Numerical Analysis of State-Constrained Optimal Control Problems for PDEs

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Abstract. We survey the results of SPP 1253 project "Numerical Analysis of State-Constrained Optimal Control Problems for PDEs ". In the first part, we consider Lavrentiev-type regularization of both distributed and boundary control. In the second part, we present a priori error estimates for elliptic control problems with finite dimensional control space and state-constraints both in finitely many points and in all points of a subdomain with nonempty interior.

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1. Introduction

Pointwise state constraints play an important role in many real world applications of PDE optimization. For instance, in optimizing the process of hot steel profiles by spraying water on their surface, the temperature differences in the steel must be bounded in order to avoid cracks. Details may be found for example in [12]. Similar restrictions apply in the production process of bulk single crystals, where the temperature in the growth apparatus must be kept between given bounds, see e.g. [27]. Even in medical applications, pointwise state constraints can be important, as for example in local hyperthermia in cancer treatment. There, the generated temperature in the patient's body must not exceed a certain limit, cf. [11].

All these problems share the mathematical difficulties associated with the presence of pointwise state constraints. One of the related challenges lies in the question of existence and regularity of Lagrange multipliers. For these reasons, we are interested in regularization methods for state constrained problems, where we focus here on time dependent parabolic problems. In particular, we address a

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Lavrentiev type regularization method. The low regularity of Lagrange multipliers also presents a challenge in the numerical analysis when e.g. trying to derive a priori discretization error estimates. We discuss this for two classes of elliptic state-constrained optimal control problems with finitely many real numbers as control variables that we discuss without regularization.

Let us survey some difficulties and questions we have been interested in with the help of two model problems. We will consider optimal control problems, respectively their regularization, of parabolic type with control u and state y, in the spatial domain $\Omega \subset \mathbb{R}^n$ and a time interval $(0,T)$. For simplicity, we introduce the time-space cylinder $Q := (0,T) \times \Omega$ and its boundary $\Sigma = (0,T) \times \partial\Omega$. We consider the distributed control problem

$$(P_D) \ \min J(y,u) := \frac{1}{2} \iint\limits_Q (y - y_d)^2 \, dxdt + \frac{\nu}{2} \iint\limits_Q u^2 \, dxdt$$

subject to the semilinear heat equation

$$\begin{aligned} \partial_t y - \Delta y + d(x,t,y) &= u && \text{in } Q \\ y(\cdot,0) &= y_0 && \text{in } \Omega \\ \partial_n y + \alpha y &= g && \text{on } \Sigma, \end{aligned}$$

the pointwise state constraints

$$y_a \le y \le y_b \quad \text{in } Q,$$

and optional control constraints

$$u_a \le u \le u_b \quad \text{in } Q.$$

Note that $d(x,t,y) \equiv 0$ can be considered to analyze a linear quadratic case. Moreover, it is possible to consider a more general elliptic differential operator A, as well as a more general objective function under certain conditions.

In addition, we are interested in boundary control problems of the form

$$(P_B) \ \min J(y,u) := \frac{1}{2} \iint\limits_Q (y - y_d)^2 \, dxdt + \frac{\nu}{2} \iint\limits_\Sigma u^2 \, dsdt$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= f && \text{in } Q \\ y(\cdot,0) &= y_0 && \text{in } \Omega \\ \partial_n y + \alpha y &= u && \text{on } \Sigma, \end{aligned}$$

and the pointwise state constraints

$$y_a \le y \le y_b \quad \text{in } Q,$$

without control constraints. All appearing data is supposed to fulfill typical regularity assumption, and the boundary $\partial\Omega$ is as smooth as desired. The nonlinearity d appearing in the state equation governing (P_D) is assumed to fulfill standard Carathéodory type conditions as well as monotonicity and smoothness, so that for given control u in either $L^\infty(Q)$ or $L^\infty(\Sigma)$ the existence of a unique corresponding state $y(u) \in W(0,T)$ is guaranteed. For a precise formulation of the given setting, we refer to [32], where linear-quadratic problems without control constraints of distributed and boundary control type have been considered, as well as to [31], which is concerned with semilinear distributed control problems with state- and control-constraints.

As mentioned above, the presence of pointwise state constraints leads to difficulties in the analysis and the numerical solution of the problems. One issue is the existence of Lagrange multipliers in order to formulate first-order necessary optimality conditions of Karush-Kuhn-Tucker type. The most common approach is to assume Slater-type conditions. To apply them to pointwise state constraints, the cone of nonnegative functions must have a nonempty interior. This requires continuity of the state functions, because in L^p-spaces with $1 \leq p < \infty$ the cone of nonnegative functions has empty interior, while for $p = \infty$ the dual space is not useful. Depending on the type of problem, however, continuity is not always guaranteed. If, for example, no bounds on the control are given and the control u belongs only to $L^2(Q)$, then the continuity of the associated state y in (P_D) is only granted for spatially one-dimensional domains, cf. for example [4] or the exposition in [39] for associated regularity results. For u in $L^2(\Sigma)$, the parabolic equation in the boundary control problem (P_B) does not generally admit a continuous state y, not even if $\Omega \subset \mathbb{R}$. Therefore, these problems are not well-posed a priorily in the sense that first order necessary optimality conditions of KKT type can be formulated in useful spaces. Of course, a problem may admit a bounded control in L^∞ and an associated continuous state, but this is not clear in advance. Even if Lagrange multipliers do exist, due to the Slater point arguments they are generally only obtained in the space of regular Borel measures. It turns out that regularization concepts are useful to obtain an optimal control problem with more regular Lagrange multipliers in L^p-spaces.

Another difficulty is hidden in the formulation of second-order-sufficient conditions (SSC), which are of interest for nonlinear optimal control problems. While they can be expected for regularized problem formulations, the purely state-constrained parabolic case remains challenging even in cases where Lagrange multipliers exist due to L^∞- bounds as control constraints. For spatio-temporal control functions and pointwise state constraints given in the whole domain Q, a satisfactory theory of SSC is so far only available for one-dimensional distributed control problems, cf. [35], [7]. As part of the research in the SPP, SSC for unregularized problems have been established for higher dimensions in the special setting with finitely many time-dependent controls that are found in practice more often than controls that can vary freely in space and time, cf. [8].

For all these reasons, regularization techniques have been a wide field of active research in the recent past and remain to be of interest. We mention for example a Moreau-Yosida regularization approach by Ito and Kunisch, [20], a Lavrentiev-regularization technique by Meyer, Rösch, and Tröltzsch, [28], or the virtual control concept by Krumbiegel and Rösch, [22], originally developed for elliptic boundary control problems during the first funding period of SPP 1253. Moreover, barrier methods, cf. [38], can be interpreted as regularization methods. We also point out comparisons between different approaches as in e.g. [1], the analysis of solution algorithms as in e.g. [15], [17], [18], or discretization error estimates from [14] or [16]. In addition, a combination of Lavrentiev regularization and interior point methods as for example in [33] has been considered. Here, Lavrentiev regularization is used to prove that the barrier method is indeed an interior point method. We lay out in Section 2 how Lavrentiev regularization techniques can be transferred to parabolic control problems, and describe an extension to parabolic boundary control problems. In addition, we comment on additional helpful properties of regularized problems that for example allow to prove a local uniqueness result of local solutions of nonlinear optimal control problems, which is an important property in the context of solving optimal control problems numerically.

A further leading question in the SPP 1253 were error estimates for the numerical approximation of state constrained control problems. Only few results on elliptic problems were known for pointwise state constraints in the whole domain. In [5], [6] convergence of finite element approximations to optimal control problems for semilinear elliptic equations with finitely many state constraints was shown for piecewise constant approximations of the control. Error estimates for elliptic state constrained distributed control functions have been derived in [9] and, with additional control constraints, in [26], [10]. Since in many applications the controls are given by finitely many real parameters, another goal of our SPP project was to investigate the error for associated state-constrained elliptic control problems. Unlike in problems with control functions, the treatment of the finitely many control parameters does not require special attention, and an error estimate without a contribution to the error due to control discretization is automatically obtained. For control *functions*, the same property is exploited by the so called variational discretization, cf., e.g., [19]. Moreover, in problems with only finite-dimensional controls, it is not exceptional that a state constraint is only active in finitely many points. If the location of these points is known approximately, it is reasonable to prescribe the constraints in these approximate points. As part of the project and a first step towards discussing state constraints in the whole domain, such problems with finitely many state constraints have therefore been considered in [25]. The resulting optimal control problems are equivalent to finite dimensional mathematical programming problems. Yet, the associated error analysis is not trivial. Maximum norm estimates for the finite element approximation of the semilinear state equation had to be derived. Then results of the stability analysis of nonlinear optimization were applicable to obtain also estimates for the Lagrange multipliers, which are a linear combination of Dirac measures. We survey the results in Section 3.1. If it is necessary to consider the constraints in a subset of Ω with nonempty interior, then the elliptic control problem with finite-dimensional controls is of semi-infinite type. We completed the discussion by considering such elliptic problems and report on this in Section 3.2.

2. Regularization of parabolic state-constrained problems

Let us mention here first that Lavrentiev regularized problems with additional control constraints require a more involved analysis than problems with pure regularized state constraints, and additional assumptions have to be imposed. Therefore, we consider these situations separately, beginning with the purely state-constrained case.

2.1. Problems without control constraints

2.1.1. Distributed control problems. We consider here a linear-quadratic distributed control problem, i.e. consider (P_D) with $d(x,t,y) \equiv 0$. The idea of Lavrentiev regularization for distributed control problems is to replace the pure state constraints by mixed control-state constraints of the form

$$y_a \leq \lambda u + y \leq y_b \quad \text{a.e. in } Q,$$

where $\lambda \in \mathbb{R}$ is a small regularization parameter. Following [29], the existence of regular Lagrange multipliers in $L^2(Q)$ for arbitrary dimension of Ω is easily shown using a simple substitution technique. The idea is to introduce the new control $w := \lambda u + y$, which yields a purely control-constrained optimal control problem. More precisely, from $w = \lambda u + y$ we obtain $u = (w-y)/\lambda$ so that the state equation can be rewritten as

$$y_t - \Delta y + d(x,t,y) + \lambda^{-1} y = \lambda^{-1} w \quad \text{in } Q,$$

and the objective function can be transformed into

$$\tilde{J}(y,w) := \frac{1}{2}\|y - y_d\|^2_{L^2(Q)} + \frac{\nu}{2\lambda^2}\|w - y - \lambda u_d\|^2_{L^2(Q)}.$$

Then, to prove the existence of regular multipliers of the transformed control constrained problem with

$$y_a \leq w \leq y_b \quad \text{in } Q$$

and hence of the Lavrentiev regularized problem is standard technique. They are also multipliers for the original state constraints, and are obtained without any Slater condition.

Theorem 2.1. *For each $\lambda > 0$, the linear-quadratic Lavrentiev regularized version of (P_D) admits a unique optimal control $\bar{u}_\lambda$ with associated state $\bar{y}_\lambda$. For arbitrary spatial dimension n, there exist Lagrange multipliers $\mu_a^\lambda, \mu_b^\lambda \in L^2(Q)$ and an adjoint state $p_\lambda \in W(0,T)$ such that:*

$$\begin{array}{rclrcl} \partial_t \bar{y}_\lambda - \Delta \bar{y}_\lambda &=& \bar{u}_\lambda & \partial_t p_\lambda - \Delta p_\lambda &=& \bar{y}^\lambda - y_d + \mu_b^\lambda - \mu_a^\lambda \\ \bar{y}_\lambda(\cdot,0) &=& y_0 & p_\lambda(\cdot,T) &=& 0 \\ \partial_n \bar{y}_\lambda + \alpha \bar{y}_\lambda &=& g & \partial_n p_\lambda + \alpha p_\lambda &=& 0, \end{array}$$

$$\begin{array}{rl} \nu \bar{u}_\lambda + p_\lambda + \lambda(\mu_b^\lambda - \mu_a^\lambda) = 0 & \text{a.e. in } Q, \\ y_a \leq \lambda \bar{u}_\lambda + \bar{y}_\lambda \leq y_b & \text{a.e. in } Q, \end{array}$$

$$\begin{aligned}(\mu_a^\lambda, y_a - \lambda\bar{u}_\lambda - \bar{y}_\lambda)_{L^2(Q)} &= 0 \qquad \mu_a^\lambda \geq 0 \qquad \textit{a.e. in } Q\\ (\mu_b^\lambda, \lambda\bar{u}_\lambda + \bar{y}_\lambda - y_b)_{L^2(Q)} &= 0 \qquad \mu_b^\lambda \geq 0 \qquad \textit{a.e. in } Q.\end{aligned}$$

For details, we refer to [32]. This simple substitution technique cannot be adapted to the case of additional control constraints. It is, however, possible to show a multiplier rule with L^2-Lagrange multipliers for such problems under a certain separability assumption, cf. [36]. We will consider this type of problem later in Section 2.2. Let us also state a convergence result for the regularized solutions, again for the linear-quadratic version of (P_D).

Theorem 2.2. *Let $\{\lambda_n\}$ be a sequence of positive real numbers converging to zero and denote by $\{u_n\}$ the sequence of associated optimal solutions of the regularized control problem. For $N = 1$, the sequence $\{u_n\}$ converges strongly in $L^2(Q)$ towards $\bar{u}$, where $\bar{u}$ is the unique optimal solution of the unregularized problem. If the optimal control of the unregularized problem is a function in $L^\infty(Q)$ this holds also for dimension $N > 1$.*

The proof has been carried out in detail [32]. From a practical point of view, the boundedness assumption seems reasonable. Indeed, knowing that $\bar{u}$ is essentially bounded, artifical inactive bounds on the control u can be introduced in advance, such that the convergence result from [31] holds. If in a practical application the optimal control is unbounded, then most likely additional bounds on u must be posed.

2.1.2. Boundary control problems. While the distributed control problem (P_D) without control constraints is at least well-formulated in one-dimensional cases, the boundary control problem (P_B) may lack the existence of Lagrange multipliers in a suitable space as long as the optimal control is possibly unbounded. It is also quite obvious that the Lavrentiev regularization approach explained above cannot directly be applied to boundary control problems, since the control u and the state y are defined on different sets. In [32], we developed a Lavrentiev-type method for parabolic boundary control problems. Our motivation to extend the Lavrentiev regularization from the distributed case to treat such problems came from [2], where a well-known benchmark problem was introduced.

To treat the state equation in a concise way, we consider the control-to-state mapping $S\colon u \mapsto y$, $S\colon L^2(\Sigma) \to L^2(Q)$. The adjoint operator S^* maps $L^2(Q)$ into $L^2(\Sigma)$. We consider only controls u in the range of S^*, i.e. we introduce an auxiliary control $v \in L^2(Q)$ and set $u = S^*v$. Clearly, this is some smoothing of u, which is motivated by the optimality conditions for the unregularized problems, where we expect $\bar{u} = G^*\mu$ with some measure μ, if it exists. Then the state $y = y(v)$ is given by $y = SS^*v$ and the state constraints can be written as $y_a \leq SS^*v \leq y_b$. Now, we can apply our Lavrentiev regularization to these constraints, i.e. we consider

$$y_a \leq \lambda v + y(v) \leq y_b.$$

The regularizing effect comes from the restriction of $(C(\bar{Q}))^*$ to $L^2(Q)$ by the ansatz $u = S^*v$. This idea also turned out useful in the elliptic case, cf. [40].

Considering a reduced formulation of the optimal control problem in the control v, it is possible to prove first order optimality conditions with regular Lagrange multipliers, cf. [32].

Theorem 2.3. *Let $\bar{v}_\lambda \in L^2(Q)$ be the optimal control for the Lavrentiev-regularized version of (P_B) with associated boundary control $\bar{u}_\lambda$. Then there exist Lagrange multipliers $\mu_a^\lambda, \mu_b^\lambda \in L^2(Q)$ and adjoint states $p_\lambda, q_\lambda \in W(0,T)$ such that:*

$$
\begin{aligned}
\partial_t \bar{y}_\lambda - \Delta \bar{y}_\lambda &= f & \qquad -\partial_t z_\lambda - \Delta z_\lambda &= \bar{v}_\lambda \\
\bar{y}_\lambda(\cdot,0) &= y_0 & z_\lambda(\cdot,T) &= 0 \\
\partial_n \bar{y}_\lambda + \alpha \bar{y}_\lambda &= u & \partial_n z_\lambda + \alpha z_\lambda &= 0
\end{aligned}
$$

$$
\begin{aligned}
-\partial_t p_\lambda - \Delta p_\lambda &= \bar{y}_\lambda - y_d + \mu_b^\lambda - \mu_a^\lambda & \qquad \partial_t q_\lambda - \Delta q_\lambda &= 0 \\
p_\lambda(\cdot,T) &= 0 & q_\lambda(\cdot,0) &= 0 \\
\partial_n p_\lambda + \alpha p_\lambda &= 0 & \partial_n q_\lambda + \alpha q_\lambda &= \nu z_\lambda + p_\lambda
\end{aligned}
$$

$$
\begin{aligned}
(\mu_a^\lambda, y_a - \lambda \bar{v}_\lambda - \bar{y}_\lambda)_{L^2(Q)} &= 0, \qquad \mu_a^\lambda \geq 0 \\
(\mu_b^\lambda, \lambda \bar{v}_\lambda + \bar{y}_\lambda - y_b)_{L^2(Q)} &= 0, \qquad \mu_b^\lambda \geq 0
\end{aligned}
$$

$$
\varepsilon \bar{v}_\lambda + q_\lambda + \lambda(\mu_b^\lambda - \mu_a^\lambda) = 0.
$$

Here, z_λ is the solution of the adjoint equation for the ansatz $u = S^* v$, i.e. $u = z_{|\Sigma}$. The optimality system also shows the drawback of this approach, since there are twice as many PDEs to be solved as in the unregularized case. Nevertheless, the numerical results are quite satisfying, as we will see in the example of the Betts and Campbell heat transfer problem. Under the reasonable assumption that the optimal control $\bar{u}$ of the unregularized problem is bounded, or at least regular enough to guarantee continuity of the state, we obtain a convergence result for the regularized solution:

Theorem 2.4 ([32]). *Let $\bar{u}$ belong to $L^s(\Sigma)$, $s > N+1$, and let there exist a Slater point $v_0 \in C(\bar{Q})$, such that*

$$
y_a + \delta \leq G(\bar{u} + S^* v_0) \leq y_b - \delta,
$$

with a given $\delta > 0$, and select the regularization parameter ε by

$$
\varepsilon = c_0 \lambda^{1+c_1}, \qquad c_0 > 0, \quad 0 \leq c_1 < 1.
$$

Moreover, let $\lambda_n \to 0$ and $\{v_n\}_{n=1}^\infty$ be the sequence of optimal controls of the regularized version of (P_B). Then the sequence $\{S^* v_n\}$ converges strongly in $L^2(\Sigma)$ towards the solution $\bar{u}$ of the unregularized problem.

Using a primal dual active set strategy, we tested this regularization technique numerically in `Matlab` for the following Robin-boundary control problem that is motivated by the Betts and Campbell heat transfer problem:

$$\min \frac{1}{2} \int_0^5 \int_0^\pi y^2 dx dt + \frac{10^{-3}}{2} \int_0^T (u_1^2 + u_\pi^2) dt$$

subject to

$$\begin{array}{rcll} \partial y_t - \Delta y & = & 0 & \text{in } (0,\pi) \times (0,5) \\ y(x,0) & = & 0 & \text{in } (0,\pi) \\ -\partial_x y(0,t) + \alpha y(0,t) & = & \alpha u_1(t) & \text{in } (0,5) \\ \partial_x y(\pi,t) + \alpha y(\pi,t) & = & \alpha u_\pi(t) & \text{in } (0,5) \end{array}$$

$$\text{as well as} \quad y(x,t) \geq \sin(x) \sin(\frac{\pi t}{5}) - 0.7 \quad \text{in } (0,\pi) \times (0,5),$$

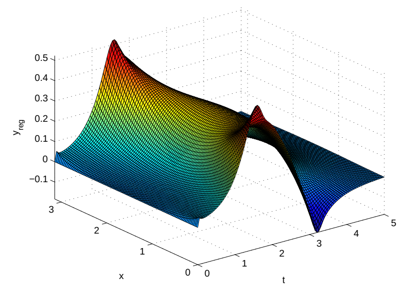
We obtained the optimal control shown in Figure 1(a), compared to the reference solution obtained by `Matlab`'s optimization routine `quadprog`, indicating that the regularization method works quite satisfying. The associated state is shown in Figure 1(b). Notice that the numerical results indicate that the optimal control is indeed bounded, and therefor this model problem is an example for problems that admit Lagrange multipliers in the unregularized case, even though this is not a priorily clear. We have also conducted experiments for the penalization technique by Ito and Kunisch, cf. [20], which yields similar results while solving only two PDEs in each iteration. We also point out the experiments in [30], were a modelling and simulation environment specialized for solving PDEs has been used. In contrast to unregularized state constraints, Lavrentiev regularization permits to make use of projection formulas for the Lagrange multipliers that are equivalent to the complementary slackness conditions and the non-negativity condition of the Lagrange multiplier, e.g.

$$\mu_a^\lambda = \max\left(0, \frac{\varepsilon}{\lambda^2}(y_a - \bar{y}_\lambda) + \frac{1}{\lambda} q_\lambda\right).$$

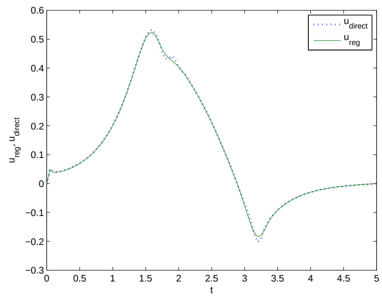
Then, the optimality system associated with the regularized version of (P_B), and in a similar way of (P_D), can be supplied in a symbolic way as a coupled system of PDEs to specialized PDE software. If all appearing functions can be handled by the software and a converged solution is returned, this is a time-efficient way to solve optimal control problems without specialized optimization routines and without much implementational effort. For the above example problem, we obtained satisfying results.

2.2. Lavrentiev regularized distributed control problems with additional control constraints

Let us now consider the semilinear version of (P_D) with additional control constraints. We have already mentioned that then Lagrange multipliers exist as regular Borel measures for the unregularized problem. However, the same dimensional limits as before are needed for a second-order analysis. After Lavrentiev regularization, a generalization of SSC to arbitrary dimensions should be possible in the



(a) Optimal state



(b) Optimal controls

spirit of [37]. Interestingly, there are some problems associated with the existence of regular Lagrange multipliers. It has been shown in [36] that regular multipliers exist under the assumption that the active sets associated with the different constraints are well-separated. If this assumption does not hold, Lagrange multipliers are only known to exist in the space $L^\infty(Q)^*$, which is even less regular than the space of regular Borel measures. Convergence of local solutions of the Lavrentiev regularized problem has been addressed in detail in [31], along with a global analysis of Moreau-Yosida regularized problems. There we also showed the following helpful result:

Theorem 2.5. *If a locally optimal control $\bar{u}_\lambda$ of the Lavrentiev regularized version of the distributed semilinear control problem (P_D) satisfies additionally a second order sufficient condition and if, for fixed λ, the active sets of the different constraints are strictly separated, then it is locally unique.*

Strict separation means that at most one constraint can be active or almost active in a given pair $(x,t) \in Q$. Local uniqueness of local solutions is important, since it excludes situations where a local minimum is an accumulation point of a sequence of other local optima. The proof is based on the verification of strong regularity. This property is also helpful for the analysis of SQP methods. We refer to [13] for an associated analysis of elliptic problems. Strong regularity implies local uniqueness of local optima. If one is only interested in local uniqueness, this can be directly deduced from a second-order sufficient condition, if the state constraints are regularized. We refer to [21], where regularization by virtual controls is considered.

3. Finite-element error analysis for state constrained elliptic control problems with finite-dimensional control space

In this section we derive error estimates for control problems with control vector $u \in \mathbb{R}^m$ in a two-dimensional polygonal convex spatial (open) domain Ω. We consider first the fully finite-dimensional optimal control problem

$$(P_F)\ \min_{u\in U_{ad}} J(y,u) := \int_\Omega L(x,y,u)\,dx$$

subject to the nonlinear state equation

$$\begin{aligned} -\Delta\, y(x) + d(x,y(x),u) &= 0 \quad \text{in} \quad \Omega \\ y(x) &= 0 \quad \text{on} \quad \Gamma = \partial\Omega, \end{aligned}$$

as well as the finitely many state constraints

$$\begin{aligned} g_i(y(x_i)) &= 0, \quad \text{for all } i = 1,\dots,k, \\ g_i(y(x_i)) &\le 0, \quad \text{for all } i = k+1,\dots,\ell \end{aligned}$$

given in points $x_i \in \Omega$, and bounds on the control,

$$u \in U_{ad} = \{u \in \mathbb{R}^m : u_a \le u \le u_b\}$$

with given vectors $u_a \le u_b$ of $\mathbb{R}^m$ that has been analysed in [25]. We assume $l \ge 1$ and set $k = 0$, if only inequality constraints are given and $k = l$, if only equality constraints are given. The precise assumptions on the appearing functions L, d, and g_i are layed out in [25]. In particular, L and d are supposed to be Hölder continuous with respect to x and d is assumed to be monotone non-decreasing with respect to y. A typical tracking type functional would fit into the given setting. The possibly nonlinear appearance of u does not cause problems with existence of an optimal solution, since u has finite dimension.

Moreover, we consider a model problem of semi-infinite type, given by

$$(P_S)\ \min_{u\in U_{ad}} J(y,u) := \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\nu}{2}|u|^2$$

subject to a linear state equation

$$\begin{aligned} -\Delta y(x) &= \sum_{i=1}^{M} u_i e_i(x) \quad \text{in } \Omega \\ y(x) &= 0 \quad\quad\quad\quad \text{on } \Gamma, \end{aligned}$$

as well as a pointwise bound $b \in \mathbb{R}$ on the state in a compact interior subdomain of Ω denoted by K,

$$y(x) \le b, \quad \forall x \in K.$$

For the precise assumptions on the given data, we refer to [24], let us just mention that the basis functions e_i, $i = 1,\dots,M$, are given in $C^{0,\beta}(\bar\Omega)$, for some $0 < \beta < 1$. The set U_{ad} is defined as in (P_F) with given bounds $u_a \in \mathbb{R} \cup \{-\infty\}$ and $u_b \in \mathbb{R} \cup \{\infty\}$, where $u_a < u_b$.

3.1. The finite dimensional control problem

We now consider the finite dimensional problem (P_F) in the equivalent reduced formulation

$$\min_{u \in U_{ad}} f(u) := J(y(u), u)$$

subject to the constraints

$$G_i(u) = 0, \quad i = 1, \dots k, \quad G_i(u) \leq 0, \quad i = k+1, \dots \ell, .$$

where G is defined as $G(u) = (g_1(y_u(x_1)), \dots, g_\ell(y_u(x_\ell)))$. Using the finite element discretization of the state equation and denoting a corresponding discrete state by $y_h(u)$, let us define

$$f_h(u) = J(y_h(u), u), \qquad G_h(u) = (g_1(y_h(u)(x_1)), \dots, g_\ell(y_h(u)(x_\ell))).$$

By these terms, we obtain an approximate problem formulation

$$(P_{F,h}) \quad \min_{u \in U_{ad}} f_h(u)$$

$$\text{subject to} \quad G_{h,i}(u) = 0, \quad i = 1, \dots k, \quad G_{h,i}(u) \leq 0, \quad i = k+1, \dots \ell.$$

Based on [6] and [34], in [25] the following result has been derived for the semilinear state equation: For all $u \in U_{ad}$, the discretized state equation has a unique discrete solution $y_h(u)$. There exists a constant c independent of h and $u \in U_{ad}$ such that, for all $u \in U_{ad}$, there holds

$$\|y(u) - y_h(u)\|_{L^2(\Omega)} + \|y(u) - y_h(u)\|_{C(K)} \leq c\, h^2 |\log h|.$$

Due to the finite-dimensional character of this problem, techniques from the perturbation analysis of parametric nonlinear programming problems can be applied. Therefore, also an error of the Lagrange multipliers can be quantified.

Theorem 3.1 ([25]). *Let, under our assumptions, $\bar{u}$ be a locally optimal control of Problem (P_F) satisfying the condition of linear independence of active constraints and the standard strong second-order condition. Then $\bar{u}$ is locally unique and there exists a sequence $\bar{u}_h$ of locally optimal controls of the corresponding finite element approximated problem $(P_{F,h})$ and a constant $C > 0$ independent of h such that the following estimate is satisfied for all sufficiently small h:*

$$|\bar{u} - \bar{u}_h| \leq C\, h^2 |\log h|.$$

3.2. A problem of semi-infinite type

Let us finally consider the problem (P_S), which we have discussed in detail in [24]. This problem combines the advantages of a finite dimensional control space with the difficulties of pointwise state constraints in a domain rather than finitely many points. In contrast to Problem (P_F), the Lagrange multipliers associated with the state constraints will be regular Borel measures rather than vectors of real numbers. In view of this, an estimate not better than $h\sqrt{|\log h|}$ would not surprise.

However, in several computations we observed a much better order of convergence. The associated analytical confirmation turned out to be interesting and surprisingly difficult. Let us briefly outline the main steps.

Due to linearity of the underlying state equation, we can apply the superposition principle to obtain a semi-infinite formulation of Problem (P_S),

$$(P_S)\ \min_{u\in U_{ad}} f(u) := \frac{1}{2}\|\sum_{i=1}^{M} u_i y_i - y_d\|^2 + \frac{\nu}{2}|u|^2$$

$$\text{subject to} \quad \sum_{i=1}^{M} u_i y_i(x) \le b, \quad \forall x \in K,$$

where y_i denotes the solution of the state equation associated with $u_i = 1$ and all other components of u taken as zero.

Our results on error estimates are based on a standard Slater condition:

Assumption 3.2. There exist a $u^\gamma \in U_{ad}$ and a real number $\gamma > 0$ such that

$$y^\gamma(x) = y_{u^\gamma}(x) \le b - \gamma \quad \forall\, x \in K.$$

Then there exists a non-negative Lagrange multiplier $\bar\mu$ in the space of regular Borel measures such that the standard Karush-Kuhn-Tucker conditions are satisfied by $\bar u$. However, collecting the state constraints in a feasible set U_{feas} given by

$$U_{feas} := \{u \in U_{ad} \colon y(u) \le b\ \forall x \in K\}\,,$$

the optimality conditions can be formulated with the help of a standard variational inequality due to linearity of the state equation. Let now y_i^h, $i = 1,\dots,M$, be the discrete states associated with y_i. We obtain the discretized problem formulation

$$(P_{S,h})\ \min_{u\in U_{ad}} f^h(u) := \frac{1}{2}\|\sum_{i=1}^{M} u_i y_i^h - y_d\|^2 + \frac{\nu}{2}|u|^2$$

$$\text{subject to} \quad \sum_{i=1}^{M} u_i y_i^h(x) \le b, \quad \forall x \in K,$$

where the pointwise state constraints are still prescribed in the whole subdomain K rather than in finitely many discrete points. Under our assumptions, we have

$$\|y_i^h - y_i\|_{L^2(\Omega)} + \|y_i^h - y_i\|_{L^\infty(K)} \quad \le \quad ch^2|\log h|.$$

thanks to an L^∞-error estimate from [34]. Clearly, this error estimate extends to any linear combinations $y_u = \sum_{i=1}^{M} u_i y_i$ and $y_u^h = \sum_{i=1}^{M} u_i y_i^h$ for any fixed $u \in \mathbb{R}^M$. As a consequence of the Slater assumption and the error estimate for the state equation, the feasible set of (P^h) is not empty for all sufficiently small $h > 0$. Therefore, there exists a unique optimal control $\bar u^h$ of Problem (P^h), with associated optimal state $\bar y^h$. Associated with $\bar y^h$, there exists a non-negative Lagrange multiplier $\mu_h \in M(\bar\Omega)$ such that the standard KKT-conditions are satisfied. Again, by introducing a feasible set U^h_{feas} analogously to U_{feas}, the first order optimality conditions can expressed as a variational inequality. Then, invoking only the Slater condition, the error estimate

$$|\bar{u} - \bar{u}^h| \leq ch\sqrt{|\log h|} \tag{3.1}$$

can be shown in a standard way. While this seems rather obvious, it is to the authors' knowledge the first time that an a priori error estimate for this problem class has been shown. As a consequence we obtain that $\bar{y}^h$ converges uniformly to $\bar{y}$ in K as h tends to zero.

We are able to improve (3.1) under additional assumptions on the structure of the active set. First of all, one obtains under quite natural assumptions that the active set of $\bar{y}$ cannot contain any open subset of K. Still, the set of active points might be fairly irregular, but it is reasonable to assume the following:

Assumption 3.3. The optimal state $\bar{y}$ is active in exactly N points $\bar{x}_1, \ldots, \bar{x}_N \in$ int K, i.e. $\bar{y}(\bar{x}_i) = b$. Moreover, there exists $\sigma > 0$ such that

$$-\langle \xi, \nabla^2 \bar{y}(\bar{x}_j)\xi\rangle \geq \sigma|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n,\ \forall j = 1, \ldots, N.$$

Notice that, in contrast to the fully finite-dimensional case, the location of these active points is not known in advance. To guarantee the existence of sequences $\bar{x}^h_j$ of active points of $\bar{y}^h$ such that $x^h_j \to \bar{x}_j$ as $h \to 0$, we assume strong activity:

Assumption 3.4. All active control and state constraints are strongly active, i.e. the associated Lagrange multipliers are strictly positive.

For simplicity, we do not consider the case of additional weakly active state constraints here. Under this assumption, we are able to show that to any active $\bar{x}_j$ there exists a sequence $\bar{x}^h_j$ of active points for $\bar{y}^h$ such that

$$|\bar{x}_j - \bar{x}^h_j| \leq ch\sqrt{|\log h|}. \tag{3.2}$$

In view of the piecewise linear form of $\bar{y}^h$, we can even assume that all $\bar{x}^h_j$ are node points, and consider a problem formulation where the constraints are only prescribed in the nodes. The proof of this inequality, which is the key estimate in deriving the improved error estimate, is quite elaborate. We refer to [24] for the key ideas.

Assumption 3.5. The number N of active state constraints is equal to the number of inactive control constraints.

Define the $N \times N$-matrix Y with entries $Y_{i_k,j_k} = y_{i_k}(\bar{x}_{j_k})$, $i_k \in \mathcal{I}_{\bar{u}}$, $j_k \in \mathcal{A}_{\bar{y}}$, where $\mathcal{I}_{\bar{u}}$ and $\mathcal{A}_{\bar{y}}$ denote the index sets of inactive control and active state constraints, respectively.

Theorem 3.6. *Let $\bar{u}$ be the optimal solution of Problem (P_S), let $\bar{u}^h$ be optimal for $(P_{S,h})$, and let Assumptions* 3.2- 3.5 *be satisfied. Moreover, let the matrix Y be* *regular. Then, there exists $h_0 > 0$ such that the following estimate is true for a $c > 0$ independent of h:*

$$|\bar{u} - \bar{u}^h| \leq ch^2|\log h| \quad \forall h \leq h_0.$$

Assumption 3.5 seems quite restrictive at first glance, and the question is interesting, whether it is indeed necessary for the optimal error estimate of the last theorem. In [23], we constructed simple analytical and numerical examples with more (inactive) controls than active constraints, where the lower order estimate (3.1) is sharp. On the other hand, the theory of semi-infinite optimization problems says that there can be at most as many strongly active constraints as there are control parameters, cf. [3]. Therefore, the analysis of Problem (P_S) is complete and the estimates are sharp.

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