbf{J}^*}) \ge 0 \quad \forall \mathbf{J} \in \mathbf{J}_{ad} \,.$$

In view of the particular ansatz (3.27), this variational inequality can be simplified:

inserting the particular form of $\mathbf{J}$, we find

$$\mathrm{Re}\int_{\Omega_C}(\mathbf{D}_{\mathbf{J}^*}+\nu e^{i\phi}J^*)\cdot e^{-i\phi}(J-J^*)\geq 0 \quad \forall J\in J_{ad}\,,$$

or

$$\int_{\Omega_C}(D_{J^*}+\nu J^*)\cdot(J-J^*)\,dx\geq 0 \quad \forall J\in J_{ad}\,, \tag{3.30}$$

with

$$D_{J^*}:=\mathrm{Re}\,(e^{-i\phi}\mathbf{D}_{\mathbf{J}^*}). \tag{3.31}$$

These inequalities with control functions appearing under the integral can be discussed further in a pointwise way (for this type of argument, see, e.g., [29, Sect. 2.8]). For instance, we have

$$J_\ell^*(\mathbf{x})=\begin{cases}-\mathrm{j}_{\max}, & \text{if } (D_{J^*}+\nu\, J^*)_\ell(\mathbf{x})>0\\ \ \ \mathrm{j}_{\max}, & \text{if } (D_{J^*}+\nu\, J^*)_\ell(\mathbf{x})<0\end{cases}$$

for almost all $\mathbf{x}\in\Omega_C$ and all $\ell\in\{1,2,3\}$. A detailed discussion for different classes of $\mathbf{J}_{ad}$ and aspects of modeling electrical current sources is contained in [30].

4. Sparse optimal controls

4.1. *Introduction to sparse controls*

In the problem of controlling the current in a package of independent wires, the whole cross section of the induction coil is densely filled with wires. However, it might happen as the result of numerical calculations that only some part of the wires is really important while the optimal current in some others is negligible. In such cases, one might be interested to find those wires that are most important for achieving the desired goal of optimization. The result would be a better geometry of the coil. This is an issue, where the method of sparse controls might be useful.

Sparsity techniques originated from the field of image processing, where L^1 distance functionals are used for some purpose. In the context of optimal control of partial differential equations, a first reference is [28], where this technique is applied to problems with linear elliptic equations. This opened an active research in this field. We refer to the contributions [6–9, 15] to the application of sparsity methods for different types of elliptic or parabolic PDEs.

To our best knowledge, the method of sparse control was not yet applied in the control of electromagnetic fields. Though the underlying analysis does not essentially differ from that in the papers mentioned above, we think it is worth presenting it for our particular setting. Our analysis follows the steps outlined in [9]. However, since our electromagnetic fields might be unbounded, we obtain a slightly weaker result. We discuss the theory of sparse controls for the model of real-valued controls introduced in Section 3.4. A generalization to complex-valued controls $\mathbf{J}$ is possible as well. However, the model and the notation would be more technical.

To present the theory of sparse controls, we continue the discussion of problem (3.29) (real controls). It has has a quadratic, hence smooth objective functional F. For sparse controls, we add to this functional a multiple of the L^1-norm of J and consider the functional

$$J \mapsto f(J) + \kappa \sum_{\ell=1}^{3} \int_{\Omega_C} |J_\ell(\mathbf{x})|, \tag{4.1}$$

where κ is the so-called sparsity parameter. For convenience, we define

$$\begin{aligned} \gamma(j) &:= \textstyle\int_{\Omega_C} |j(\mathbf{x})| \\ g(J) &:= \textstyle\sum_{\ell=1}^{3} \gamma(J_\ell) = \sum_{\ell=1}^{3} \int_{\Omega_C} |J_\ell(\mathbf{x})|. \end{aligned}$$

This motivates the following optimal control problem with sparsity parameter κ:

$$\min_{J \in J_{ad}} \{f(J) + \kappa\, g(J)\}. \tag{4.2}$$

Again, the existence of an optimal control $J^* \in J_{ad}$ follows by standard arguments. If $\nu > 0$, then the objective functional of (4.2) is strictly convex and hence the optimal control is unique.

We shall sketch below that the sparsity parameter influences the size of the support of the optimal control of the problem (4.2). The larger κ is, the smaller is the support of the optimal control.

To understand this effect that is meanwhile well studied (see, for instance, [6], [7–9, 15, 28]), we first have to set up the associated system of necessary optimality conditions. To this aim, we need the subdifferential $\partial\gamma(j)$ of the convex but non-differentiable functional $\gamma : L^1_{\mathbb{R}}(\Omega_C) \to \mathbb{R}$ at an arbitrary but fixed $j \in L^1_{\mathbb{R}}(\Omega_C)$.

This subdifferential is the set of all elements $\lambda \in L^\infty_{\mathbb{R}}(\Omega_C)$ such that

$$\gamma(v) \geq \gamma(j) + \int_{\Omega_C} \lambda(\mathbf{x})\,(v(\mathbf{x}) - j(\mathbf{x})) \quad \forall v \in L^1_{\mathbb{R}}(\Omega_C). \tag{4.3}$$

It is hence defined by

$$\partial\gamma(j) := \{\lambda \in L^\infty_{\mathbb{R}}(\Omega_C) : \text{ (4.3) is satisfied}\}.$$

The following representation is known for $\partial\gamma(j)$ (see, e.g., [21, Sect. 4.5.1]):

$$\partial\gamma(j) := \Big\{\lambda \in L^\infty_{\mathbb{R}}(\Omega_C) : \lambda \text{ satisfies (4.5) below}\Big\}, \tag{4.4}$$

$$\lambda(\mathbf{x}) = \begin{cases} 1, & \text{if } j(\mathbf{x}) > 0 \\ [-1, 1], & \text{if } j(\mathbf{x}) = 0 \\ -1, & \text{if } j(\mathbf{x}) < 0\,. \end{cases} \tag{4.5}$$

After some easy computation, the subdifferential of g is obtained as

$$\partial g(J) = \{\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in L^\infty_{\mathbb{R}}(\Omega_C)^3 : \lambda_\ell \in \partial\gamma(J_\ell),\ \ell = 1, 2, 3\}; \tag{4.6}$$

notice that $g(J) = \sum_{\ell=1}^{3} \gamma(J_\ell)$.

4.2. *Necessary optimality conditions*

For the case $\kappa = 0$, where the functional (4.2) is differentiable, we derived the variational inequality (3.30) as necessary condition. In the case $\kappa > 0$, the variational inequality (3.30) has to be complemented by the subdifferential of g. The following result is obtained:

THEOREM 4.1 (Necessary conditions for sparse optimal controls) *Let J^* be the optimal control for the problem* (4.2) *and let $\mathbf{J}^* := e^{i\phi} J^*$. Then there exists a unique adjoint state $(\mathbf{W}_{\mathbf{J}^*}, \eta_{\mathbf{J}^*}, \boldsymbol{\alpha}_{\mathbf{J}^*})$ solving the adjoint equation* (3.5) *and a function $\Lambda^* \in \partial g(J^*) \subset L^\infty_{\mathbb{R}}(\Omega_C)^3$ such that the variational inequality*

$$\int_{\Omega_C} (D_{J^*} + \nu J^* + \kappa\, \Lambda^*) \cdot (J - J^*) \geq 0 \quad \forall J \in J_{ad} \tag{4.7}$$

is satisfied, where D_{J^*} is defined by (3.31).

Proof. The main line of proof is more or less standard in convex optimization. However, it is not completely obvious how the associated ideas should be merged to derive our result in the case of optimal control. We therefore detail the proof for convenience of the reader.

Due to our notation, J^* minimizes the functional

$$\Phi := f + \kappa g$$

in the set J_{ad}. In a first step, we derive an auxiliary variational inequality by differentiating only the smooth part f of Φ. For all $0 \leq s \leq 1$ and arbitrary fixed $J \in J_{ad}$, we have

$$\begin{aligned} 0 &\leq \frac{\Phi(J^* + s(J - J^*)) - \Phi(J^*)}{s} \\ &\leq \frac{f(J^* + s(J - J^*)) - f(J^*)}{s} + \kappa\,(g(J) - g(J^*)) \end{aligned}$$

because g is convex. Passing to the limit $s \downarrow 0$, it follows

$$0 \leq f'(J^*)(J - J^*) + \kappa\, g(J) - \kappa\, g(J^*) \quad \forall J \in J_{ad}.$$

This variational inequality is a standard result for minimizing the sum of a convex and of a differentiable functional (see [12, Chap. II, Prop. 2.2]). It can be re-written as

$$f'(J^*)\, J^* + \kappa\, g(J^*) \leq f'(J^*)\, J + \kappa\, g(J) \quad \forall J \in J_{ad}.$$

In other words, we have

$$J^* \in \arg \min_{J \in J_{ad}} \{ f'(J^*)\, J + \kappa\, g(J) \}. \tag{4.8}$$

Next, we include the constraint $J \in J_{ad}$ in the objective functional. To this aim, we introduce the indicator function

$$\Psi_{J_{ad}}(J) = \begin{cases} 0, & J \in J_{ad} \\ \infty, \text{else}. \end{cases}$$

We also define the linear part of the functional above by

$$\varphi : J \mapsto f'(J^*)\, J = \int_{\Omega_C} (D_{J^*} + \nu J^*) \cdot J.$$

Thanks to (4.8), J^* is the minimizer of the convex optimization problem

$$J^* = \arg\min\{\varphi(J) + \kappa g(J) + \Psi_{J_{ad}}(J)\},$$

and hence J^* must satisfy the associated necessary optimality condition

$$0 \in \partial(\varphi + \kappa g + \Psi_{J_{ad}})(J^*). \tag{4.9}$$

The subdifferential of $\partial\Psi_{J_{ad}}$ is equal to the normal cone $N_{J_{ad}}$ at J^*, where

$$N_{J_{ad}}(J^*) = \Big\{ z \in L^\infty_{\mathbb{R}}(\Omega_C)^3 : \int_{\Omega_C} z \cdot (J - J^*)\, dx \le 0 \quad \forall J \in J_{ad} \Big\},$$

if $J^* \in J_{ad}$. For $J^* \notin J_{ad}$, we have $N_{J_{ad}}(J^*) = \emptyset$. Applying the theorem of Moreau–Rockafellar (see [21, Sect. 4.2.2]), we find

$$\begin{aligned} \partial(\varphi + \kappa g + \Psi_{J_{ad}})(J^*) &= \partial\varphi(J^*) + \kappa\, \partial g(J^*) + \partial\Psi_{J_{ad}}(J^*) \\ &= (D_{J^*} + \nu J^*) + \kappa\, \partial g(J^*) + N_{J_{ad}}(J^*); \end{aligned}$$

notice that the assumptions of the Moreau–Rockafellar theorem are satisfied, because the functional $\varphi + \kappa g$ is continuous on the whole space $L^2_{\mathbb{R}}(\Omega_C)$. By (4.9), we have

$$-(D_{J^*} + \nu J^*) \in \kappa\, \partial g(J^*) + N_{J_{ad}}(J^*),$$

i.e., there exist $\Lambda^* \in \partial g(J^*) \subset L^\infty_{\mathbb{R}}(\Omega_C)^3$ (notice that $g : L^1_{\mathbb{R}}(\Omega_C)^3 \to \mathbb{R}$, hence the properties of ∂g remain true, if the argument J even belongs to $L^2_{\mathbb{R}}(\Omega_C)^3$) and $Z^* \in N_{J_{ad}}(J^*)$ such that

$$-(D_{J^*} + \nu J^* + \kappa \Lambda^*) = Z^* \in N_{J_{ad}}(J^*).$$

By definition of $N_{J_{ad}}(J^*)$ this means

$$-\int_{\Omega_C} (D_{J^*} + \nu J^* + \kappa \Lambda^*) \cdot (J - J^*) \le 0 \quad \forall J \in J_{ad},$$

the inequality being equivalent to (4.7). $\square$

Let us describe a few consequences of this theorem. The main one is the sparsity of the optimal control J^*.

Corollary 4.2 (Sparsity) *Assume $\nu > 0$ and $\kappa > 0$ and let J^* be optimal for the problem (4.2). Then, for $\ell = 1, 2, 3$,*

$$J_\ell^*(\mathbf{x}) = 0 \text{ if and only if } \kappa \geq |(D_{J^*})_\ell(\mathbf{x})| \tag{4.10}$$

holds for a.a. $\mathbf{x} \in \Omega_C$. For almost all $\mathbf{x} \in \Omega_C$, the element $\Lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in \partial g(J^*)$ is given by the projection formula

$$\lambda_\ell^*(\mathbf{x}) = \mathbb{P}_{[-1,1]}\{-\frac{1}{\kappa}(D_{J^*})_\ell(\mathbf{x})\}, \ \ell \in \{1, 2, 3\}. \tag{4.11}$$

Proof. Let us fix $\ell \in \{1, 2, 3\}$. To avoid an extensive use of subscripts, let us write for short $D^* := D_{J^*}$. First, we show the implication $\kappa \geq |D_\ell^*(\mathbf{x})| \Rightarrow J_\ell^*(\mathbf{x}) = 0$. Assume the contrary, i.e., $J_\ell^*(\mathbf{x}) \neq 0$. It follows from (4.7) that

$$\begin{aligned} J_\ell^*(\mathbf{x}) > 0 &\Rightarrow (D_\ell^* + \nu J_\ell^* + \kappa\, \lambda_\ell^*)(\mathbf{x}) \leq 0 \\ J_\ell^*(\mathbf{x}) < 0 &\Rightarrow (D_\ell^* + \nu J_\ell^* + \kappa\, \lambda_\ell^*)(\mathbf{x}) \geq 0. \end{aligned}$$

If $J_\ell^*(\mathbf{x}) > 0$, then we have $\lambda_\ell^*(\mathbf{x}) = 1$ and the first case above implies

$$D_\ell^*(\mathbf{x}) + \nu J_\ell^*(\mathbf{x}) + \kappa \leq 0,$$

hence

$$0 < \nu J_\ell^*(\mathbf{x}) \leq -D_\ell^*(\mathbf{x}) - \kappa$$

must hold; this yields

$$\kappa < -D_\ell^*(\mathbf{x})\,, \tag{4.12}$$

a contradiction. Analogously, if $J_\ell^*(\mathbf{x}) < 0$, then $\lambda_\ell^*(\mathbf{x}) = -1$ and $D_\ell^*(\mathbf{x}) + \nu J_\ell^*(\mathbf{x}) - \kappa \geq 0$ must hold. This leads to

$$\kappa < D_\ell^*(\mathbf{x})\,, \tag{4.13}$$

a contradiction. Altogether, we have proved that

$$\kappa \geq |D_\ell^*(\mathbf{x})| \quad \Rightarrow J_\ell^*(\mathbf{x}) = 0\,.$$

Next, we verify the converse implication $J_\ell^*(\mathbf{x}) = 0 \Rightarrow \kappa \geq |D_\ell^*(\mathbf{x})|$. From the variational inequality (4.7) we deduce for almost all $\mathbf{x} \in \Omega_C$

$$0 = (D_\ell^* + \nu J_\ell^* + \kappa\, \lambda_\ell^*)(\mathbf{x}) = (D_\ell^* + \kappa\, \lambda_\ell^*)(\mathbf{x})\,,$$

hence $|D_\ell^*(\mathbf{x})| = \kappa|\lambda_\ell^*(\mathbf{x})|$. By the definition of the subdifferential, we have $|\lambda_\ell^*(\mathbf{x})| \leq 1$. Therefore, $\kappa \geq |D_\ell^*(\mathbf{x})|$ must be satisfied.

Finally, let us confirm the projection formula (4.11). For $J_\ell^*(\mathbf{x}) = 0$, we found $D_\ell^*(\mathbf{x}) + \kappa\lambda_\ell^*(\mathbf{x}) = 0$, i.e. $\lambda_\ell^*(\mathbf{x}) = -\kappa^{-1} D_\ell^*(\mathbf{x})$. Since $|\lambda_\ell^*(\mathbf{x})| \leq 1$, this implies (4.11). For $J_\ell^*(\mathbf{x}) > 0$ we have derived the inequality (4.12) that yields

$$\lambda_\ell^*(\mathbf{x}) = 1 < -\frac{D_\ell^*(\mathbf{x})}{\kappa}.$$

Again, this complies with (4.11). Analogously, we invoke (4.13), if $J_\ell^*(\mathbf{x}) < 0$. $\square$

In view of this result, we can expect that for increasing κ the support of the optimal control functions J_ℓ^* becomes smaller. This is expressed by the following conclusion.

COROLLARY 4.3 *Assume $\nu > 0$ and denote by J_κ^* the optimal control of the problem* (4.2) *for given $\kappa > 0$. Then there holds*

$$\lim_{\kappa\to\infty} \operatorname{meas}\{\mathbf{x} \in \Omega_C : |(J_\kappa^*)_\ell(\mathbf{x})| > 0\} = 0 \quad \forall \ell \in \{1,2,3\}. \tag{4.14}$$

Proof. By the definition (3.28) of J_{ad}, we have a bound $c_1 > 0$ such that

$$\|J\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \le c_1 \quad \forall J \in J_{ad}.$$

This bound remains valid for all associated $\mathbf{J} = e^{i\phi} J$, $J \in J_{ad}$, of the (complex) space $L^2(\Omega_C)^3$. The control-to-state mapping $\mathbf{J} \mapsto (\mathbf{H_J}, \psi_\mathbf{J}, \alpha_\mathbf{J})$ defined by the state equation (2.9) is continuous from $L^2(\Omega_C)^3$ to $H(\text{curl};\Omega_C)\times(H^1(\Omega_I)/\mathbb{C})\times\mathbb{C}^g$. Therefore, the mapping $\mathbf{J} \mapsto \mathbf{E_J} = \sigma^{-1}(\operatorname{curl}\mathbf{H_J} - \mathbf{J})$ is continuous in $L^2(\Omega_C)^3$.

In the adjoint equation (3.5), considered for $\widehat{\mathbf{J}} := \mathbf{J}$, the terms $\mathbf{H_J} - \mathbf{H}_d$, $\nabla\psi_\mathbf{J} - \nabla\psi_d$, $\alpha_\mathbf{J} - \alpha_d$, and $\sigma^{-1}(\operatorname{curl}\mathbf{H_J} - \mathbf{J}) - \mathbf{E}_d$ appear. In view of the continuity properties stated above, these terms depend continuously on $\mathbf{J}$ in the spaces $H(\text{curl};\Omega_C)$, $L^2(\Omega_C)^3$, $\mathbb{C}^g$, and $L^2(\Omega_C)^3$, respectively.

Therefore, also the mapping $\mathbf{J} \mapsto (\mathbf{W_J}, \eta_\mathbf{J}, \boldsymbol{\beta}_\mathbf{J})$ is continuous from $L^2(\Omega_C)^3$ to the associated spaces. Consequently, in this way, the boundedness of the admissible set J_{ad} implies the boundedness of the set of all adjoint states $(\mathbf{W_J}, \eta_\mathbf{J}, \boldsymbol{\beta}_\mathbf{J})$ in $H(\text{curl};\Omega_C) \times (H^1(\Omega_I)/\mathbb{C}) \times \mathbb{C}^g$ that can be generated by $J \in J_{ad}$.

By the definition (3.12) of $\mathbf{D_J}$, the set of all possible functions $\mathbf{D_J}$ that are generated by the controls $J \in J_{ad}$ is bounded in $L^2(\Omega_C)^3$. Taking the real part of $e^{-i\phi}\mathbf{D_J}$, this implies the existence of $c_3 > 0$ such that

$$\|D_J\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \le c_3 \quad \forall J \in J_{ad}. \tag{4.15}$$

In particular, this holds true for $D^* = D_{J^*}$, that is related to the optimal control J^*.

After having found this bound, we argue by contradiction and assume that (4.14) is not true. Then there exist $\ell_0 \in \{1,2,3\}$, $\delta > 0$, and a sequence $\kappa_n \to \infty$ as $n \to \infty$, such that

$$\operatorname{meas}(S_n) \ge \delta \quad \forall n \in \mathbb{N},$$

where

$$S_n = \{\mathbf{x} \in \Omega_C : |(J_{\kappa_n}^*)_{\ell_0}(\mathbf{x})| > 0\}.$$

Let us write for short $D_n^* := D_{J_{\kappa_n}^*}$ and $D_{n,\ell_0}^* := (D_{J_{\kappa_n}^*})_{\ell_0}$. From Corollary 4.2, condition (4.10), we deduce that

$$\kappa_n < |D_{n,\ell_0}^*(\mathbf{x})| \quad \text{a.e. in } S_n.$$

Now we find

$$c_3 \geq \|D_n^*\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \geq \left(\int_{S_n} |D^*_{n,\ell_0}(\mathbf{x})|^2 \right)^{1/2} \geq \kappa_n \left(\mathrm{meas}(S_n)\right)^{1/2} \geq \kappa_n \sqrt{\delta}\,.$$

This is a contradiction to (4.15), since $\kappa_n \to \infty$ as $n \to \infty$. □

Remark 3 By the particular form of J_{ad}, we might expect a stronger result. The set of all possible control functions J is bounded in $L^\infty_{\mathbb{R}}(\Omega_C)^3$ by the constant $j_{\max}$. If we were able to deduce from this fact that all associated functions D_J are bounded in $L^\infty_{\mathbb{R}}(\Omega_C)^3$ by a joint constant as well, then we would obtain the existence of some κ_0 such that the optimal controls must vanish whenever $\kappa > \kappa_0$. However, to our knowledge such a boundedness result for the state functions in the space $L^\infty_{\mathbb{R}}(\Omega_C)^3$ is not available.

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References

[1] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, and A. Valli. Construction of a finite element basis of the first de Rham cohomology group and numerical solution of 3D magnetostatic problems. *SIAM J. Numer. Anal.*, 51(4):2380–2402, 2013.

[2] A. Alonso Rodríguez and A. Valli. *Eddy current approximation of Maxwell equations.* Springer-Verlag Italia, Milan, 2010.

[3] L. Arnold and B. von Harrach. A unified variational formulation for the parabolic-elliptic eddy current equations. *SIAM J. Appl. Math.*, 72(2):558–576, 2012.

[4] G. Bärwolff and M. Hinze. Optimization of semiconductor melts. *ZAMM Z. Angew. Math. Mech.*, 86(6):423–437, 2006.

[5] J. Cantarella, D. DeTurck and H. Gluck. Vector calculus and the topology of domains in 3-space. *Amer. Math. Monthly*, 109(5):409–442, 2002.

[6] E. Casas, C. Clason, and K. Kunisch. Approximation of elliptic control problems in measure spaces with sparse solutions. *SIAM J. Control Optim.*, 50(4):1735–1752, 2012.

[7] E. Casas, C. Clason, and K. Kunisch. Parabolic control problems in measure spaces with sparse solutions. *SIAM J. Control Optim.*, 51(1):28–63, 2013.

[8] E. Casas, R. Herzog, and G. Wachsmuth. Optimality conditions and error analysis of semilinear elliptic control problems with L^1 cost functional. *SIAM J. Optim.*, 22(3):795–820, 2012.

[9] E. Casas, C. Ryll, and F. Tröltzsch. Sparse optimal control of the Schlögl and FitzHugh-Nagumo systems. *Comput. Methods Appl. Math.*, 13:415–442, 2014.

[10] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 2.* Springer-Verlag, Berlin, 2000.

[11] P.E. Druet, O. Klein, J. Sprekels, F. Tröltzsch, and I. Yousept. Optimal control of three-dimensional state-constrained induction heating problems with nonlocal radiation effects. *SIAM J. Control Optim.*, 49(4):1707–1736, 2011.

[12] I. Ekeland and R. Temam. *Analyse convexe et problèmes variationnels.* Dunod, Gauthier-Villars, 1974.

[13] R. Griesse and K. Kunisch. Optimal control for a stationary MHD system in velocity-current formulation. *SIAM J. Control Optim.*, 45(5):1822–1845, 2006.

[14] M. Gunzburger and C. Trenchea. Analysis and discretization of an optimal control problem for the time-periodic MHD equations. *J. Math. Anal. Appl.*, 308(2):440–466, 2005.

[15] R. Herzog, G. Stadler, and G. Wachsmuth. Directional sparsity in optimal control of partial differential equations. *SIAM J. Control Optim.*, 50(2):943–963, 2012.

[16] M. Hinze. Control of weakly conductive fluids by near wall Lorentz forces. *GAMM-Mitt.*, 30(1):149–158, 2007.

[17] D. Hömberg and J. Sokołowski. Optimal shape design of inductor coils for surface hardening. *Numer. Funct. Anal. Optim.*, 42(3):1087–1117, 2003.

[18] D. Hömberg and S. Volkwein. Control of laser surface hardening by a reduced-order approach using proper orthogonal decomposition. *Math. Comput. Modelling*, 38(10):1003–1028, 2003.

[19] L. S. Hou and A. J. Meir. Boundary optimal control of MHD flows. *Appl. Math. Optim.*, 32(2):143–162, 1995.

[20] L. S. Hou and S. S. Ravindran. Computations of boundary optimal control problems for an electrically conducting fluid. *J. Comput. Phys.*, 128(2):319–330, 1996.

[21] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems.* North-Holland Publishing Co., Amsterdam, 1979.

[22] M. Kolmbauer. *The multiharmonic finite element and boundary element method for simulation and control of eddy current problems.* PhD Thesis, Johannes Kepler University, Linz, 2012.

[23] M. Kolmbauer and U. Langer. A robust preconditioned MinRes solver for distributed time-periodic eddy current optimal control problems. *SIAM J. Sci. Comput.*, 34(6):B785–B809, 2012.

[24] P. Monk. *Finite element methods for Maxwell's equations.* Oxford University Press, New York, NY, 2003.

[25] S. Nicaise, S. Stingelin, and F. Tröltzsch. On two optimal control problems for magnetic fields. *Comput. Methods Appl. Math.*, 2014.

[26] S. Nicaise, S. Stingelin, and F. Tröltzsch. Optimal control and model reduction for two magnetization processes. *Discrete Contin. Dyn. Syst.-S*, 2015.

[27] S. S. Ravindran. Real-time computational algorithm for optimal control of an MHD flow system. *SIAM J. Sci. Comput.*, 26(4):1369–1388, 2005.

[28] G. Stadler. Elliptic optimal control problems with L^1-control cost and applications for the placement of control devices. *Comput. Optim. Appl.*, 44(2):159–181, 2009.

[29] F. Tröltzsch. *Optimal control of partial differential equations.* American Mathematical Society, Providence, RI, 2010.

[30] F. Tröltzsch, A. Valli. Modeling and control of low-frequency electromagnetic fields in multiply connected conductors. *Submitted to "Proceedings of the 27th IFIP Conference"*, 2015.

[31] I. Yousept. Optimal control of Maxwell's equations with regularized state constraints. *Comput. Optim. Appl.*, 52(2):559–581, 2012.

[32] I. Yousept and F. Tröltzsch. PDE-constrained optimization of time-dependent 3D electromagnetic induction heating by alternating voltages. *ESAIM M2AN*, 46(4):709–729, 2012.