On an Optimal Shape Design Problem Connected with the Heating of Elastic Bodies

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Abstract

An optimal shape design problem arising in linear thermoelasticity is considered. The objective is to determine which initial shape a workpiece undergoing a prescribed (known) thermal treatment must have in order that its final shape after the treatment has a desired form. The problem is studied for a simplified two–dimensional situation. Upon proving the well–posedness of the associated state equations, the directional differentiability of the solution operator with respect to the control variable is shown, and the first–order necessary conditions of optimality are derived.

AMS(MOS) Subject Classifications: 49 B 22, 73 U 05, 93 C 20.

1 Introduction

During the past decade the field of optimal shape design has attracted much interest. In particular, many papers have been devoted to the optimal shaping of objects having properties that can be described by elliptic partial differential equations or variational inequalities. We refer, for instance, to the monographs by Pironneau [6] and by Haslinger–Neittaanmäki [2], and to the references therein. In contrast to this, merely a few investigations have been devoted to instationary problems. In this connection, we mention the contributions by Cannarsa–da Prato–Zolesio [1], Sokolowski [7], Hoffmann–Sokolowski [3], Sokolowski–Sprekels [8] and Okhezin [5].

In this paper, we consider a problem which is, in a sense, intermediate between

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stationary and dynamical shape design problems: the shape control itself does not depend on time, while the equations describing the state of the system do. More specifically, we shall study the following problem (which is of considerable interest in the applications): Suppose that an isotropic and homogeneous solid body is subjected to a prescribed (known) thermal treatment. Due to the temperature change, the body body undergoes a thermoelastic deformation, that is, the induced thermal stresses force the body to change its shape in time. The following question arises: Which *initial* shape must the body be given in order that its *final* shape after the thermal treatment resembles a desired prescribed form as closely as possible?
A very simplified problem of this type will be discussed here: let $s \in C^{1,\nu}[0,d]$ be given, and suppose that initially, at $t = 0$, the body occupies the domain (see Fig. 1)

$$\Omega(s) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < d\,,\, 0 < x_2 < s(x_1)\}\,. \tag{1}$$

The set $\Gamma_3(s) = \{x_2 = s(x_1)\}$ denotes the part of the boundary $\Gamma(s)$ of $\Omega(s)$ which is to be shaped.

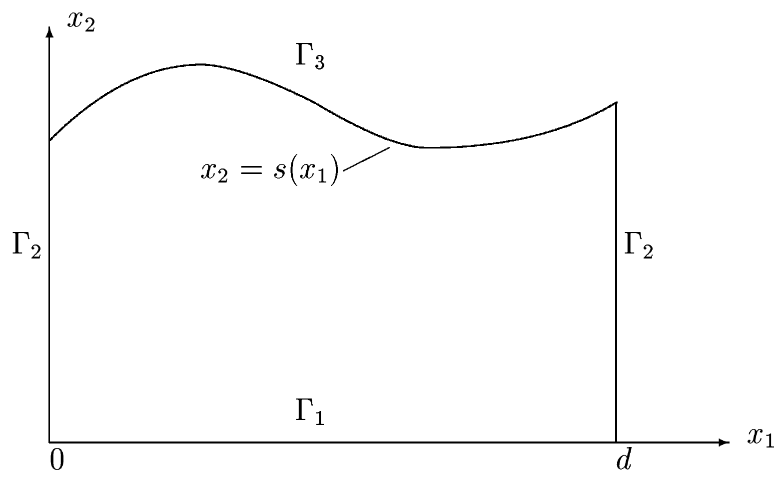

Figure 1: Shape of the domain $\Omega(s)$.

The function s plays the role of the *control variable*. Let $\underline{s}\,, \bar{s}\,, \bar{s}'\,, c$ denote fixed positive constants, and let $\nu \in (0,1)$ and $s_a, s_b \in \mathbb{R}$ be given. We assume that s satisfies the conditions

$$0 < \underline{s} \le s(\tau) \le \bar{s}\,, \quad |s'(\tau)| \le \bar{s}'\,, \quad \forall\, \tau \in [0,d]\,, \tag{2}$$

$$|s'(\tau_1) - s'(\tau_1)| \le c\,|\tau_1 - \tau_2|^{\nu}\,, \quad \forall\, \tau \in [0,d]\,, \tag{3}$$

$$s(0) = s_a\,, \quad s(d) = s_b\,, \tag{4}$$

$$\underline{s} \le s_a \le \bar{s}\,, \quad \underline{s} \le s_b \le \bar{s}\,. \tag{5}$$

Thus, $s \in U^{ad}$, where $U^{ad} = \{s \in C^{1,\nu}[0,d] : s \text{ satisfies } (1.2)–(1.5)\}$ is the *set of admissible controls*. Note that U^{ad} forms a nonempty, convex and compact subset of $C^1[0,d]$.
Next, let $\theta = \theta(t,x)\,, t \in [0,T]\,, x = (x_1,x_2) \in \bar{\Omega}(s)$, denote the *temperature*. We assume that θ satisfies the parabolic problem

$$\begin{aligned}
\rho\, c_v\, \theta_t(t,x) &= \kappa\, \Delta_x \theta(t,x)\,, \quad \text{in } (0,T)\times\Omega(s)\,, && (6)\\
\theta(0,x) &= \theta_0(x)\,, \quad \text{in } \bar{\Omega}(s)\,, && (7)\\
-\kappa\frac{\partial\theta}{\partial n}(t,x) &= 0\,, \quad \text{on } (0,T)\times(\Gamma_1\cup\Gamma_2)\,, && (8)\\
-\kappa\frac{\partial\theta}{\partial n}(t,x) &= \alpha\,(\theta(t,x) - g(t,x_1))\,, \quad \text{on } (0,T)\times\Gamma_3(s)\,, && (9)
\end{aligned}$$

where θ_0 and g are the (given) initial temperature and the temperature of the surrounding medium at $\Gamma_3(s)$, respectively; the (positive) constants $\rho\,, c_v\,, \kappa$ stand for mass density, specific heat and heat conductivity, in that order, while $\alpha \ge 0$ is a constant. Moreover, we denote by $\sigma_{ij} = \sigma_{ij}(u(t,x))$ and $\varepsilon_{ij} = \varepsilon_{ij}(u(t,x))$ the components of the *stress* and *(linearized) strain tensors*, respectively, where $u = (u^1(t,x), u^2(t,x))$ is the *displacement vector*. Assuming a linear thermoelastic behaviour, we obtain for the *quasistatic regime*

$$\Delta u + (\lambda+\mu)\,\nabla\,(\operatorname{div} u) + \rho F - \beta\,\nabla\theta = 0\,, \quad \text{in } (0,T)\times\Omega(s)\,, \quad , \tag{10}$$

where $\mu > 0\,, \lambda > 0$ are Lamé's constants, $\beta > 0$ is a constant and F is vector of the (distributed) body forces. In addition, we prescribe the boundary conditions

$$u = 0\,, \quad \text{on } (0,T)\times\Gamma_1\,, \qquad n_j\,\sigma_{ij} = 0\,, \quad \text{on } (0,T)\times\Gamma_3(s)\,, \tag{11}$$

as well as

$$\begin{array}{lll}
\text{either} & & \\
\text{(a)} & u = 0\,, \quad \text{on } (0,T)\times\Gamma_2\,, & \\
\text{or} & & \\
\text{(b)} & u^1 = 0\,, \quad \sigma_{12} = 0\,, \quad \text{on } (0,T)\times\Gamma_2\,. &
\end{array} \tag{12}$$

Here, $n = (n_1,n_2)$ is the outward unit normal vector at $\Gamma_3(s)$. We have also used the summation convention.
Remark: Note that the system (6)–(12) is not the full system of linear thermoelasticity. Indeed, besides assuming the quasistatic form for the balance of linear momentum, the energy balance (6) is assumed in the form of the linear heat equation, so that the source term accounting for the elastic energy is missing. However, in solids this term is usually small in comparison with the other terms appearing in the energy balance, and it is a common practice to neglect it.

The *optimal shape design problem* is to minimize the cost functional

$$
\begin{aligned}
J(s) = J_1(s) + J_2(s) \quad &= \quad \int_{\Omega(s)} q_1(x)\, |u(T,x) - q_2(x)|^2 \, dx \\
&+ \quad \int_0^d q_3(x_1)(s(x_1) + u^2(T, x_1, s(x_1)) - \hat{s}(x_1))^2 dx_1 \qquad (13)
\end{aligned}
$$

over U^{ad}, subject to (6)–(12). Here, q_1 and q_2 are given functions satisfying $q_1 \in L^\infty(\Omega(s))$, $q_1 \geq 0$ and $q_2 \in L^2(\Omega(s)) \times L^2(\Omega(s))$, for all $s \in U^{ad}$; q_1 serves as a weight function. Moreover, $q_3 \in L^\infty(0,d)$ and a desired final shape function $\hat{s} \in L^2(0,d)$ are given.
In the sequel, we shall always work with the weak formulation of the initial-boundary value problems (6)–(9) and (10)–(12), respectively. To facilitate the exposition, we shall always assume that the physical constants $\rho\,, c_v\,, \alpha\,, \beta\,, \kappa$ are normalized to unity. This has no bearing on the mathematical analysis. We also introduce the abbreviating notations

$$
(u,v)_\Omega = \int_\Omega (u(x))^T\, v(x)\, dx\,, \quad (u,v)_\Gamma = \int_\Gamma (u(x))^T\, v(x)\, dS_x\,, \qquad (14)
$$

$$
H = L^2(Q)\,, \quad \mathbf{H} = H \times H\,, \quad W = H^1(Q)\,. \qquad (15)
$$

Moreover, $\|\cdot\|_{0,Q}$ and $\|\cdot\|_{1,Q}$ will always denote the norm of the spaces $L^2(Q)$ and $H^1(Q)$, respectively.
We shall write for convenience $\sigma_{ij} = \tau_{ij} - \delta_{ij}\,\beta\,\theta$. With these notations, the weak formulation of (6)–(9) and (10)–(12), respectively, is given by

$$
\begin{aligned}
&(\theta_t(t)\,,\, w)_{\Omega(s)} \;+\; (\nabla\theta(t)\,,\, \nabla w)_{\Omega(s)} \;+\; (\theta(t)\,,\, w)_{\Gamma_3(s)} = (g(t)\,,\, w)_{\Gamma_3(s)}\,, \\
&\qquad\qquad\qquad \text{for all } w \in H^1(\Omega(s)) \text{ and a.e. } t \in (0,T)\,, \qquad (16)
\end{aligned}
$$

$$
\theta(0) = \theta_0\,, \qquad (17)
$$

$$
\begin{aligned}
&\int_{\Omega(s)} \tau_{ij}(u(t))\, \varepsilon_{ij}(v)\, dx = (F(t) \;-\; \nabla\theta(t)\,,\, v)_{\Omega(s)} + (\theta(t)\, n\,,\, v)_{\Gamma(s)}\,, \\
&\qquad\qquad\qquad \text{for all } v \in V(s) \text{ and a.e. } t \in (0,T)\,, \qquad (18)
\end{aligned}
$$

where $V(s) = \{v \in (H^1(\Omega(s)))^2 : v = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}$, if the boundary conditions (11,12(a)) are given, and $V(s) = \{v \in (H^1(\Omega(s)))^2 : v = 0 \text{ on } \Gamma_1\,,\ v^1 = 0 \text{ on } \Gamma_2\}$, in the case (11,12(b)). The quantities $u\,, \tau$ and ε are connected through the relations

$$
\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) \qquad (19)
$$

$$
\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}. \qquad (20)
$$

We assume once and for all that $\theta_0 \in H^2(0,d)$, $g \in C^1([0,T]; L^2(0,d))$ and $F \in (C([0,T]; L^2(0,d)))^2$, for all admissible $s \in U^{ad}$.

The remainder of the paper is organized as follows. In Section 2, we transform the optimal shape design problem into a control problem for the coefficients of an associated system of partial differential equations on a fixed domain. In Section 3, the unique solvability of the corresponding state equations is proved, and the stability of the solution with respect to variations of s is shown. Section 4 brings a derivation of the first order necessary conditions of optimality.

2 Transformation onto Fixed Domain

To cope with the variation of the domain, we transform the optimal shape design problem into an optimal control problem for the coefficients of an associated coupled system of partial differential equations acting on a fixed domain. To this end, we put $\xi_1 = x_1\,, \xi_2 = x_2/s(x_1)$. Then, with

$$x = \Phi(\xi) = \begin{pmatrix} \xi_1 \\ \xi_2\, s(\xi_1) \end{pmatrix} \quad , \quad \xi = \Phi^{-1}(x) = \begin{pmatrix} x_1 \\ x_2/s(x_1) \end{pmatrix} \quad , \tag{1}$$

the domain Ω is transformed under the application of Φ^{-1} into the rectangle $Q = (0,d) \times (0,1)$. The Jacobian of Φ satisfies, for all $\xi \in Q$,

$$\Phi'(\xi) = \begin{pmatrix} 1 & 0 \\ \xi_2 s'(\xi_1) & s(\xi_1) \end{pmatrix}, \qquad \det\, \Phi'(\xi) = s(\xi_1) \geq \underline{s} > 0\,. \tag{2}$$

Let $\theta(x) = \theta(\Phi(\xi)) =: \tilde{\theta}(\xi)$. Then $\nabla_x \theta(x) = (\Phi'(\xi)^{-1})^T\, \nabla_\xi \tilde{\theta}(\xi)$, whence

$$\nabla_x \theta = \begin{pmatrix} 1 & -\xi_2\, s'(\xi_1)/s(\xi_1) \\ 0 & 1/s(\xi_1) \end{pmatrix} \nabla_\xi \tilde{\theta}\,, \tag{3}$$

as well as

$$\langle \nabla_x \theta\,,\, \nabla_x \theta \rangle = \langle \nabla_\xi \tilde{\theta}\,,\, D(s)\, \nabla_\xi \tilde{\theta} \rangle\,, \tag{4}$$

where

$$D(s)(\xi) = (s(\xi_1))^{-2} \begin{pmatrix} (s(\xi_1))^2 & -\xi_2\, s(\xi_1)\, s'(\xi_1) \\ -\xi_2\, s(\xi_1)\, s'(\xi_1) & 1 + (\xi_2\, s'(\xi_1))^2 \end{pmatrix}. \tag{5}$$

In the sequel, we shall simplify the notation and omit the tilde, that is, we denote $\theta(\xi) := \tilde{\theta}(\xi)$.
Using the relation $dx = \det \Phi'(\xi)\, d\xi = s(\xi_1)\, d\xi$, and introducing $w(\xi) := s(\xi_1)\, w(\xi)$ as new test function (otherwise, s would appear coupled with $\theta_t(t)$), we find after a routine calculation that (1.18)–(1.19) is transformed into

$$(\theta_t(t)\,,\, w)_Q \,+\, a(s;\theta(t),w) = \gamma(s;g(t),w)\,, \quad \forall w \in H^1(Q)\,, \tag{6}$$

$$\theta(0) = \theta_0\,. \tag{7}$$

Here, $a = a_1 + a_2$, with

$$a_1(s;\theta,w) = (\nabla\theta\,,\, D(s)\,\nabla w)_Q \,+\, \int_{\Gamma_3} \theta\, w\, (s(\xi_1))^{-1} \sqrt{1 + (s'(\xi_1))^2}\, d\xi_1\,, \tag{8}$$

and with the nonsymmetric expression

$$a_2(s;\theta,w) = -\,(\nabla\theta\,,\,w\,d(s))_Q\,, \tag{9}$$

where

$$d(s)(\xi) = (s(\xi_1))^{-1}\,D(s)(\xi)\,(s'(\xi_1),0)^T = (s'(\xi_1)/s(\xi_1), -\xi_2\,(s'(\xi_1)/s(\xi_1))^2)^T\,. \tag{10}$$

Moreover,

$$\gamma(s;w_1,w_2) = \int_{\Gamma_3} (s(\xi_1))^{-1}\sqrt{1+(s'(\xi_1))^2}\,w_1\,w_2\,d\xi_1\,. \tag{11}$$

Here, and throughout, the pieces of the boundary of Q are numbered in the same way as the pieces of $\Gamma(s)$.
The transformation of (1.20) requires a little more effort. Let, for convenience, $q(\xi) := -\,\xi_2\,s'(\xi_1)/s(\xi_1)$. Introducing $u(x) = u(\Phi(\xi)) =: \tilde{u}(\xi)$, we find

$$\begin{aligned}
\varepsilon_{11}(u) &= \frac{\partial u^1}{\partial x_1} = \frac{\partial \tilde{u}^1}{\partial \xi_1} + q\frac{\partial \tilde{u}^1}{\partial \xi_2} =: \tilde{\varepsilon}_{11}(\tilde{u})\,,\\
\varepsilon_{22}(u) &= \frac{\partial u^2}{\partial x_2} = \frac{1}{s}\frac{\partial \tilde{u}^2}{\partial \xi_2} =: \tilde{\varepsilon}_{22}(\tilde{u})\,,\\
\varepsilon_{12}(u) &= \frac{1}{2}\left(\frac{\partial u^1}{\partial x_2} + \frac{\partial u^2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{1}{s}\frac{\partial \tilde{u}^1}{\partial \xi_2} + \frac{\partial \tilde{u}^2}{\partial \xi_1} + q\,\frac{\partial \tilde{u}^2}{\partial \xi_2}\right) =: \tilde{\varepsilon}_{12}(\tilde{u})\,.
\end{aligned} \tag{12}$$

The stresses are obtained by the linear transformation (1.23) from ε, hence

$$\begin{pmatrix} \tilde{\tau}_{11}\\ \tilde{\tau}_{22}\\ \tilde{\tau}_{12} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & 0\\ \lambda & 2\mu+\lambda & 0\\ 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \tilde{\varepsilon}_{11}\\ \tilde{\varepsilon}_{22}\\ \tilde{\varepsilon}_{12} \end{pmatrix}. \tag{13}$$

Moreover,

$$\int_{\Omega(s)} \tau_{ij}(u)\,\varepsilon_{ij}(v)\,dx = \int_Q \tilde{\tau}_{ij}(\tilde{u})\,\tilde{\varepsilon}_{ij}(\tilde{v})\,s(\xi_1)\,d\xi\,. \tag{14}$$

Since $s(\xi_1)$ appears also in the transformation of the expression $(u_{tt}(t),v)_{\Omega(s)}$, we introduce $\bar{v}(\xi) := s(\xi_1)\,\tilde{v}(\xi)$ as new test function. After a routine calculation, we obtain

$$\begin{aligned}
\int_{\Omega(s)} \tau_{ij}(u)\,\varepsilon_{ij}(v)\,dx &= \int_Q \tilde{\tau}_{ij}(\tilde{u})\,\tilde{\varepsilon}_{ij}(\bar{v})\,d\xi\\
&\quad - \int_Q \left(\tilde{\tau}_{11}(\tilde{u})\bar{v}^1 + \tilde{\tau}_{12}(\tilde{u})\bar{v}^2\right)\,s'(\xi_1)\,(s(\xi_1))^{-1}\,d\xi\\
&=: b_1(s;\tilde{u},\bar{v}) \,+\, b_2(s;\tilde{u},\bar{v}) \,=:\, b(s;\tilde{u},\bar{v})\,.
\end{aligned} \tag{15}$$

In the sequel, we shall omit the tilde on $\tilde{u}$ and the bar on $\bar{v}$.
As can easily be seen, the boundary conditions (1.14) are transformed into completely analogous conditions for $\tilde{\tau}$ and $\tilde{u}$, respectively; that is, we have $\tilde{u} = 0$

on Γ_1, $n_j \tilde{\tau}_{ij} = 0$ on Γ_3, and either $\tilde{u} = 0$ on Γ_2 or $\tilde{u}^1 = 0$ and $\tilde{\tau}_{12} = 0$ on Γ_2. These boundary conditions define a subspace V of $((H^1(Q))^2$ in the obvious way.
In summary, the transformed equations for the elastic deformations are given by

$$\begin{aligned} (u_{tt}(t)\,,\, v)_Q + b(s; u(t), v) &= (F(t)\,,\, v)_Q + (G(s)\,\nabla\theta(t)\,,\, v)_Q\,, \quad \forall\, v \in V, && (16)\\ u(0) &= u_0\,, && (17)\\ u_t(0) &= u_1\,, && (18) \end{aligned}$$

where

$$G(s)(\xi) = -\frac{1}{s(\xi_1)} \begin{pmatrix} s(\xi_1) & 0 \\ -\,\xi_2\, s'(\xi_1) & 1 \end{pmatrix}. \tag{19}$$

To obtain useful a priori estimates for θ and u, we derive coercivity estimates for a and b.

Lemma 2.1 *There are constants $\alpha_0 > 0\,, \alpha_1 > 0$, independent of $s \in U^{ad}$, such that*

$$a(s;\theta,\theta) \geq \alpha_1\, \|\theta\|^2_{1,Q} - \alpha_0 \|\theta\|^2_{0,Q}\,, \quad \forall\, \theta \in H^1(Q) \text{ and } \forall\, s \in U^{ad}. \tag{20}$$

Proof: We have $a = a_1 + a_2$. It is not difficult to see that the matrix function $D(s)$ is uniformly positive and uniformly bounded with respect to $s \in U^{ad}$; that is, there exist suitable constants $c_1 > 0\,, c_2 > 0$, independent of $s \in U^{ad}$, which satisfy

$$\langle x\,,\, D(s)(\xi)\, x\rangle \geq c_1\, |x|^2\,, \quad |D(s)(\xi)\, x| \leq c_2\, |x|\,, \quad \forall\, x \in \mathbb{R}^2 \text{ and } \forall\, \xi \in Q. \tag{21}$$

Indeed, choosing $\alpha \in (0,1)$ in a suitable way, we obtain

$$\begin{aligned} \langle x\,,\, D(s)(\xi)\, x\rangle &= s(\xi_1)^{-2} \left(s^2(\xi_1)\, x_1^2 - 2\,\alpha^{-1}\, \xi_2\, s'(\xi_1)\, \alpha\, s(\xi_1)\, x_1\, x_2 + (1 + (\xi_2\, s'(\xi_1))^2) x_2^2 \right) \\ &\geq (1-\alpha^2)\, s^2(\xi_1)\, x_1^2 + \left(1 + (1-\alpha^{-2})\, (\xi_2 s'(\xi_1))^2\right) x_2^2 \\ &\geq c_1 \left(x_1^2 + x_2^2 \right) \end{aligned} \tag{22}$$

with some $c_1 > 0$ which is independent of s. Consequently,

$$\begin{aligned} a_1(s;\theta,\theta) &= (\nabla\theta\,,\, D(s)\,\nabla\theta)_Q + \int_{\Gamma_3} \theta^2\, (s(\xi_1))^{-1} \sqrt{1 + (s'(\xi_1))^2}\, d\xi_1 \\ &\geq c_1 \int_Q |\nabla\theta|^2\, d\xi\,. \end{aligned} \tag{23}$$

Moreover, owing to (2.10) and Young's inequality, there are constants $c_3 > 0\,, c_4 > 0$, independent of s, such that

$$|a_2(s;\theta,\theta)| \leq c_3 \int_Q |\nabla\theta||\theta|\, d\xi \leq \varepsilon \int_Q |\nabla\theta|^2\, d\xi + c_4 \int_Q \theta^2\, d\xi\,, \tag{24}$$

where, independently of s, $\varepsilon > 0$ can be chosen arbitrarily small. Hence

$$a(s;\theta,\theta) \geq (c_1 - \varepsilon) \int_Q |\nabla\theta|^2 \, d\xi \; - \; c_4 \int_Q \theta^2 \, d\xi \,, \tag{25}$$

from which the assertion follows. $\square$

The estimation for $b(s;u,u)$ requires a little more effort.

Lemma 2.2 *There are constants $\beta_0 > 0$, $\beta_1 > 0$ such that*

$$b(s;u,u) \geq \beta_1 \|u\|_{1,Q}^2 \; - \; \beta_0 \|u\|_{0,Q}^2 \,, \quad \forall\, u \in V \text{ and } \forall\, s \in U^{ad} \,. \tag{26}$$

Proof: At first, we have $\tau_{ij}(u)\,\varepsilon_{ij}(u) = \langle \varepsilon \,,\, E\varepsilon \rangle$, where $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ and where E is given by

$$E = \begin{pmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & 4\mu \end{pmatrix} . \tag{27}$$

This matrix is positive definite. Hence, with a certain $\delta > 0$,

$$\tau_{ij}\,\varepsilon_{ij} \geq 2\,\delta\,|\varepsilon|^2 \geq \delta(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2) = \delta \sum_{i,j=1}^{2} \varepsilon_{ij}^2 \,. \tag{28}$$

Since $\tilde{\tau}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are related in the same way, it follows

$$\tilde{\tau}_{ij}(u)\,\tilde{\varepsilon}_{ij}(u) \geq \delta \sum_{i,j=1}^{2} \tilde{\varepsilon}_{ij}^2 \,. \tag{29}$$

Next, using (1.2) and (2.21), we have for all $y \in H^1(\Omega(s))$

$$\begin{aligned} \|y\|_{1,\Omega(s)}^2 &= \int_{\Omega(s)} (|\nabla y|^2 + y^2)\, dx \\ &= \int_Q \langle \nabla\tilde{y} \,,\, D(s)\,\nabla\tilde{y} \rangle \, s(\xi_1)\, d\xi + \int_Q \tilde{y}^2\, s(\xi_1)\, d\xi \\ &\geq c_1\, \underline{s} \int_Q |\nabla\tilde{y}|^2\, d\xi + \underline{s} \int_Q \tilde{y}^2 d\xi \\ &\geq \delta_1\, \|\tilde{y}\|_{1,Q}^2 \,, \end{aligned} \tag{30}$$

where $\delta_1 > 0$ is independent of $s \in U^{ad}$. Clearly, the analogous relation

$$\|u\|_{1,\Omega}^2 \geq \delta_2\, \|\tilde{u}\|_{1,Q}^2 \,, \quad \forall\, u \in V(s) \,, \tag{31}$$

holds as well. After these preparations, we can estimate $b = b_1 + b_2$. We obtain

$$\begin{aligned} b_1(s;\tilde{u},\tilde{u}) &= \int_Q \tilde{\tau}_{ij}(\tilde{u})\,\tilde{\varepsilon}_{ij}(\tilde{u})\, s(\xi_1)\, (s(\xi_1))^{-1}\, d\xi \\ &\geq (\bar{s})^{-1} \int_Q \tilde{\tau}_{ij}(\tilde{u})\,\tilde{\varepsilon}_{ij}(\tilde{u})\, s(\xi_1)\, d\xi \\ &= (\bar{s})^{-1} \int_{\Omega(s)} \tau_{ij}(u)\,\varepsilon_{ij}(u)\, dx \,, \end{aligned} \tag{32}$$

by (2.14). Korn's inequality (note that $\text{meas}\,(\Gamma_1) > 0$) yields

$$b_1(s;\tilde{u},\tilde{u}) \geq \delta_3\,\|u\|^2_{1,\Omega(s)}\,. \tag{33}$$

Using (2.31), we find that

$$b_1(s;\tilde{u},\tilde{u}) \geq \delta_4\,\|\tilde{u}\|^2_{1,Q}\,. \tag{34}$$

Finally, in view of Young's inequality, it is not difficult to see that

$$|b_2(s;\tilde{u},\tilde{u})| \leq \delta_5\,\|\tilde{u}\|^2_{0,Q} + \varepsilon\,\|\tilde{u}\|^2_{1,Q}\,, \tag{35}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Since all the constants in this proof are independent of $s \in U^{ad}$, the assertion follows. $\square$

3 Well–Posedness of the State Equations

Let W' denote the dual space of $W = H^1(Q)$. For almost every $t \in (0,T)$, the right–hand sides of (2.6) and (2.16) define linear continuous functionals $f_1(s;t) \in W'$ and $f_2(s;t) \in \mathbf{H}' \cong \mathbf{H}$, respectively, through the relations

$$\begin{aligned} f_1(s;t)(w) &:= \gamma(s;g(t),w)\,, \quad \forall\, w \in W\,, \\ f_2(s;t)(v) &:= (F(t) + G(s)\,\nabla\theta(t)\,,\,v)_Q\,, \quad \forall\, v \in \mathbf{H}\,. \end{aligned} \tag{1}$$

We have the following result.

Theorem 3.1 *Under the general assumptions on the data of the problem, there exists to any $s \in U^{ad}$ a unique solution pair $(\theta\,,\,u)$ to the system* (2.6)–(2.7) , (2.16)–(2.18) , *satisfying the conditions*

$$\theta \in C([0,T];H) \cap L^2(0,T;W) \cap H^1(0,T;W')\,, \tag{2}$$

$$u \in C([0,T];V) \cap C^1([0,T];\mathbf{H}) \cap H^2(0,T;V')\,. \tag{3}$$

Moreover, there exists some constant $\bar{C} > 0$ which does not depend on $s \in U^{ad}$ and satisfies

$$\max_{0\le t\le T} \|\theta(t)\|^2_{0,Q} + \int_0^T \|\theta(t)\|^2_{1,Q}\,dt \leq \bar{C}\left(\|\theta_0\|^2_{0,Q} + \int_0^T \|f_1(s;t)\|^2_{W'}\,dt\right)\,, \tag{4}$$

$$\max_{0\le t\le T}\left(\|u(t)\|^2_{1,Q} + \|u_t(t)\|^2_{0,Q}\right) \leq \bar{C}\left(\|u_0\|^2_{1,Q} + \|u_1\|^2_{0,Q} + \int_0^T \|f_2(s;t)\|^2_{0,Q}\,dt\right)\,, \tag{5}$$

$$\int_0^T \|\theta_t(t)\|^2_{W'}\,dt \leq \bar{C}\left(\|\theta_0\|^2_{0,Q} + \int_0^T \|f_1(s;t)\|^2_{W'}\,dt\right)\,, \tag{6}$$

$$\int_0^T \|u_{tt}(t)\|^2_{V'}\,dt \leq \bar{C}\left(\|u_0\|^2_{1,Q} + \|u_1\|^2_{0,Q} + \int_0^T \|f_2(s;t)\|^2_{0,Q}\,dt\right)\,. \tag{7}$$

Proof: In view of the Lemmas 2.1 and 2.2, it follows from the standard theory of linear parabolic problems (cf. Lions–Magenes [4]) that (2.6)–(2.7) admits a unique solution θ satisfying (3.2). Then $\nabla\theta \in L^2(0,T;\mathbf{H})$, and thus $F + G(s)\,\nabla\theta \in L^2(0,T;\mathbf{H})$. Hence, from the standard theory of linear hyperbolic problems (cf. Lions–Magenes [4]) we can infer that (2.16)–(2.18) has a unique solution u satisfying (3.3). The estimates (3.4)–(3.7) follow immediately. $\square$
In the sequel, $C_k > 0$, $k \in \mathbb{N}$, will always denote constants that may depend on T and the data of the system, but not on $s \in U^{ad}$. We have the following stability result.

Theorem 3.2 *The mapping $\mathcal{S}$, which assigns to each $s \in U^{ad}$ the solution pair $(\theta\,,\,u)$, is Lipschitz continuous from $C^1[0,d]$ into $(C([0,T];H) \cap L^2(0,T;W) \cap H^1(0,T;W'))$
$\times\,(C([0,T];V) \cap C^1([0,T];H) \cap H^2(0,T;V'))$.*

Proof: Let $s_i \in U^{ad}$, $(\theta_i, u_i) = \mathcal{S}(s_i)$, $i = 1,2$, and $(\bar\theta,\bar u) = (\theta_1 - \theta_2, u_1 - u_2)$. Then $(\bar\theta,\bar u)$ is a solution to the system

$$
\begin{aligned}
(\bar\theta_t(t)\,,\,w)_Q + a(s_1;\bar\theta(t),w) &= a(s_2;\theta_2(t),w) - a(s_1;\theta_2(t),w) \\
&\quad + \gamma(s_1;g(t),w) - \gamma(s_2;g(t),w)\,, \quad \forall\, w \in W\,, && (8)\\
(\bar u_{tt}(t)\,,\,v)_Q + b(s_1;\bar u(t),v) &= b(s_2;u_2(t),v) - b(s_1;u_2(t),v) \\
&\quad + (G(s_1)\nabla\bar\theta(t)\,,\,v)_Q \\
&\quad + ((G(s_1) - G(s_2))\nabla\theta_2(t)\,,\,v)_Q\,, \quad \forall\, v \in V\,, && (9)\\
\bar\theta(0) = 0\,, &\qquad \bar u = \bar u_t(0) = 0\,. && (10)
\end{aligned}
$$

For almost every $t \in (0,T)$, the right–hand side of (3.8) defines a linear continuous functional $F_1(t)$ on W,

$$
\begin{aligned}
F_1(t)(w) &= \int_Q \langle \nabla\theta_2(t,\xi)\,,\,(D(s_2)(\xi) - D(s_1)(\xi))\,\nabla w(\xi)\rangle\,d\xi \\
&+ \int_0^d \Big(s_2(\xi_1)^{-1}\sqrt{1+(s_2'(\xi_1))^2} - s_1(\xi_1)^{-1}\sqrt{1+(s_1'(\xi_1))^2}\Big) \\
&\qquad \cdot\,\theta_2(t,\xi_1,1)\,w(\xi_1,1)\,d\xi_1 \\
&- \int_Q \langle \nabla\theta_2(t,\xi)\,,\,d(s_2)(\xi) - d(s_1)(\xi)\rangle\,w(\xi)\,d\xi \\
&+ \int_0^d \Big(s_1(\xi_1)^{-1}\sqrt{1+(s_1'(\xi_1))^2} - s_2(\xi_1)^{-1}\sqrt{1+(s_2'(\xi_1))^2}\Big) \\
&\qquad \cdot\,g(t,\xi_1,1)\,w(\xi_1,1)\,d\xi_1. && (11)
\end{aligned}
$$

The set U^{ad} is bounded in $C^1[0,d]$. Moreover, the entries of $D(s)$ and $d(s)$ and the mapping $s \mapsto s^{-1}\sqrt{1+(s')^2}$ are Lipschitz continuous from $C^1[0,d]$ into $C[0,d]$. Therefore, it is easy to check that

$$
|F_1(t)w| \le C_1\,\|\theta_2(t)\|_{1,Q}\,\|s_1 - s_2\|_{C^1}\,\|w\|_{1,Q}, \qquad (12)
$$

whence, using (3.4),

$$\int_0^T \|F_1(t)w\|_{1,Q}^2\,dt \le C_2\,\|s_1 - s_2\|_{C^1}^2\,. \tag{13}$$

Consquently, by Theorem 3.1,

$$\max_{0\le t\le T} \|\bar\theta(t)\|_{0,Q}^2 + \int_0^T \Big(\|\bar\theta(t)\|_{1,Q}^2 + \|\bar\theta_t(t)\|_{W'}^2\Big)\,dt \le C_3\,\|s_1 - s_2\|_{C^1}^2\,. \tag{14}$$

Equation (3.9) can be handled in a similar way. We have

$$\begin{aligned}\int_0^T |(G(s_1)\nabla\bar\theta(t)\,,\,v)_Q|^2\,dt &\le C_4\int_0^T \|\bar\theta(t)\|_{1,Q}^2\,dt\,\|v\|_{0,Q}^2\\ &\le C_5\,\|s_1-s_2\|_{C^1}^2\,\|v\|_{0,Q}^2\,.\end{aligned} \tag{15}$$

Then, as above,

$$\max_{0\le t\le T}\Big(\|u(t)\|_{1,Q}^2 + \|u_t(t)\|_{0,Q}^2\Big) + \int_0^T \|u_{tt}(t)\|_{V'}^2\,dt \le C_6\,\|s_1 - s_2\|_{C^1}^2\,. \tag{16}$$

With this, the assertion is proved. □

Next, we construct the directional derivative of $\mathcal{S}$ with respect to s. To this end, let $s_o \in U^{ad}$ be fixed. We introduce the derivatives

$$a'(s_o,\sigma;\theta,w) = \lim_{\lambda\downarrow 0}\lambda^{-1}(a(s_o+\lambda\sigma;\theta,w) - a(s_o;\theta,w))\,, \tag{17}$$

$$b'(s_o,\sigma;u,v) = \lim_{\lambda\downarrow 0}\lambda^{-1}(b(s_o+\lambda\sigma;u,v) - b(s_o;u,v))\,, \tag{18}$$

and, analogously, $\gamma'(s_o,\sigma;g,v)$, where $\sigma\in C^{1,\nu}[0,d]$ is arbitrary but fixed. One needs only some formal calculations to confirm that

$$\begin{aligned}a'(s_o,\sigma;\theta,w) &= \int_Q\langle\nabla\theta\,,\,(D'(s_o)\sigma)\nabla w\rangle\,d\xi - \int_Q\langle\nabla\theta\,,\,d'(s_o)\sigma\rangle\,w\,d\xi\\ &\quad + \int_{\Gamma_3}\theta\,w\,l'(s_o)\sigma\,d\xi\,,\end{aligned} \tag{19}$$

where

$$\begin{aligned}D'(s_o)\sigma &= s_o^{-2}\begin{pmatrix}0 & \xi_2\,s_o'\\ \xi_2\,s_o' & -2\,s_o^{-1}\,(1+(\xi_2 s_o')^2)\end{pmatrix}\sigma\\ &\quad + s_o^{-2}\begin{pmatrix}0 & -\,\xi_2\,s_o\\ -\,\xi_2\,s_o & 2\,\xi_2\,s_o'\end{pmatrix}\sigma'\,,\end{aligned} \tag{20}$$

$$d'(s_o)\sigma = s_o^{-2}\begin{pmatrix}s_o'\\ 2\,\xi_2\,s_o^{-1}\,(s_o')^2\end{pmatrix}\sigma + s_o^{-2}\begin{pmatrix}s_o\\ -2\,\xi_2\,s_o'\end{pmatrix}\sigma'\,, \tag{21}$$

$$l'(s_o)\sigma = -s_o^{-2}\sqrt{1+(s_o')^2}\;\sigma + \frac{s_o^{-1}\,s_o'}{\sqrt{1+(s_o')^2}}\,\sigma'\,. \tag{22}$$

Moreover,

$$\gamma'(s_o,\sigma;g(t),w) = \int_{\Gamma_3} g(t)\,w\,l'(s_o)\sigma\,d\xi\,, \tag{23}$$

and, with the matrix E defined in (2.27),

$$\begin{aligned} b'(s_o,\sigma;u,v) \;=\; & \int_Q \langle \varepsilon'(s_o,u)\sigma\,,\,E\,\varepsilon(v)\rangle\,d\xi \;+\; \int_Q \langle \varepsilon(u)\,,\,E\,\varepsilon'(s_o,u)\sigma\rangle\,d\xi \\ & - \int_Q s_o^{-1}s_o'\,\langle\,(v^1,0,\frac{1}{2}\,v^2)^T\,,\,E\,\varepsilon'(s_o,u)\sigma\,\rangle\,d\xi \\ & - \int_Q (-s_o^{-2}s_o'\sigma + s_o^{-1}\sigma')\,\langle\,(v^1,0,\frac{1}{2}\,v^2)^T\,,\,E\,\varepsilon(u)\,\rangle\,d\xi\,, \end{aligned} \tag{24}$$

where $\varepsilon(u) = (\tilde\varepsilon_{11}(u),\tilde\varepsilon_{22}(u),\tilde\varepsilon_{12}(u))^T$, and

$$\varepsilon'(s_o,u)\sigma \;=\; \begin{pmatrix} \xi_2\,s_o^{-2}\,s_o'\,\frac{\partial u^1}{\partial\xi_2} \\ -\,s_o^{-2}\,\frac{\partial u^2}{\partial\xi_2} \\ \frac{1}{2}\,s_o^{-2}\left(\frac{\partial u^1}{\partial\xi_2}\;-\;s_o'\,\frac{\partial u^2}{\partial\xi_2}\right) \end{pmatrix}\sigma \;+\; \begin{pmatrix} -\,\xi_2\,s_o^{-1}\,\frac{\partial u^1}{\partial\xi_2} \\ 0 \\ -\,\frac{1}{2}\,\xi_2\,s_o^{-1}\,\frac{\partial u^2}{\partial\xi_2} \end{pmatrix}\sigma'\,. \tag{25}$$

Finally,

$$G'(s_o)\,\sigma \;=\; \begin{pmatrix} 0 & 0 \\ -\,\xi_2\,s_o^{-2}\,s_o' & s_o^{-2} \end{pmatrix}\sigma \;+\; \begin{pmatrix} 0 & 0 \\ \xi_2\,s_o^{-1} & 0 \end{pmatrix}\sigma'\,. \tag{26}$$

We have the following differentiability result.

Theorem 3.3 *Let $s_o \in U^{ad}$ and $\sigma \in C^{1,\nu}[0,d]$. Then the operator $\mathcal{S}$ is weakly differentiable in the direction σ as mapping from $C^{1,\nu}[0,d]$ into the Banach space*

$$\begin{aligned} B \;=\; & \big(L^2(0,T;H)\cap L^2(0,T;W)\cap H^1(0,T;W')\big) \\ & \times\big(L^2(0,T;V)\cap H^1(0,T;\mathbf{H})\cap H^2(0,T;V')\big)\,. \end{aligned} \tag{27}$$

The (weak) directional derivative $(\delta\theta,\delta u)$ of $\mathcal{S}$ in the direction σ is given as the (unique) solution to the initial–boundary value problem

$$\begin{aligned} (\,(\delta\theta)_t(t)\,,\,w\,)_Q \;+\; a(s_o;(\delta\theta)(t),w) \;=\; & \gamma'(s_o,\sigma;g(t),w) \\ -\,a'(s_o,\sigma;\theta_o(t),w)\,,\quad & \forall\,w\in W \;\; \textit{and a. e.}\;\; t\in(0,T)\,, \end{aligned} \tag{28}$$

$$\begin{aligned} (\,(\delta u)_{tt}(t)\,,\,v\,)_Q \;\;+\;\; b(s_o;(\delta u)(t),v) \;=\; & ((G'(s_o)\sigma)\,\nabla\theta_o(t)\,,\,v)_Q \\ +\;\;(G(s_o)\,\nabla(\delta\theta)(t)\,,\,w)_Q \;-\; & b'(s_o,\sigma;u_o(t),v)\,, \\ & \forall\,v\in V \;\; \textit{and a. e.}\;\; t\in(0,T)\,, \end{aligned} \tag{29}$$

$$(\delta\theta)(0)=0\,,\quad (\delta u)(0)=0\,,\quad (\delta u)_t(0)=0\,. \tag{30}$$

Proof: At first, note that the right–hand sides of (3.28) and (3.29) belong to $L^2(0,T;H)$ and $L^2(0,T;\mathbf{H})$, respectively. By the standard theory of linear parabolic and hyperbolic equations (cf. Lions–Magenes [4]), the solution pair $(\delta\theta,\delta u)$ satisfies (3.2) and (3.3). Next, let $(\theta_\lambda,u_\lambda)=\mathcal{S}(s_\lambda)$, where $s_\lambda=s_o+\lambda\,\sigma$, for $0\le\lambda<\bar\lambda$. By Theorem 3.2,

$$\sup_{0\le\lambda<\bar\lambda}\left\|\lambda^{-1}\,(\theta_\lambda-\theta_o\,,\,u_\lambda-u_o)\right\|_B\le C<+\infty\,. \tag{31}$$

Hence, a subsequence, still denoted $\{\lambda^{-1}\,(\theta_\lambda-\theta_o\,,\,u_\lambda-u_o)\}$, converges weakly in B to some limit point $(\tilde\theta,\tilde u)\in B$. Obviously, $z_\lambda:=\lambda^{-1}\,(\theta_\lambda-\theta_o)$ is a solution to

$$\begin{aligned}\frac{d}{dt}(z_\lambda(t)\,,\,w)_Q\,+\,a(s_o;z_\lambda(t),w)&=-\,\lambda^{-1}\left(a(s_o+\lambda\sigma;\theta_\lambda(t),w)-a(s_o;\theta_\lambda(t),w)\right)\\&\quad+\lambda^{-1}\left(\gamma(s_o+\lambda\sigma;g(t),w)-\gamma(s_o;g(t),w)\right),\quad\forall\,w\in W\ \text{and a. e.}\ t\in(0,T)\,,\end{aligned}\tag{32}$$

$$z_\lambda(0)=0\,. \tag{33}$$

Letting $\lambda\to 0+$, we find that $\tilde\theta$ solves (3.28), and $\tilde\theta(0)=0$. Analogous reasoning shows that $(\tilde\theta,\tilde u)$ solves (3.29), and also $\tilde u(0)=\tilde u_t(0)=0$. By the unique solvability of the system (3.28)–(3.30), it follows that $(\tilde\theta,\tilde u)=(\delta\theta,\delta u)$ and, in addition, that the whole sequence $\{\lambda^{-1}\,(\theta_\lambda-\theta_o,u_\lambda-u_0)\}$ converges weakly in B to $(\delta\theta,\delta u)$. The assertion is proved. □

4 Existence of an optimal solution and first order necessary optimality conditions

The transformed optimal shape design problem is to minimize

$$\begin{aligned}\tilde J(s)=\tilde J_1(s)+\tilde J_2(s)\;&=\;\int_Q\tilde q_1(\xi)|u(T,\xi)-\tilde q_2(\xi)|^2 s(\xi_1)\,d\xi\\&+\;\int_0^d q_3(\xi_1)(s(\xi_1)+u^2(T,\xi_1,1)-\hat s(\xi_1))^2\,d\xi_1\end{aligned}\tag{1}$$

subject to $s\in U^{ad}$ and to the system (2.6)–(2.7), (2.16)–(2.18) defining the state (θ,u), where $\tilde q_i(\xi):=q_i(\Phi(\xi)),i=1,2$.

Theorem 4.1 *There exists at least one optimal solution $s_o\in U^{ad}$ of the optimal shape design problem.*

Proof: U^{ad} is a compact subset of $C^1[0,d]$. The mapping $S:s\mapsto(\theta,u)$ is continuous from $C^1[0,d]$ into the state-space under consideration (theorem 3.2), and $\tilde J$ is continuous on this space. Thus the result follows from theorem 3.2 and the Weierstrass theorem. □

As before, let (θ_o, u_o) denote the (transformed) optimal state corresponding to s_o. The directional derivative of $\tilde{J}$ at s_o is

$$\begin{aligned}\lim_{\lambda \downarrow 0} \lambda^{-1}(J(s_o + \lambda\sigma) - J(s_o)) &= \int_Q \{c_o(\xi)\sigma(\xi_1) + d_o(\xi)^T(\delta u)(T,\xi)\}\, d\xi \\ &+ \int_0^d c_1(\xi_1)\{\sigma(\xi_1) + (\delta u^2)(T,\xi_1,1)\}\, d\xi_1, \quad (2)\end{aligned}$$

where δu is defined by (3.28)–(3.30) and

$$\begin{aligned}c_o(\xi) &= \tilde{q}_1(\xi)|u_o(T,\xi) - \tilde{q}_2(\xi)|^2 &(3)\\ d_o(\xi) &= 2\tilde{q}_1(\xi)(u_o(T,\xi) - \tilde{q}_2(\xi)) &(4)\\ c_1(\xi_1) &= 2q_3(\xi_1)(s_o(\xi_1) + u_o^2(T,\xi_1,1) - \hat{s}(\xi_1)) &(5)\end{aligned}$$

Lemma 4.2 *(Linearization) Let s_o be optimal for the shape design problem and (θ_o, u_o) be the (transformed) optimal state. Then*

$$\begin{aligned}&\int_Q \{c_o(\xi)\sigma(\xi_1) + d_o(\xi)^T u(T,\xi)\}\, d\xi \\ &+ \int_0^d c_1(\xi_1)\{\sigma(\xi_1) + u^2(T,\xi_1,1)\}\, d\xi_1 \geq 0 \qquad (6)\end{aligned}$$

for all $\sigma \in U^{ad} - s_o$ and θ, u, satisfying the system

$$\begin{aligned}(\theta_t(t)\,,\, w)_Q + a(s_o;\theta(t),w) &= \gamma'(s_o,\sigma;g(t),w) - a'(s_o,\sigma;\theta_o(t),w) \qquad (7)\\ \forall w \in W \text{ and a.e. } t \in (0,T),& \\ (u_{tt}(t)\,,\, v)_Q + b(s_o;u(t),v) &= ((G'(s_o)\sigma)\nabla\theta_o(t)\,,\, v)_Q \\ &\quad -b'(s_o,\sigma;u_o(t),v) + (G(s_o)\nabla\theta(t)\,,\, v)_Q \quad (8)\\ \forall v \in V \text{ and a.e. } t \in (0,T),&\end{aligned}$$

and the homogeneous initial conditions

$$\theta(0) = 0, u(0) = 0, u_t(0) = 0. \qquad (9)$$

Proof: U^{ad} is convex, hence $s_o + \lambda(s - s_o) =: s_\lambda$ belongs to U^{ad} for all $\lambda \in (0,1)$. Let $\sigma = s - s_o, u_\lambda = S(s_\lambda)$. Then

$$\lim_{\lambda \downarrow 0} \lambda^{-1}(J(s_o + \lambda\sigma) - J(s_o)) \geq 0. \qquad (10)$$

Therefore, the right hand side of (4.2) must be nonnegative. Note that

$$\delta u(T) = \lim_{\lambda \downarrow 0} \lambda^{-1}(u_\lambda(T) - u_o(T))$$

and its trace on Γ are well defined, as this limit exists in $C([0,T];V)$. The statement of the lemma follows after setting $\theta := \delta\theta, u := \delta u$. □

Now we introduce the *adjoint state* $(p,\Psi) \in C([0,T];H) \cap L^2(0,T;W) \cap H^1(0,T;W') \times C([0,T];V) \cap C^1([0,T];\mathbf{H}) \cap H^2(0,T;V')$ as the solution of the *adjoint system*

$$\begin{aligned}
-(p_t(t)\,,\,w)_Q + a(s_o;w,p(t)) &= -(\Psi_t(t)\,,\,G(s_o)\nabla w)_Q && (11)\\
\forall w \in W \text{ and a.e. } t \in (0,T), & \\
p(T) &= 0 && (12)\\
(\Psi_{tt}(t)\,,\,v)_Q + b(s_o;v,\Psi(t)) &= (d_o\,,\,v)_Q + (\underset{\sim}{c}_1\,,\,v)_{\Gamma_3} && (13)\\
\forall v \in V \text{ and a.e. } t \in (0,T), & \\
\Psi(T) &= 0 && (14)\\
\Psi_t(T) &= 0, && (15)
\end{aligned}$$

where $\underset{\sim}{c}_1 = (0,c_1)^T$. The existence of (p,Ψ) follows from theorem 3.1. Note that $\Psi_t \in C([0,T],H)$.

Theorem 4.3 *If s_o is optimal for the optimal shape design problem, then for all $s \in U^{ad}$*

$$\begin{aligned}
&\int_Q c_o(\xi)(s(\xi_1) - s_o(\xi_1))\,d\xi + \int_0^d c_1(\xi_1)(s(\xi_1) - s_o(\xi_1))\,d\xi_1\\
&+\int_0^T \{\gamma'(s_o, s-s_o; g(t), p(t)) - a'(s_o, s-s_o; \theta_o(t), p(t))\\
&-((G'(s_o)(s-s_o))\nabla\theta_o(t)\,,\,\Psi_t(t))_Q + b'(s_o, s-s_o; u_o(t), \Psi_t(t))\}\,dt \geq 0 \qquad (16)
\end{aligned}$$

Proof: We insert $w = \theta(t), v = -u_t(t)$ in the system (4.11)– (4.15) and $w = -p(t), v = \Psi_t(t)$ in (4.7)–(4.9). Next, we add the equations (4.7), (4.8), and (4.11), subtract (4.13) and integrate the resulting expression over $[0,T]$. Then

$$\begin{aligned}
&\int_0^T \{-(\theta_t(t)\,,\,p(t))_Q - (p_t(t),\theta(t))_Q + (u_{tt}\,,\,\Psi_t(t))_Q + (\Psi_{tt}(t)\,,\,u_t(t))_Q\}\,dt\\
&+\int_0^T \{b(s_o;u(t),\Psi_t(t)) + b(s_o;u_t(t),\Psi(t))\}\,dt\\
=\;&\int_0^T \{-\gamma'(s_o, s-s_o; g(t), p(t)) + a'(s_o, s-s_o; \theta_o(t), p(t))\\
&+((G'(s_o)(s-s_o))\nabla\theta_o(t)\,,\,\Psi_t(t))_Q - b'(s_o, s-s_o; u_o(t), \Psi_t(t))\\
&+(d_o\,,\,u_t(t))_Q + (\underset{\sim}{c}_1\,,\,u_t(t))_{\Gamma_3}\}\,dt. \qquad (17)
\end{aligned}$$

Integrating by parts the first, third, and fifth expression we arrive at

$$\begin{aligned}
&-(\theta(t)\,,\,p(t))_Q|_0^T + (u_t(t)\,,\,\Psi_t(t))_Q|_0^T + b(s_o;u(t),\Psi(t))|_0^T\\
&= \int_0^T \{-\gamma' + a' + ((G'(s-s_o)\nabla\theta_o, \Psi_t)_Q - b' + (d_o, u_t)_Q + (\underset{\sim}{c}_1, u_t)_{\Gamma_3}\}\,dt \qquad (18)
\end{aligned}$$

Owing to the homogeneous initial and final time conditions the left hand side of (4.18) is vanishing, hence

$$\int_0^T \{\gamma' - a' + b' - ((G'(s-s_o)\nabla\theta_o, \Psi_t)_Q\}\,dt = (d_o, u(T))_Q + (\underset{\sim}{c}_1, u(T))_{\Gamma_3}. \quad (19)$$

Inserting (4.19) in the inequality (4.6) yields the relation (4.16). $\square$

Remark: In the case $q_3 = 0$, where the boundary integral J_2 is missing, the necessary conditions can be simplified.

We put $y(t) := \Psi_t(t)$. Then the integral over $[0,T]$ in (4.16) admits the form

$$\begin{aligned}\textstyle\int_0^T\{...\}dt &= \textstyle\int_0^T\{\gamma'(s_o, s-s_o; g(t), p(t)) - a'(s_o, s-s_o, \theta_o(t), p(t)) \\ &\quad -((G'(s_o)(s-s_o))\nabla\theta_o(t))\,,\, y(t))_Q + b'(s_o, s-s_o; u_o(t), y(t))\}\,dt,\end{aligned}$$

and y satisfies the adjoint equation

$$\begin{aligned}(y_{tt}(t)\,,\,v)_Q + b(s_o; v, y(t)) &= 0 \qquad (20)\\ \forall v\in V \text{ and a.e. } t\in(0,T), & \\ y(T) &= 0 \qquad (21)\\ y_t(T) &= d_o. \qquad (22)\end{aligned}$$

(Differentiation of (4.13) gives $\quad (y_{tt}, v)_Q + b(s_o; v, y) = 0 \quad$, (4.15) implies $y(T) = 0$. Moreover, (4.13) taken at $t = T$, leads to

$$(\Psi_{tt}(T)\,,\,v)_Q = (d_o\,,\,v)_Q \quad \forall v \in V \qquad (23)$$

(note that $\underline{c}_1 = 0$). Now (4.22) is a simple consequence).

Analyzing the *variational inequality* (4.16) we see that it admits the form

$$\int_0^d \{\phi_1(\xi_1)(s(\xi_1) - s_o(\xi_1)) + \phi_2(\xi_1)(s'(\xi_1) - s_o'(\xi_1))\}\,d\xi_1 \geq 0 \;\; \forall s \in U^{ad}, \qquad (24)$$

where ϕ_1, ϕ_2 are certain functions depending on θ_o, u_o, Ψ, and p. Thus s_o must solve a certain control problem for a linear ordinary differential equation with constraints given by (1.1)–(1.4).

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