SECOND ORDER NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR OPTIMIZATION PROBLEMS AND APPLICATIONS TO CONTROL THEORY *

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Abstract. The paper deals with a class of nonlinear optimization problems in a function space, where the solution is restricted by pointwise upper and lower bounds and by finitely many equality and inequality constraints of functional type. Second order necessary and sufficient optimality conditions are established, where the cone of critical directions is arbitrarily close to the form which is expected from the optimization in finite dimensional spaces. The results are applied to some optimal control problems for ordinary and partial differential equations.

Key words. Necessary and sufficient optimality conditions, control of differential equations, state constraints

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1. Introduction. Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < +\infty$. In this paper we will study the following optimization problem

$$\text{(P)} \begin{cases} \text{Minimize } J(u) \\ u_a(x) \leq u(x) \leq u_b(x) \text{ a.e. } x \in X, \\ G_j(u) = 0, \ 1 \leq j \leq m_1, \\ G_j(u) \leq 0, \ m_1 + 1 \leq j \leq m, \end{cases}$$

where $u_a, u_b \in L^\infty(X)$ and $J, G_j : L^\infty(X) \longrightarrow \mathbb{R}$ are given functions with differentiability properties to be fixed later. We will state necessary and sufficient optimality conditions for a local minimum of (P). Our main goal is to reduce the classical gap between the necessary and sufficient conditions for optimization problems in Banach spaces. We shall prove some optimality conditions very close to the ones for finite dimensional optimization problems. In the case of finite dimensions, strongly active inequality constraints are considered in the critical cone by associated linearized equality constraints. Roughly speaking, this is what we are able to extend to infinite dimensions. Due to the lack of compactness, the direct proof of the sufficiency theorem known for finite dimensions cannot be transferred to the case of general Banach spaces. Our direct method of proof is able to overcome this difficulty. To our best knowledge, this result has not yet been presented in literature. Of course, the bound constraints $u_a(x) \leq u(x) \leq u_b(x)$ introduce some additional difficulties in the study because they constitute an infinite number of constraints. In Section 2 we introduce a slightly stronger regularity assumption than that one considered in the Kuhn-Tucker theorem, which allows us to deal with the bound constraints.

In Section 4 we discuss the application of our general results to different types of optimal control problems. We consider the control of ordinary differential equations as well as that of partial differential equations of elliptic and parabolic type.

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2. Necessary Optimality Conditions. In this section we will assume that $\bar{u}$ is a local solution of (P), which means that there exists a real number $r > 0$ such that for every feasible point of (P), with $\|u - \bar{u}\|_{L^\infty(X)} < r$, we have that $J(\bar{u}) \le J(u)$.

For every $\varepsilon > 0$, we denote set of *ε-inactive constraints* by

$$X_\varepsilon = \{x \in X : u_a(x) + \varepsilon \le \bar{u}(x) \le u_b(x) - \varepsilon\}.$$

We make the following regularity assumption

$$\left\{ \begin{array}{l} \exists \varepsilon_{\bar{u}} > 0 \text{ and } \{h_j\}_{j \in I_0} \subset L^\infty(X), \text{ with } \operatorname{supp} h_j \subset X_{\varepsilon_{\bar{u}}}, \text{ such that} \\ G_i'(\bar{u}) h_j = \delta_{ij}, \quad i, j \in I_0, \end{array} \right. \tag{2.1}$$

where

$$I_0 = \{j \le m \,|\, G_j(\bar{u}) = 0\}.$$

I_0 is the set of indices corresponding to active constraints. We also denote the set of non active constraints by I_-

$$I_- = \{j \le m \,|\, G_j(\bar{u}) < 0\}.$$

Obviously (2.1) is equivalent to the independence of the derivatives $\{G_j'(\bar{u})\}_{j \in I_0}$ in $L^\infty(X_\varepsilon)$. Under this assumption we can derive the first order necessary conditions for optimality satisfied by $\bar{u}$. For the proof the reader is referred to Bonnans and Casas [1] or Clarke [5].

THEOREM 2.1. *Let us assume that* (2.1) *holds and* J *and* $\{G_j\}_{j=1}^m$ *are of class* C^1 *in a neighbourhood of* $\bar{u}$. *Then there exist real numbers* $\{\bar{\lambda}_j\}_{j=1}^m \subset \mathbb{R}$ *such that*

$$\bar{\lambda}_j \ge 0, \quad m_1 + 1 \le j \le m, \ \bar{\lambda}_j = 0 \ \textit{if } j \in I_-; \tag{2.2}$$

$$\langle J'(\bar{u}) + \sum_{j=1}^m \bar{\lambda}_j G_j'(\bar{u}), u - \bar{u} \rangle \ge 0 \quad \textit{for all } u_a \le u \le u_b. \tag{2.3}$$

Since we want to establish some optimality conditions useful for the study of control problems, we need to take into account the two-norm discrepancy; for this question see for instance Ioffe [8] and Maurer [9]. Then we have to impose some additional assumptions on the functions J and G_j.

(A1) There exist functions $f, g_j \in L^2(X)$, $1 \le j \le m$, such that for every $h \in L^\infty(X)$

$$J'(\bar{u})h = \int_X f(x)h(x)d\mu(x) \ \text{ and } \ G_j'(\bar{u})h = \int_X g_j(x)h(x)d\mu(x), \ 1 \le j \le m. \tag{2.4}$$

(A2) If $\{h_k\}_{k=1}^\infty \subset L^\infty(X)$ is bounded, $h \in L^\infty(X)$ and $h_k(x) \to h(x)$ a.e. in X, then

$$[J''(\bar{u}) + \sum_{j=1}^m \bar{\lambda}_j G_j''(\bar{u})]h_k^2 \to [J''(\bar{u}) + \sum_{j=1}^m \bar{\lambda}_j G_j''(\bar{u})]h^2. \tag{2.5}$$

If we define

$$L(u, \lambda) = J(u) + \sum_{j=1}^m \lambda_j G_j(u) \text{ and } d(x) = f(x) + \sum_{j=1}^m \bar{\lambda}_j g_j(x), \tag{2.6}$$

then

$$(2.7)\quad \frac{\partial L}{\partial u}(\bar{u},\bar{\lambda})h = [J'(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G'_j(\bar{u})]h = \int_X d(x)h(x)d\mu(x) \quad \forall h \in L^\infty(X).$$

From (2.3) we deduce that

$$(2.8)\qquad d(x) = \begin{cases} 0 & \text{for a.e. } x \in X \text{ where } u_a(x) < \bar{u}(x) < u_b(x), \\ \geq 0 & \text{for a.e. } x \in X \text{ where } \bar{u}(x) = u_a(x), \\ \leq 0 & \text{for a.e. } x \in X \text{ where } \bar{u}(x) = u_b(x). \end{cases}$$

Associated with d we set

$$(2.9)\qquad X^0 = \{x \in X : |d(x)| > 0\}.$$

Given $\{\bar{\lambda}_j\}_{j=1}^m$ by Theorem 2.1 we define the *cone of critical directions*

$$(2.10)\quad C^0_{\bar{u}} = \{h \in L^\infty(X) \text{ satisfying (2.11) and } h(x) = 0 \text{ for a.e. } x \in X^0\},$$

with

$$(2.11)\qquad \begin{cases} G'_j(\bar{u})h = 0 \text{ if } (j \leq m_1) \text{ or } (j > m_1,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j > 0); \\[1ex] G'_j(\bar{u})h \leq 0 \text{ if } j > m_1,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j = 0; \\[1ex] h(x) = \begin{cases} \geq 0 & \text{if } \bar{u}(x) = u_a(x); \\ \leq 0 & \text{if } \bar{u}(x) = u_b(x). \end{cases} \end{cases}$$

In the following theorem we state the necessary second order optimality conditions.

THEOREM 2.2. *Assume that* (2.1), **(A1)** *and* **(A2)** *hold,* $\{\bar{\lambda}_j\}_{j=1}^m$ *are the Lagrange multipliers satisfying* (2.2) *and* (2.3) *and* J *and* $\{G_j\}_{j=1}^m$ *are of class* C^2 *in a neighbourhood of* $\bar{u}$. *Then the following inequality is satisfied*

$$(2.12)\qquad \frac{\partial^2 L}{\partial u^2}(\bar{u},\bar{\lambda})h^2 \geq 0 \quad \forall h \in C^0_{\bar{u}}.$$

To prove this theorem we will make use of the following lemma

LEMMA 2.3. *Let us assume that* (2.1) *holds and* J *and* $\{G_j\}_{j=1}^m$ *are of class* C^2 *in a neighbourhood of* $\bar{u}$. *Let* $h \in L^\infty(X)$ *satisfy* $G'_j(\bar{u})h = 0$ *for every* $j \in I$, *where* I *is an arbitrary subset of* I_0. *Then there exist a number* $\varepsilon_h > 0$ *and* C^2*-functions* $\gamma_j : (-\varepsilon_h, +\varepsilon_h) \longrightarrow \mathbb{R}$, $j \in I$, *such that*

$$(2.13)\qquad \begin{cases} G_j(u_t) = 0 \ j \in I, \ \text{and } G_j(u_t) < 0 \ j \notin I_0, \quad \forall |t| \leq \varepsilon_h; \\ \gamma_j(0) = \gamma'_j(0) = 0, \ j \in I, \end{cases}$$

with

$$u_t = \bar{u} + th + \sum_{j \in I} \gamma_j(t)h_j,$$

$\{h_j\}_{j\in I}$ *given by* (2.1).

Proof. Let k be the cardinal number of I and let us define $\omega : \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ by

$$\omega(t, \rho) = (G_j(\bar{u} + th + \sum_{i \in I} \rho_i h_i))_{j \in I}.$$

Then ω is of class C^2 in a neighbourhood of $(0,0)$,

$$\frac{\partial \omega}{\partial t}(0,0) = (G'_j(\bar{u})h)_{j \in I} = 0 \quad \text{and} \quad \frac{\partial \omega}{\partial \rho}(0,0) = (G'_j(\bar{u})h_i)_{i,j \in I} = \text{Identity}.$$

Therefore we can apply the implicit function theorem and deduce the existence of $\varepsilon > 0$ and functions $\gamma_j : (-\varepsilon, +\varepsilon) \longrightarrow \mathbb{R}$ of class C^2, $j \in I$, such that

$$\omega(t, \gamma(t)) = \omega(0,0) = 0 \ \forall t \in (-\varepsilon, +\varepsilon) \quad \text{and} \quad \gamma(0) = 0,$$

where $\gamma(t) = (\gamma_j(t))_{j \in I}$. Furthermore, by differentiation in the previous identity we get

$$\frac{\partial \omega}{\partial t}(0,0) + \frac{\partial \omega}{\partial \rho}(0,0)\gamma'(0) = 0 \Longrightarrow \gamma'(0) = 0.$$

Taking into account the continuity of γ and G_j and that $\gamma(0) = 0$, we deduce the existence of $\varepsilon_h \leq \varepsilon$ such that (2.13) holds for every $t \in (-\varepsilon_h, +\varepsilon_h)$. ☐

Proof of Theorem 2.2. Let us take $h \in C^0_{\bar{u}}$ satisfying

$$h(x) = 0 \quad \text{if } u_a(x) < \bar{u}(x) < u_a(x) + \varepsilon \ \text{ or } \ u_b(x) - \varepsilon < \bar{u}(x) < u_b(x) \tag{2.14}$$

for some $\varepsilon \in (0, \varepsilon_{\bar{u}}]$. We introduce

$$I = \{1, \ldots, m_1\} \cup \{j : m_1 + 1 \leq j \leq m,\ G_j(\bar{u}) = 0 \text{ and } G'_j(\bar{u})h = 0\}. \tag{2.15}$$

I includes all equality constraints, all strongly active inequality constraints and, depending on h, possibly some of the weakly active inequality constraints. Then we are under the assumptions of Lemma 2.3. Let us set

$$u_t = \bar{u} + th + \sum_{j \in I} \gamma_j(t) h_j, \quad t \in (-\varepsilon_h, \varepsilon_h).$$

From Lemma 2.3 we know that $G_j(u_t) = 0$ if $j \in I$ and $G_j(u_t) < 0$ if $j \notin I_0$, provided that $t \in (-\varepsilon_h, +\varepsilon_h)$. From (2.11) we deduce that $G_j(\bar{u}) = 0$ and $G'_j(\bar{u})h < 0$ for $j \in I_0 \setminus I$. Therefore we have that $G_j(u_t) < 0$ for every $j \notin I$ and $t \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$ small. On the other hand, the assumptions on h along with the additional condition (2.14) and the fact that $\operatorname{supp} h_j \subset X_{\varepsilon_{\bar{u}}}$ imply that $u_a(x) \leq u_t(x) \leq u_b(x)$ for $t \geq 0$ small enough. Consequently, by taking $\varepsilon_0 > 0$ sufficiently small, we get that u_t is a feasible control for (P) for every $t \in [0, \varepsilon_0)$. Now we know $G_j(u_t) = 0$ for $j \in I$ and $\bar{\lambda}_j = 0$ for $j \notin I_0$ (cf. (2.2). According to (2.11) we require $G'_j(\bar{u})h = 0$ for active inequalities with $\bar{\lambda}_j > 0$, hence if i belongs to $I_0 \setminus I$, then $\bar{\lambda}_j = 0$ must hold. This leads to

$$\sum_{j=1}^{m} \bar{\lambda}_j G_j(u_t) = 0 \ \forall t \in [0, \varepsilon_0).$$

Therefore the function $\phi : [0, +\epsilon_0) \longrightarrow \mathbb{R}$ given by

$$\phi(t) = J(u_t) + \sum_{j=1}^{m} \bar{\lambda}_j G_j(u_t)$$

has a local minimum at 0 and, taking into account that $\gamma_j'(0) = 0$,

$$\phi'(0) = (J'(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j'(\bar{u}))(h + \sum_{j \in I} \gamma_j'(0) h_j) =$$

$$(J'(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j'(\bar{u}))h = \int_X d(x)h(x)d\mu(x) = 0.$$

The last identity follows from the fact that h vanishes on X^0.

Since the first derivative of ϕ is zero we have the following second order necessary optimality condition

$$0 \le \phi''(0) = [J''(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j''(\bar{u})]h^2 +$$

$$[J'(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j'(\bar{u})](\sum_{i \in I} \gamma_i''(0) h_i) = [J''(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j''(\bar{u})]h^2 +$$

$$\sum_{i \in I} \gamma_i''(0) \int_X d(x) h_i(x) d\mu(x) = [J''(\bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j G_j''(\bar{u})]h^2 = \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda})h^2.$$

Here we have used **(A1)**. Now let us consider $h \in L^\infty(X)$ satisfying (2.11), but not (2.14), i.e. h is any critical direction. The main idea in this case is to approach h by functions h_ε, which belong to the critical cone $C_{\bar{u}}^0$ and satisfying (2.14) as well. Then for every $\varepsilon > 0$, we define $A_\varepsilon = X_\varepsilon \cup \{x \in X : \bar{u}(x) = u_a(x) \text{ or } \bar{u}(x) = u_b(x)\}$. This is the complement of the set of points x satisfying (2.14). Put

$$h_\varepsilon = h\chi_{A_\varepsilon} + \sum_{i \in I} \left(\int_{X \setminus A_\varepsilon} g_i(x) h(x) d\mu(x) \right) h_i = h\chi_{A_\varepsilon} + \hat{h},$$

where χ_{A_ε} is the characteristic function of A_ε and I is given by (2.15). We verify that h_ε belongs to $C_{\bar{u}}^0$, while $h\chi_{A_\varepsilon}$ possibly is not contained in this cone.

Thus for every $j \in I$, using (2.1) and taking $0 < \varepsilon < \varepsilon_{\bar{u}}$, we have

$$\begin{aligned} G_j'(\bar{u})h_\varepsilon &= \int_X g_j(x)(h\chi_{A_\varepsilon})(x)d\mu(x) + \int_X g_j(x)\hat{h}(x)d\mu(x) \\ &= \int_{A_\varepsilon} g_j(x)h(x)d\mu(x) \\ &\quad + \sum_{i \in I} \left(\int_{X \setminus A_\varepsilon} g_i(x)h(x)d\mu(x) \right) \int_X g_j(x)h_i(x)d\mu(x) \\ &= \int_{A_\varepsilon} g_j(x)h(x)d\mu(x) + \sum_{i \in I} \left(\int_{X \setminus A_\varepsilon} g_i(x)h(x)d\mu(x) \right) \delta_{ji} \\ &= \int_X g_j(x)h(x)d\mu(x) = G_j'(\bar{u})h = 0. \end{aligned}$$

In case of $j \in I_0 \setminus I$, then $G_j'(\bar{u})h < 0$. Then it is enough to take ε sufficiently small to get $G_j'(\bar{u})h_\varepsilon < 0$.

Thus, reminding that $\operatorname{supp} h_j \subset X_{\varepsilon_{\bar{u}}}$, we have that h_ε satisfies the conditions (2.11) and (2.14), therefore (2.12) holds for each h_ε, $\varepsilon > 0$ small enough.

Finally, it is clear that $h_\varepsilon(x) \to h(x)$ a.e. in X as $\varepsilon \to 0$. Therefore, assumption **(A2)** allows us to pass to the limit in the second order optimality conditions satisfied for every h_ε and to conclude (2.12) □

3. Sufficient Optimality Conditions. In this section $\bar{u}$ is a given feasible element for the problem (P). Motivated again by the considerations on the two-norm discrepancy we have to make some assumptions involving the $L^\infty(X)$ and $L^2(X)$ norms,

(A3) There exists a positive number $r > 0$ such that J and $\{G_j\}_{j=1}^m$ are of class C^2 in the $L^\infty(X)$-ball $B_r(\bar{u})$ and for every $\eta > 0$ there exists $\varepsilon \in (0, r)$ such that for each $u \in B_r(\bar{u})$, $\|v - \bar{u}\|_{L^\infty(X)} < \varepsilon$, $h, h_1, h_2 \in L^\infty(X)$ and $1 \le j \le m$ we have

$$\left\{ \begin{array}{l} \left| \left[\dfrac{\partial^2 L}{\partial u^2}(v, \bar{\lambda}) - \dfrac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda}) \right] h^2 \right| \le \eta \, \|h\|_{L^2(X)}^2, \\[2ex] |J'(u)h| \le M_{0,1} \|h\|_{L^2(X)}, \quad |G_j'(u)h| \le M_{j,1} \|h\|_{L^2(X)}, \\[2ex] |J''(u)h_1 h_2| \le M_{0,2} \|h_1\|_{L^2(X)} \|h_2\|_{L^2(X)}, \\[2ex] |G_j''(u)h_1 h_2| \le M_{j,2} \|h_1\|_{L^2(X)} \|h_2\|_{L^2(X)}. \end{array} \right. \tag{3.1}$$

Analogously to (2.9) and (2.10) we define for every $\tau > 0$

$$X^\tau = \{x \in X : |d(x)| > \tau\} \tag{3.2}$$

and

$$C_{\bar{u}}^\tau = \{h \in L^\infty(X) \ \text{ satisfying } \ (2.11) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in X^\tau\}. \tag{3.3}$$

The next theorem provides the second order sufficient optimality conditions of (P). Though they seem to be different from the classical ones, we will prove later that they are equivalent; see Theorem 3.2 and Corollary 3.3.

THEOREM 3.1. *Let $\bar{u}$ be a feasible point for problem* (P) *verifying the first order necessary conditions* (2.2) *and* (2.3)*, and let us suppose that assumptions* (2.1), **(A1)** *and* **(A3)** *hold. Let us also assume that for every $h \in L^\infty(X)$ satisfying* (2.11) *we have*

$$\frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \ge \delta_1 \|h\|_{L^2(X \setminus X^\tau)}^2 - \delta_2 \|h\|_{L^2(X^\tau)}^2 \tag{3.4}$$

for some $\delta_1 > 0$, $\delta_2 \ge 0$ and $\tau > 0$ given. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that $J(\bar{u}) + \delta \, \|u - \bar{u}\|_{L^2(X)}^2 \le J(u)$ for every feasible point u for (P)*, with $\|u - \bar{u}\|_{L^\infty(X)} < \varepsilon$.*

Proof. (i) *Condition* (3.4) *is stable w.r. to perturbations of $\bar{u}$*:

Without loss of generality, we will assume that $\delta_2 > 0$. From Assumption **(A3)** we deduce the existence of $r_0 \in (0, r)$ such that for every $h \in L^\infty(X)$ and $\|v - \bar{u}\|_{L^\infty(X)} < r_0$

$$\left| \left[\frac{\partial^2 L}{\partial u^2}(v, \bar{\lambda}) - \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda}) \right] h^2 \right| \le \min\left\{ \frac{\delta_1}{2}, \delta_2 \right\} \|h\|_{L^2(X)}^2.$$

From this inequality and (3.4) it follows easily

$$\frac{\partial^2 L}{\partial u^2}(v,\bar\lambda)h^2 \geq \frac{\delta_1}{2}\|h\|^2_{L^2(X\setminus X^\tau)} - 2\delta_2\|h\|^2_{L^2(X^\tau)} \tag{3.5}$$

for every h satisfying (2.11) and $\|v-\bar u\|_{L^\infty(X)} < r_0$.

(ii) *Some technical definitions*:

Let us set

$$M = M_{0,2} + \sum_{j=1}^m |\bar\lambda_j| M_{j,2} \quad \text{and} \quad \rho = \min\left\{1, \frac{\delta_1}{16M}\right\}, \tag{3.6}$$

$$C_1 = \max\left\{\frac{\delta_1}{4}, 2\delta_2\right\} + \frac{3M}{2} + \frac{4M^2}{\delta_1}, \quad C_2 = \frac{C_1}{2}\max_{j\in I_0}\|h_j\|^2_{L^2(X)}\left(\sum_{j=1}^m M_{j,2}\right)^2, \tag{3.7}$$

$$C_3 = 2C_1 m\mu(X)^{1/2}\max_{j\in I_0}\|h_j\|^2_{L^2(X)}\max_{1\leq j\leq m} M_{j,1}. \tag{3.8}$$

Finally we take

$$\varepsilon = \min\left\{r_0, \sqrt{\frac{\delta_1}{64C_2\mu(X)}}, \frac{8\tau}{\delta_1+16\delta_2}, \frac{\rho}{C_3}\min_{j\in I_+, j>m_1}\bar\lambda_j\right\}, \tag{3.9}$$

where

$$I_+ = \{1,\ldots,m\}\cup\{j>m_1 : G_j(\bar u) = 0 \text{ and } \bar\lambda_j > 0\},$$

(iii) *Approximation of $u-\bar u$ by elements of the critical cone*:

Let u be a feasible point for problem (P), with $\|u-\bar u\|_{L^\infty(X)} < \epsilon$. Then $u-\bar u$ will not in general belong to the critical cone. Therefore, we use the representation $u-\bar u = h+h_0$, where h is in the critical cone and h_0 is some small correction.

Let us introduce the set of indices

$$I_u = \{j\in I_0 : G_j'(\bar u)(u-\bar u) > 0 \text{ or } [G_j'(\bar u)(u-\bar u) < 0 \text{ and } j\in I_+]\}.$$

This is the set of indices, where we need to correct $G_j'(\bar u)(u-\bar u)$, since the conditions of the critical cone are not met. We need this for equality constraints if $G_j'(\bar u)(u-\bar u)\neq 0$. Moreover this happens, if for an active inequality constraint we have $G_j'(\bar u)(u-\bar u) > 0$. Finally, we need this for strongly active inequality constraints, if $G_j'(\bar u)(u-\bar u) < 0$ holds instead of $G_j'(\bar u)(u-\bar u) = 0$. We define for all $j\in I_u$

$$\alpha_j = G_j'(\bar u)(u-\bar u),\ h_0 = \sum_{j\in I_u}\alpha_j h_j \text{ and } h = u-\bar u-h_0, \tag{3.10}$$

where the elements h_j are introduced in assumption (2.1). Then h satisfies (2.11). This is seen as follows:

$$G_j'(\bar u)h_0 = \sum_{i\in I_u}\alpha_i G_j'(\bar u)h_i = \sum_{i\in I_u}\alpha_i\delta_{ji}.$$

If $j \notin I_u$, then $\delta_{ji} = 0\ \forall i \in I_u$, hence

$$G'_j(\bar u)h = G'_j(\bar u)(u-\bar u) - G'_j(\bar u)h_0 = G'_j(\bar u)(u-\bar u) = \begin{cases} 0 & \text{if } j \le m_1 \\ \le 0 & \text{if } j > m_1. \end{cases}$$

(the last inequality follows from $j \notin I_u$). So $G'_j(\bar u)h$ fulfils the conditions of the critical cone. If $j \in I_u$, then

$$G'_j(\bar u)h = G'_j(\bar u)(u-\bar u) - \alpha_j\delta_{jj} = \alpha_j - \alpha_j = 0$$

and $G'_j(\bar u)h$ fulfils the conditions of the critical cone, too.

Let us now estimate h_0 in $L^2(X)$. For every $j \in I_u$ there exists $v_j = \bar u + \theta_j(u-\bar u)$, with $0 < \theta_j < 1$, such that

$$(3.11)\quad 0 \ge G_j(u) = G_j(\bar u) + G'_j(\bar u)(u-\bar u) + \frac{1}{2}G''_j(v_j)(u-\bar u)^2 = \alpha_j + \frac{1}{2}G''_j(v_j)(u-\bar u)^2.$$

If $\alpha_j \ge 0$ we deduce from (3.11) and (3.1) that

$$(3.12)\qquad |\alpha_j| = \alpha_j \le \frac{1}{2}|G''_j(v_j)(u-\bar u)^2| \le \frac{1}{2}M_{j,2}\|u-\bar u\|^2_{L^2(X)}.$$

If $\alpha_j < 0$ and $G_j(u) = 0$, we get

$$(3.13)\qquad |\alpha_j| = -\alpha_j = \frac{1}{2}G''_j(v_j)(u-\bar u)^2 \le \frac{1}{2}M_{j,2}\|u-\bar u\|^2_{L^2(X)}.$$

Let us denote

$$I_u^- = \{j \in I_u : G_j(u) < 0 \text{ and } \alpha_j < 0\}.$$

This is the set of all indices, where we do not obtain an estimate of α_j having the order $\|u-\bar u\|^2_{L^2(x)}$. We should notice at this point that $\bar\lambda_j > 0$ holds for all $j \in I_u^-$. (Since u must be feasible, j stands for an inequality constraint. Therefore, $0 > \alpha_j = G'_j(\bar u)(u-\bar u)$ and $j \in I_u$ implies $j \in I_+$.) Then we have

$$(3.14)\quad \|h_0\|_{L^2(X)} \le \max_{j\in I_0}\|h_j\|_{L^2(X)}\left[\frac{1}{2}\left(\sum_{j=1}^m M_{j,2}\right)\|u-\bar u\|^2_{L^2(X)} + \sum_{j\in I_u^-}|\alpha_j|\right].$$

(iv) *Estimation of $J(u)-J(\bar u)$*:

Using (2.6), (2.7), (3.6), (3.10) and (3.11) we have for some $v = \bar u + \theta(u-\bar u)$, with $0 < \theta < 1$,

$$\begin{aligned} J(u) &= J(u) + \sum_{j=1}^{m_1}\bar\lambda_j G_j(u) + \sum_{j=m_1+1}^{m}\bar\lambda_j G_j(u) - \sum_{j=m_1+1}^{m}\bar\lambda_j G_j(u) \\ &= L(u,\bar\lambda) - \sum_{j=m_1+1}^{m}\bar\lambda_j G_j(u) \\ &\ge L(u,\bar\lambda) - \sum_{j\in I_u^-}\bar\lambda_j G_j(u) \ge L(u,\bar\lambda) - \rho\sum_{j\in I_u^-}\bar\lambda_j G_j(u) \end{aligned}$$

since $\rho < 1$. Therefore,

$$J(u) \geq L(u,\bar{\lambda}) - \rho \sum_{j \in I_u^-} \bar{\lambda}_j G_j(u) = L(\bar{u},\bar{\lambda}) + \frac{\partial L}{\partial u}(\bar{u},\bar{\lambda})(u-\bar{u}) + \frac{1}{2}\frac{\partial^2 L}{\partial u^2}(v,\bar{\lambda})(u-\bar{u})^2 -$$

$$\rho \sum_{j \in I_u^-} \bar{\lambda}_j \alpha_j - \frac{\rho}{2} \sum_{j \in I_u^-} \bar{\lambda}_j G_j''(v_j)(u-\bar{u})^2 = J(\bar{u}) + \int_X d(x)(u(x)-\bar{u}(x))d\mu(x) +$$

$$\frac{1}{2}\frac{\partial^2 L}{\partial u^2}(v,\bar{\lambda})h^2 + \frac{\partial^2 L}{\partial u^2}(v,\bar{\lambda})hh_0 + \frac{1}{2}\frac{\partial^2 L}{\partial u^2}(v,\bar{\lambda})h_0^2 +$$

$$\rho \sum_{j \in I_u^-} \bar{\lambda}_j |\alpha_j| - \frac{\rho}{2} \sum_{j \in I_u^-} \bar{\lambda}_j G_j''(v_j)(u-\bar{u})^2.$$

Now from (2.8), (2.11), (3.1), (3.5) and (3.6) it follows

$$J(u) \geq J(\bar{u}) + \tau \int_{X^\tau} |u(x)-\bar{u}(x)| d\mu(x) + \frac{\delta_1}{4}\|h\|^2_{L^2(X\setminus X^\tau)} - \delta_2 \|h\|^2_{L^2(X^\tau)} -$$

$$M\|h_0\|_{L^2(X)}\|h\|_{L^2(X)} - \frac{M}{2}\|h_0\|^2_{L^2(X)} + \rho \sum_{j \in I_u^-} \bar{\lambda}_j |\alpha_j| - \frac{\rho}{2}\left(\sum_{j \in I_u^-} \bar{\lambda}_j M_{j,2}\right)\|u-\bar{u}\|^2_{L^2(X)} \geq$$

$$J(\bar{u}) + \frac{\tau}{\varepsilon}\|u-\bar{u}\|^2_{L^2(X^\tau)} + \frac{\delta_1}{8}\|u-\bar{u}\|^2_{L^2(X\setminus X^\tau)} - \frac{\delta_1}{4}\|h_0\|^2_{L^2(X\setminus X^\tau)} - 2\delta_2\|u-\bar{u}\|^2_{L^2(X^\tau)} -$$

$$2\delta_2\|h_0\|^2_{L^2(X^\tau)} - M\|h_0\|_{L^2(X)}\left(\|u-\bar{u}\|_{L^2(X)} + \|h_0\|_{L^2(X)}\right) - \frac{M}{2}\|h_0\|^2_{L^2(X)} +$$

$$\rho \sum_{j \in I_u^-} \bar{\lambda}_j |\alpha_j| - \frac{\rho}{2} M \|u-\bar{u}\|^2_{L^2(X)}. \tag{3.15}$$

Using the definition of ε by (3.9) we have

$$\frac{\tau}{\varepsilon} - 2\delta_2 \geq \frac{\delta_1}{8}. \tag{3.16}$$

On the other hand

$$M\|h_0\|_{L^2(X)}\|u-\bar{u}\|_{L^2(X)} = 2\left[\frac{\sqrt{\delta_1}}{4}\|u-\bar{u}\|_{L^2(X)}\right]\left[\frac{2M}{\sqrt{\delta_1}}\|h_0\|_{L^2(X)}\right] \leq$$

$$\frac{\delta_1}{16}\|u-\bar{u}\|^2_{L^2(X)} + \frac{4M^2}{\delta_1}\|h_0\|^2_{L^2(X)}. \tag{3.17}$$

From the definitions of C_1 and ρ given in (3.7) and (3.6) along with (3.15), (3.16) and (3.17) we get

$$J(u) \geq J(\bar{u}) + \frac{\delta_1}{8}\|u - \bar{u}\|^2_{L^2(X)} - C_1\|h_0\|^2_{L^2(X)} -$$

$$\frac{\delta_1}{16}\|u - \bar{u}\|^2_{L^2(X)} + \rho \sum_{j\in I_u^-} \bar{\lambda}_j|\alpha_j| - \frac{\delta_1}{32}\|u - \bar{u}\|^2_{L^2(X)} =$$

$$J(\bar{u}) + \frac{\delta_1}{32}\|u - \bar{u}\|^2_{L^2(X)} - C_1\|h_0\|^2_{L^2(X)} + \rho \min_{j\in I_+, j>m_1} \bar{\lambda}_j \sum_{j\in I_u^-} |\alpha_j|. \tag{3.18}$$

(v) *Two auxiliary estimates and final result*
From (3.7), (3.9) and (3.14) we get on using $(a+b)^2 \leq 2\ (a^2+b^2)$

$$C_1\|h_0\|^2_{L^2(X)} \leq C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left[\frac{1}{2}\left(\sum_{j=1}^m M_{j,2}\right)^2 \|u - \bar{u}\|^4_{L^2(X)} + 2\left(\sum_{j\in I_u^-} |\alpha_j|\right)^2\right]$$

$$= C_2\|u - \bar{u}\|^4_{L^2(X)} + 2C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left(\sum_{j\in I_u^-} |\alpha_j|\right)^2 \leq$$

$$C_2\varepsilon^2\mu(X)\|u - \bar{u}\|^2_{L^2(X)} + 2C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left(\sum_{j\in I_u^-} |\alpha_j|\right)^2 \leq$$

$$\frac{\delta_1}{64}\|u - \bar{u}\|^2_{L^2(X)} + 2C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left(\sum_{j\in I_u^-} |\alpha_j|\right)^2. \tag{3.19}$$

The definition of α_j given by (3.10) along with the assumption (3.1) imply

$$|\alpha_j| \leq M_{j,1}\|u - \bar{u}\|_{L^2(X)} \leq M_{j,1}\varepsilon\sqrt{\mu(X)}. \tag{3.20}$$

From (3.8) and the above inequality we deduce

$$2C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left(\sum_{j\in I_u^-} |\alpha_j|\right) \leq C_3\varepsilon. \tag{3.21}$$

Definition (3.9) and (3.21) lead to

$$\rho \min_{j\in I_+, j>m_1} \bar{\lambda}_j - 2C_1 \max_{j\in I_0} \|h_j\|^2_{L^2(X)} \left(\sum_{j\in I_u^-} |\alpha_j|\right) \geq 0. \tag{3.22}$$

Finally combining (3.18), (3.19) and (3.22) we conclude the desired result

$$J(u) \geq J(\bar{u}) + \frac{\delta_1}{64}\|u - \bar{u}\|^2_{L^2(X)}.$$

$\square$

Now we prove the equivalence between the sufficient optimality conditions stated in Theorem 3.1 and the classical ones.

THEOREM 3.2. *Let $\bar{u}$ be a feasible point of* (P) *satisfying* (2.2) *and* (2.3). *Let $C_{\bar{u}}$ be the set of elements $h \in L^\infty(X)$ satisfying* (2.11) *and $C^\tau_{\bar{u}}$ be given by* (3.3). *Let us suppose that assumptions* (2.1), **(A1)** *and* **(A3)** *hold. Let $\tau > 0$ be given. Then the following statements are equivalent*

$$\exists \delta > 0 \, : \, \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \geq \delta \|h\|^2_{L^2(X)} \quad \forall h \in C^\tau_{\bar{u}}, \tag{3.23}$$

$$\exists \delta_1 > 0, \, \delta_2 \geq 0 \, : \, \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \geq \delta_1 \|h\|^2_{L^2(X \setminus X^\tau)} - \delta_2 \|h\|^2_{L^2(X^\tau)} \quad \forall h \in C_{\bar{u}}. \tag{3.24}$$

Proof. It is obvious that (3.24) implies (3.23), since $h = 0$ in X^τ if $h \in C^\tau_{\bar{u}}$. Therefore, it is enough to take $\delta = \delta_1$. Let us prove the opposite implication. Let $h \in C_{\bar{u}}$. We set $h_\tau = h\chi_{X^\tau}$, where χ_{X^τ} is the characteristic function of X^τ, and

$$I_h = \{j \in I_0 : G'_j(\bar{u})(h - h_\tau) > 0 \ \text{ or } \ [G'_j(\bar{u})(h - h_\tau) < 0 \ \text{ and } \ G'_j(\bar{u})h = 0]\}.$$

We define

$$\alpha_j = G'_j(\bar{u})(h - h_\tau) \ \ \forall j \in I_h, \ \ \hat{h} = \sum_{j \in I_h} \alpha_j h_j \ \text{ and } \ h_0 = h - h_\tau - \hat{h},$$

where the functions h_j are given by (2.1).

Let us see that $h_0 \in C^\tau_{\bar{u}}$. Since $\operatorname{supp} h_j \subset X_{\varepsilon_{\bar{u}}}$ and $h - h_\tau = h(1 - \chi_{X^\tau})$, we have that $h_0(x) = 0$ for $x \in X^\tau$. Now we distinct between the cases $j \in I_h$ and $j \in I_0 \setminus I_h$.

If $j \in I_h$, then

$$G'_j(\bar{u})h_0 = G'_j(\bar{u})(h - h_\tau) - \sum_{i \in I_h} \alpha_i G'_j(\bar{u})h_i = G'_j(\bar{u})(h - h_\tau) - \alpha_j = 0.$$

If $j \in I_0 \setminus I_h$, then from the definition of I_h we obtain that $G'_j(\bar{u})h_0 = G'_j(\bar{u})(h - h_\tau) \leq 0$.

If this inequality reduces to an equality, $G'_j(\bar{u})(h - h_\tau) = 0$, then h_0 verifies the condition to be in $C^\tau_{\bar{u}}$. In the remaining case that $j \in I_0 \setminus I_h$ but $G'_j(\bar{u})(h - h_\tau) < 0$, using again the definition of I_h, we deduce that $G'_j(\bar{u})h < 0$. ($G'_j(\bar{u})h = 0$ and $G'_j(\bar{u})(h - h_\tau) < 0$ would give $j \in I_h$.) Consequently, since $h \in C_{\bar{u}}$, we have that $j > m_1$ and $\bar{\lambda}_j = 0$ (otherwise $h \in C^\tau_{\bar{u}}$ and $\bar{\lambda}_j > 0$ would imply $G'_j(\bar{u})h = 0$). Then the inequality $G'_j(\bar{u})h_0 < 0$ also means that h_0 verifies the condition to be in $C^\tau_{\bar{u}}$.

We now prove that

$$\|\hat{h}\|_{L^2(X)} \leq C_0 \|h_\tau\|_{L^2(X)}, \tag{3.25}$$

where

$$C_0 = \sum_{j \in I_0} \|g_j\|_{L^2(X)} \|h_j\|_{L^2(X)},$$

g_j given in (2.4). Indeed, if $\alpha_j > 0$ then

$$|\alpha_j| = \alpha_j = G_j'(\bar u)(h - h_\tau) = G_j'(\bar u)h - G_j'(\bar u)h_\tau \le -G_j'(\bar u)h_\tau \le \|g_j\|_{L^2(X)}\|h_\tau\|_{L^2(X)}.$$

If $\alpha_j < 0$, then from the definition of I_h we have that $G_j'(\bar u)h = 0$, therefore

$$|\alpha_j| = -\alpha_j = -G_j'(\bar u)(h - h_\tau) = G_j'(\bar u)h_\tau \le \|g_j\|_{L^2(X)}\|h_\tau\|_{L^2(X)}.$$

Combining the previous two inequalities and the definition of $\hat h$ we get (3.25).

Finally, taking M as in (3.6), we obtain from (3.23) and (3.25)

$$\frac{\partial^2 L}{\partial u^2}(\bar u,\bar\lambda)h^2 = \frac{\partial^2 L}{\partial u^2}(\bar u,\bar\lambda)h_0^2 + \frac{\partial^2 L}{\partial u^2}(\bar u,\bar\lambda)(h_\tau+\hat h)^2 + 2\frac{\partial^2 L}{\partial u^2}(\bar u,\bar\lambda)h_0(h_\tau+\hat h) \ge$$

$$\delta\|h_0\|^2_{L^2(X)} - M\|h_\tau+\hat h\|^2_{L^2(X)} - 2M\|h_0\|_{L^2(X)}\|h_\tau+\hat h\|_{L^2(X)} \ge$$

$$\frac{\delta}{2}\|h-h_\tau\|^2_{L^2(X)} - \delta\|\hat h\|^2_{L^2(X)} - 2M\left(\|h_\tau\|^2_{L^2(X)} + \|\hat h\|^2_{L^2(X)}\right) -$$

$$2M\left(\|h-h_\tau\|_{L^2(X)} + \|\hat h\|_{L^2(X)}\right)\left(\|h_\tau\|_{L^2(X)} + \|\hat h\|_{L^2(X)}\right) \ge$$

$$\frac{\delta}{2}\|h-h_\tau\|^2_{L^2(X)} - C_0^2\delta\|h_\tau\|^2_{L^2(X)} - 2M(C_0^2+1)\|h_\tau\|^2_{L^2(X)} -$$

$$2M(C_0+1)\left(\|h-h_\tau\|_{L^2(X)} + C_0\|h_\tau\|_{L^2(X)}\right)\|h_\tau\|_{L^2(X)} \ge \frac{\delta}{4}\|h-h_\tau\|^2_{L^2(X)}$$

$$-\left\{C_0^2\delta + 2M(C_0^2+1) + \frac{4M^2(C_0+1)^2}{\delta} + 2M(C_0+1)C_0\right\}\|h_\tau\|^2_{L^2(X)} =$$

$$\delta_1\|h\|^2_{L^2(X\setminus X^\tau)} - \delta_2\|h\|^2_{L^2(X^\tau)}$$

where obviously $\delta_1 > 0$ and $\delta_2 \ge 0$ are independent of $h \in C_{\bar u}$. □

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

COROLLARY 3.3. *Let $\bar u$ be a feasible point for problem* (P) *satisfying* (2.2) *and* (2.3) *and let us suppose that assumptions* (2.1), **(A1)** *and* **(A3)** *hold. Let us also assume that*

$$\frac{\partial^2 L}{\partial u^2}(\bar u,\bar\lambda)h^2 \ge \delta\|h\|^2_{L^2(X)} \quad \forall h \in C^\tau_{\bar u}. \tag{3.26}$$

for some $\delta > 0$ and $\tau > 0$ given. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar u) + \alpha\|u-\bar u\|^2_{L^2(X)} \le J(u)$ for every feasible point u for (P)*, with $\|u-\bar u\|_{L^\infty(X)} < \varepsilon$.*

REMARK 3.4. If we compare the sufficient optimality condition (3.4) with the necessary one (2.12), we notice the existence of a gap between both coming from two facts. Firstly the constant δ_1 is strictly positive in (3.4) and it can be zero in (2.12), which is the classical situation even in finite dimension. The second fact is that we can not replace, in general, $C^\tau_{\bar u}$, with $\tau > 0$, for $C^0_{\bar u}$ in (3.26), as it is done in (2.12). This is motivated by the presence of an infinite number of constraints. The following example, due to J.C. Dunn [6], demonstrates the impossibility of taking $\tau = 0$ in (3.26). Let us consider $X = [0,1]$, $\mathcal{S}$ the σ-algebra of Lebesgue measurable sets of $[0,1]$ and let μ be the Lebesgue measure in $[0,1]$. Now we take $J : L^2([0,1]) \longrightarrow \mathbb{R}$ defined by

$$J(u) = \int_0^1 [2a(x)u(x) - \text{sign}(a(x))u(x)^2]dx,$$

with $a(x) = 1 - 2x$. The optimization problem is

$$\begin{cases} \text{Minimize } J(u) \\ u \in L^\infty([0,1]), \text{ with } u(x) \geq 0 \text{ a.e. } x \in [0,1]. \end{cases}$$

Let us set $\bar u(x) = \max\{0, -a(x)\}$. Then we have

$$J'(\bar u)h = \int_0^1 2[a(x) - \text{sign}(a(x))\bar u(x)]h(x)dx =$$

$$\int_0^1 d(x)h(x)dx = \int_0^{1/2} 2a(x)h(x)dx \geq 0$$

for all $h \in L^2([0,1])$, with $h(x) \geq 0$. If we also assume that $h(x) = 0$ for $x \in X^0$ we have

$$J''(\bar u)h^2 = -\int_0^1 2\ \text{sign}(a(x))h^2(x)dx = 2\int_{1/2}^1 h^2(x)dx - 2\int_0^{1/2} h^2(x)dx = 2\|h\|^2_{L^2(X)},$$

where, following the notation introduced in (2.9), we have

$$X^0 = \{x \in [0,1] : |d(x)| > 0\} = [0, 1/2).$$

Thus we have that (3.26) holds with $\delta = 2$ and $\tau = 0$. However $\bar u$ is not a local minimum in $L^\infty([0,1])$. Indeed, let us take for $0 < \varepsilon < \frac{1}{2}$

$$u_\varepsilon(x) = \begin{cases} \bar u(x) + 3\varepsilon & \text{if } x \in [\frac{1}{2} - \varepsilon, \frac{1}{2}] \\ \bar u(x) & \text{otherwise.} \end{cases}$$

Then we have

$$J(u_\varepsilon) - J(\bar u) = \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}} [6\varepsilon(1-2x) - 9\varepsilon^2]dx = -3\varepsilon^3 < 0.$$

4. Application to some optimal control problems.

4.1. An abstract control problem. Let, in addition to the measure space (X, S, μ), Y and Z be real Banach spaces, let $A : Y \to Z$ be a linear continuous operator, and let $B : Y \times L^\infty(X) \to Z$ be an operator of class C^2. Moreover,

F, $F_j : Y \times L^\infty(X) \to \mathbb{R}$ are functionals of class C^2, $j = 1, .., m$. Consider the optimal control problem

$$
(\mathrm{OC}) \begin{cases} \text{Minimize } F(y,u) \\ Ay + B(y,u) = 0 \\ u_a(x) \le u(x) \le u_b(x) \text{ a.e. } x \in X, \\ F_j(y,u) = 0, \ 1 \le j \le m_1, \\ F_j(y,u) \le 0, \ m_1 + 1 \le j \le m, \end{cases}
$$

where the control u is taken from $L^\infty(X)$. We assume that for all $u \in L^\infty(X)$ the equation $Ay + B(y,u) = 0$ admits a unique solution $y \in Y$, so that a control-state mapping $G : u \mapsto y$ is defined. Moreover, the inverse operator $(A + \dfrac{\partial B}{\partial y}(y,u))^{-1} : Z \to Y$ is assumed to exist for all $(y,u) \in Y \times L^\infty(X)$ as a linear continuous operator. Then the implicit function theorem yields that G is of class C^2 from $L^\infty(X)$ to Y. The first and second order derivatives $G'(u)$ and $G''(u)$ are given as follows: Define $y = G(u)$, $z_h = G'(u)h$, and $z_{h_1 h_2} := G''(u)[h_1, h_2] := (G''(u)h_1)h_2$. Then z_h is the unique solution of

$$
Az + \frac{\partial B}{\partial y}(y,u)\, z + \frac{\partial B}{\partial u}(y,u)\, h = 0, \tag{4.1}
$$

while $z_{h_1 h_2}$ is uniquely determined by

$$
\begin{aligned} Az + \frac{\partial B}{\partial y}(y,u)\, z \quad &= \quad -\{\frac{\partial^2 B}{\partial y^2}(y,u)[z_{h_1}, z_{h_2}] + \frac{\partial^2 B}{\partial y \partial u}(y,u)[z_{h_1}, h_2] \\ &\qquad + \frac{\partial^2 B}{\partial u \partial y}(y,u)[h_1, z_{h_2}] + \frac{\partial^2 B}{\partial u^2}(y,u)[h_1, h_2]\}. \end{aligned} \tag{4.2}
$$

We omit the proof, which can easily be transferred from that of Theorem 2.3 in [3]. The abstract control problem (OC) fits in the optimization problem (P) by

$$
J(u) := F(G(u), u), \qquad G_j(u) := F_j(G(u), u).
$$

In this way, we obtain necessary and/or sufficient conditions for local solutions $(\bar{y}, \bar{u})$ of (OC) by application of the Theorems 2.1, 2.2, 3.1 and Corollary 3.3 provided that the corresponding assumptions (2.1), (A1)–(A3) are satisfied. We tacitly assume this in the sequel and formulate these results in a way, which is convenient for optimal control problems. A *Lagrange function* $\mathcal{L} = \mathcal{L}(y, u, \varphi, \lambda)$ is associated with (OC) by

$$
\mathcal{L}(y, u, \varphi, \lambda) = F(y,u) - \langle \varphi\, ,\, Ay + B(y,u) \rangle + \sum_{j=1}^{m} \lambda_j\, F_j(y,u), \tag{4.3}
$$

where $\varphi \in Z^*$, and $\langle \cdot, \cdot \rangle$ denotes the duality between Z and Z^*. Notice that we must distinct between L for (P) and $\mathcal{L}$ for (OC). We have

$$
J'(\bar{u})h = \frac{\partial F}{\partial y}(\bar{y}, \bar{u}) G'(\bar{u})h + \frac{\partial F}{\partial u}(\bar{y}, \bar{u})h
$$

and obtain similar expressions for $G_j(\bar{u})h$. Therefore, (2.6) yields

$$
\begin{aligned} \frac{\partial L}{\partial u}(\bar{u}, \bar{\lambda})h \quad &= \quad (\frac{\partial F}{\partial y}(\bar{y}, \bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j\, \frac{\partial F_j}{\partial y}(\bar{y}, \bar{u}))G'(\bar{u})h + \\ &\qquad (\frac{\partial F}{\partial u}(\bar{y}, \bar{u}) + \sum_{j=1}^{m} \bar{\lambda}_j\, \frac{\partial F_j}{\partial u}(\bar{y}, \bar{u}))h. \end{aligned} \tag{4.4}
$$

Define an *adjoint state* $\varphi \in Z^*$ by

$$(4.5)\qquad (\frac{\partial F}{\partial y}(\bar y,\bar u)+\sum_{j=1}^{m}\bar\lambda_j\,\frac{\partial F_j}{\partial y}(\bar y,\bar u))\,y=\langle\bar\varphi\,,\,Ay+\frac{\partial B}{\partial y}(\bar y,\bar u)\,y\rangle\quad\forall y\in Y.$$

We assume that $\bar\varphi$ is well defined by (4.5), which is true in our applications. Notice that (4.5) is equivalent to $\partial\mathcal{L}/\partial y\,(\bar y,\bar u,\bar\varphi,\bar\lambda)\,y=0$ for all $y\in Y$ that is $\partial\mathcal{L}/\partial y\,(\bar y,\bar u,\bar\varphi,\bar\lambda)=0$ in the sense of Y^*. Insert $y=z_h=G'(\bar u)h$ in (4.5), then y solves (4.1), and the right hand side of (4.5) is equal to $-\langle\bar\varphi\,,\,\dfrac{\partial B}{\partial u}(\bar y,\bar u)\,h\rangle$. Substituting this for the first item in (4.4) we find that

$$(4.6)\qquad \frac{\partial L}{\partial u}(\bar u,\bar\lambda)h=\frac{\partial}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)h$$

for all $h\in L^\infty(X)$. If (A1) is satisfied, then we deduce from (2.7) that $d(x)$ expresses the derivative $\partial\mathcal{L}/\partial u$, i.e.

$$(4.7)\qquad \frac{\partial\mathcal{L}}{\partial u}\,(\bar y,\bar u,\bar\varphi,\bar\lambda)\,h=\int_X d(x)h(x)d\mu(x).$$

COROLLARY 4.1. *Define J and G_j, $j=1,..,m$, as above and let $\bar u$ with associated state $\bar y$ be a local solution of (OC). If the regularity assumption* (2.1) *is fulfilled, then there are Lagrange multipliers $\bar\lambda_j$, $j=1,..,m$, such that* (2.2), (2.3) *are satisfied. Assume further that $\bar\varphi\in Z^*$ is uniquely determined by* (4.5). *Then* (2.3) *is equivalent with*

$$(4.8)\qquad \frac{\partial\mathcal{L}}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)(u-\bar u)\ge 0\qquad\forall\,u_a\le u\le u_b.$$

If additionally (A1) is satisfied, then $\dfrac{\partial\mathcal{L}}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)$ can be identified with a real function $d=d(x)$, and (4.8) *admits the form*

$$(4.9)\qquad \int_X d(x)(u(x)-\bar u(x))\ge 0\qquad\forall\,u_a\le u\le u_b.$$

Proof. The statement follows from Theorem 2.1: The variational inequality (4.8) is obtained from (2.3) by (2.6) and (4.6). If (A1) is satisfied, then (4.8) and (4.7) imply (4.9) □

Let us now apply the second order conditions to the control system. We have to express $\partial^2L/\partial u^2$ in terms of $\mathcal{L}$. From

$$L(u,\lambda)=F(G(u),u)+\sum_{j=1}^{m}\lambda_j\,F_j(G(u),u)$$

we get after some straightforward computations

$$(4.10)\qquad \begin{aligned}\frac{\partial^2L}{\partial u^2}(\bar u,\bar\lambda)[h_1,h_2] &= (F''(\bar y,\bar u)+\sum_{j=1}^{m}\bar\lambda_j\,F_j''(\bar y,\bar u))[(y_1,h_1),(y_2,h_2)]\\ &\quad+(\frac{\partial F}{\partial y}(\bar y,\bar u)+\sum_{j=1}^{m}\bar\lambda_j\,\frac{\partial F_j}{\partial y}(\bar y,\bar u))G''(\bar u)[h_1,h_2],\end{aligned}$$

where $y_i = G'(\bar u)h_i = z_{h_i}$, $i = 1, 2$. We know that $G''(\bar u)[h_1, h_2] = z_{h_1h_2}$, where $z = z_{h_1h_2}$ is the solution of (4.2), hence this term can be reduced to z_{h_1} and z_{h_2}. By definition of $\bar\varphi$, (4.2), and (4.5),

$$\begin{aligned}(\frac{\partial F}{\partial y} + \sum_{j=1}^{m} \bar\lambda_j \frac{\partial F_j}{\partial y}) z_{h_1h_2} &= \langle \bar\varphi\,,\, Az_{h_1h_2} + \frac{\partial B}{\partial y} z_{h_1h_2}\,\rangle \\ &= -\langle \bar\varphi\,,\, B''(\bar y, \bar u)[(z_{h_1}, h_1), (z_{h_2}, h_2)]\rangle\end{aligned}$$

is obtained. Insert this in (4.10), then $y_i = z_{h_i}$ and $z_{h_1h_2} = G''(\bar u)[h_1, h_2]$ give

$$\begin{aligned}\frac{\partial^2 L}{\partial u^2}(\bar u, \bar\lambda)[h_1, h_2] &= (F''(\bar y, \bar u) + \sum_{j=1}^{m} \bar\lambda_j F_j''(\bar y, \bar u))[(y_1, h_1), (y_2, h_2)] \\ &\quad -\langle \bar\varphi\,,\, B''(\bar y, \bar u)[(y_1, h_1), (y_2, h_2)]\rangle \\ &= \mathcal{L}''_{(y,u)}(\bar y, \bar u, \bar\varphi, \bar\lambda)[(y_1, h_1), (y_2, h_2)].\end{aligned} \tag{4.11}$$

Notice that in (4.11) the increments (y_i, h_i) cannot be chosen independently, since y_i and h_i are coupled through $y_i = G'(\bar u)h_i = z_{h_i}$. Hence the definition of z_{h_i} shows that the pairs $(y, h) = (y_i, h_i)$ have to solve the *linearized equation*

$$Ay + \frac{\partial B}{\partial y}(\bar y, \bar u)\, y + \frac{\partial B}{\partial u}\, h = 0. \tag{4.12}$$

COROLLARY 4.2. *Assume that* (2.1)*, (A1), and (A2) are satisfied and that* $\bar\varphi \in Z^*$ *is uniquely defined by* (4.5)*. Then*

$$\mathcal{L}''_{(y,u)}(\bar y, \bar u, \bar\varphi, \bar\lambda)(y, h)^2 \ge 0 \tag{4.13}$$

holds for all $(y, h) \in Y \times L^\infty(X)$*, which satisfy the linearized equation* (4.12) *and the relations*

$$\begin{aligned}&\frac{\partial F_j}{\partial y}(\bar y, \bar u)\, y + \frac{\partial F_j}{\partial u}(\bar y, \bar u)\, h = 0 \quad && \textit{if } (j \le m_1) \\ & && \textit{or } (j > m_1,\ F_j(\bar y, \bar u) = 0 \textit{ and } \bar\lambda_j > 0); \\ &\frac{\partial F_j}{\partial y}(\bar y, \bar u)\, y + \frac{\partial F_j}{\partial u}(\bar y, \bar u)\, h \le 0 \quad && \textit{if } j > m_1,\ F_j(\bar y, \bar u) = 0 \textit{ and } \bar\lambda_j = 0;\end{aligned} \tag{4.14}$$

$$h(x) = \begin{cases} \ge 0 & \textit{if } \bar u(x) = u_a(x); \\ \le 0 & \textit{if } \bar u(x) = u_b(x); \end{cases} \tag{4.15}$$

$$h(x) = 0 \quad \textit{if} \quad x \in X^0. \tag{4.16}$$

The second order sufficient optimality conditions are given by the

COROLLARY 4.3. *Let* $(\bar y, \bar u)$ *fulfill all constraints of (OC) and, together with* $\bar\varphi$ *and* $\bar\lambda_j$*,* $j = 1, .., m$*, the first order optimality conditions stated in Corollary 4.1. Assume that* (2.1)*,* **(A1)***, and* **(A3)** *hold true. If there exist* $\tau > 0$*,* $\delta_1 > 0$*, and* $\delta_2 > 0$ *such that*

$$\mathcal{L}''_{(y,u)}(\bar y, \bar u, \bar\varphi, \bar\lambda)(y, h)^2 \ge \delta_1 \|h\|^2_{L^2(X \setminus X^\tau)} - \delta_2 \|h\|^2_{L^2(X^\tau)} \tag{4.17}$$

holds for all $(y, h) \in Y \times L^\infty(X)$*, which satisfy the linearized equation* (4.12) *and the relations* (4.14), (4.15)*. Then the conclusions of Theorem 3.1 hold true, hence* $\bar{u}$ *is a local solution of (OC). Here, the set* X^τ *is defined by* (3.2)*. The same conclusion is true, if the condition*

$$\mathcal{L}''_{(y,u)}(\bar{y}, \bar{u}, \bar{\varphi}, \bar{\lambda})(y, h)^2 \geq \delta\, \|h\|^2_{L^2(X)} \tag{4.18}$$

holds instead of (4.17) *with some* $\delta > 0$*, where* $h(x) = 0 \quad \forall x \in X^\tau$ *for some* $\tau > 0$*, and (y,h) are subject to* (4.12), (4.14)*, and* (4.15).

4.2. Optimal control of ODEs. In this section we discuss an optimal control problem governed by an ordinary differential equation. We concentrate on a very simplified setting to give the reader an easy insight in the application of the theory. For further problems and open questions we refer to the survey by Hartl, Sethi and Vickson [7]. Define

$$\begin{array}{rcl} F(y,u) & = & \psi(y(T)) + \int_0^T f_0(t, y(t), u(t))dt \\ F_j(y,u) & = & \int_0^T f_j(t, y(t), u(t))dt, \end{array}$$

$j = 1, .., m$, and regard the optimal control problem

$$\text{(ODE)} \left\{ \begin{array}{ll} \text{Minimize } F(y,u) & \\ y'(t) + b(t, y(t), u(t)) = 0 & \text{a.e. } t \in (0,T), \\ y(0) = 0, & \\ & \\ u_a(t) \leq u(t) \leq u_b(t) & \text{a.e. } t \in (0,T), \\ F_j(y,u) = 0, & 1 \leq j \leq m_1, \\ F_j(y,u) \leq 0, & m_1 + 1 \leq j \leq m. \end{array} \right.$$

Here, T is a fixed time. To reduce the amount of technicalities, let us discuss only real-valued functions y and u. The vector-valued case can be handled analogously. For the same reason, we assume that the functions ψ, f_j, and b are of class C^2 on $\mathbb{R}$ and $[0,T] \times \mathbb{R} \times [\min u_a, \max u_b]$, respectively, although weaker Carathéodory type conditions would suffice. We introduce the state space $Y = \{y \in W^{1,\infty}(0,T) \,|\, y(0) = 0\}$ and put

$$(Ay)(t) = y'(t), \qquad (B(y,u))(t) = b(t, y(t), u(t)).$$

A is continuous from Y to $Z = L^\infty(0,T)$, and B is of class C^2 from $Y \times L^\infty(0,T)$ to Z. In this way, (ODE) is related to (OC) as a particular case, where $X = [0,T]$, and μ is the Lebesgue measure, $d\mu = dt$. For convenience, the variable $t \in X$ is substituted for the variable x, which was used in the former sections.

Let $(\bar{y}, \bar{u}) \in Y \times L^\infty(0,T)$ be our *reference solution*, a given candidate for optimality. For (ODE), the Lagrange function

$$\mathcal{L}(y, u, \varphi, \lambda) = F(y,u) - \int_0^T \varphi\,(y' + b(t,y,u))dt + \sum_{j=1}^m \lambda_j\, F_j(y,u) \tag{4.19}$$

is introduced, where $\varphi \in W^{1,\infty}(0,T)$ will be defined by the adjoint equation below. In an obvious way this φ generates a linear functional belonging to Z^*, but it has more regularity than arbitrary functionals of this space.

REMARK 4.4. *Given the inhomogeneous initial condition $y(0) = y_0$, we have to work with the space $Y = W^{1,\infty}(0,T)$ and must include the initial condition in the definition of A. Then the additional term $\varphi_0(y(0) - y_0)$ would appear in* (4.19). *This requires some more notational effort. However, the optimality conditions are not changed. Therefore, w.l.o.g. we confine ourselves to a homogeneous initial condition.*

Having in mind the particular form of φ, we see that here (4.5) is nothing more than the definition of the *adjoint equation*

$$
\begin{aligned}
-\varphi' + \frac{\partial b}{\partial y}(t,\bar y,\bar u)\,\varphi &= \frac{\partial f_0}{\partial y}(t,\bar y,\bar u) + \sum_{j=1}^m \bar\lambda_j\,\frac{\partial f_j}{\partial y}(t,\bar y,\bar u)\\
\varphi(T) &= \psi'(y(T)).
\end{aligned}
\tag{4.20}
$$

It is obvious that (4.20) admits a unique solution $\bar\varphi \in W^{1,\infty}(0,T)$. In section 5 we show that (A1) is satisfied for (ODE). We obtain the following derivatives of the Lagrange function:

$$
\frac{\partial \mathcal{L}}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)h = \int_0^T (\frac{\partial f_0}{\partial u} - \bar\varphi\,\frac{\partial b}{\partial u} + \sum_{j=1}^m \bar\lambda_j\,\frac{\partial f_j}{\partial u}\,h\,dt
\tag{4.21}
$$

(all derivatives taken at $(\bar y,\bar u)$), hence $\partial\mathcal{L}/\partial u$ can be identified with $d \in L^\infty(0,T)$,

$$
d(t) = (\frac{\partial f_0}{\partial u} - \bar\varphi\,\frac{\partial b}{\partial u} + \sum_{j=1}^m \bar\lambda_j\,\frac{\partial f_j}{\partial u})(t).
\tag{4.22}
$$

The second derivative of $\mathcal{L}$ is

$$
\begin{aligned}
&\mathcal{L}''_{(y,u)}(\bar y,\bar u,\bar\varphi,\bar\lambda)[(y_1,h_1),(y_2,h_2)] = \psi''(\bar y(T))y_1(T)y_2(T)\\
&\quad + \textstyle\int_0^T \{(y_1,h_1)(f_0''(\bar y,\bar u) - \bar\varphi\, b''(\bar y,\bar u) + \sum\limits_{j=1}^m \bar\lambda_j\, f_j''(\bar y,\bar u))(y_2,h_2)^\top\}\,dt,
\end{aligned}
\tag{4.23}
$$

where f_0'', b'', f_j'' stand for 2×2-Hessian matrices taken at $(t,\bar y(t),\bar u(t))$. It is easy to verify that (A2) is satisfied.

The *first order necessary optimality conditions* are stated in Corollary 4.1. In particular, the following variational inequality has to be satisfied:

$$
\int_X d(t)(u(t) - \bar u(t))\,dt \ge 0
\tag{4.24}
$$

for all $u_a \le u(t) \le u_b$, hence $\bar u(t) = u_a$ where $d(t) > 0$, and $\bar u(t) = u_b$ where $d(t) < 0$. (These points form the set X^0.) No information is obtained where d is zero. Roughly speaking, this is the set, where higher order conditions are needed.

The *second order necessary conditions* are formulated in Corollary 4.2. We have to specify the linearized equation (4.12) and the form of the derivatives in the relations (4.14). The *linearized equation* is

$$
\begin{aligned}
y' + \frac{\partial b}{\partial y}(t,\bar y,\bar u)\,y + \frac{\partial b}{\partial u}(t,\bar y,\bar u)\,h &= 0,\\
y(0) &= 0,
\end{aligned}
\tag{4.25}
$$

while

$$
\frac{\partial F_j}{\partial y}(\bar y,\bar u)\,y + \frac{\partial F_j}{\partial u}(\bar y,\bar u)\,h = \int_X \{\frac{\partial f_j}{\partial y}(t,\bar y,\bar u)\,y + \frac{\partial f_j}{\partial u}(t,\bar y,\bar u)\,h\}\,dt.
\tag{4.26}
$$

4.3. Optimal boundary control of an elliptic equation. As a further application, we consider an elliptic control problem. For convenience, we discuss a simplified version and refer for further reading to [4].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{0,1}$. Let ν denote the outward unit normal vector at Γ and ∂_ν be the associated normal derivative. Define

$$\begin{aligned} F(y,u) &= \textstyle\int_\Omega \gamma_0(x,y(x))dx + \int_\Omega \psi_0(x,y(x))d\mu_0(x) + \int_\Gamma f_0(x,y(x),u(x))dS(x) \\ F_j(y,u) &= \textstyle\int_\Omega \gamma_j(x,y(x))dx + \int_\Omega \psi_j(x,y(x))d\mu_j(x) + \int_\Gamma f_j(x,y(x),u(x))dS(x), \end{aligned}$$

$j = 1,..,m$. We assume that the functions $\gamma_j = \gamma_j(x,y)$, $\psi_j = \psi_j(x,y)$, and $f_j = f_j(x,y,u)$ are of class C^2 on $\bar\Omega \times \mathbb{R}$ and $\bar\Omega \times \mathbb{R}^2$, respectively. Moreover, real Borel measures μ_j are given on Ω. Here, μ is the Lebesgue surface measure induced on Γ, $d\mu = dS$. The appearance of the measures μ_j in the functionals will heavily influence the verification of the assumptions (A1)-(A3). Therefore, the easier case $\Psi_j = 0$, $j = 1,..,m$, is of interest as well.

Regard the optimal control problem

$$\text{(ELL)} \begin{cases} \text{Minimize } F(y,u) & \\ -\Delta y + y = 0 & \text{in } \Omega, \\ \partial_\nu y + b(x,y,u) = 0 & \text{on } \Gamma, \\ & \\ u_a(x) \le u(x) \le u_b(x) & \text{a.e. on } \Gamma, \\ F_j(y,u) = 0, & 1 \le j \le m_1, \\ F_j(y,u) \le 0, & m_1 + 1 \le j \le m. \end{cases}$$

In this setting, the *boundary control* u is looked upon in the space $L^\infty(\Gamma)$, hence $X = \Gamma$, while the *state* y belongs to $Y = \{y \in H^1(\Omega) \,|\, -\Delta y + y \in L^q(\Omega),\ \partial_\nu y \in L^p(\Gamma)\}$ ($q > N/2$ and $p > N-1$ are given fixed). Endowed Y with the graph norm, it is known that $Y \subset C(\bar\Omega)$, the embedding being continuous. Assume that $b = b(x,y,u)$ satisfies the same conditions as the f_j. Additionally, we require that $(\partial b/\partial y)(x,y,u) \ge 0$ on $\Gamma \times \mathbb{R} \times [\min u_a, \max u_b]$. Define

$$A : Y \to L^q(\Omega) \times L^p(\Gamma) \quad \text{and} \quad B : Y \times L^\infty(\Gamma) \to L^q(\Omega) \times L^p(\Gamma)$$

by

$$(Ay) = \begin{pmatrix} -\Delta y + y \\ \partial_\nu y \end{pmatrix} \quad \text{and} \quad B(y,u)(x) = \begin{pmatrix} 0 \\ b(x,y(x),u(x)) \end{pmatrix}.$$

The equation $Ay + B(y,u) = 0$, which is equivalent to our elliptic boundary value problem, admits for each $u \in L^\infty(\Gamma)$ exactly one solution $y \in Y$. The mapping $u \mapsto y$ is of class C^2 from $L^\infty(\Gamma)$ to Y. Now we proceed in the same way as in the preceding section. The Lagrange function is

$$\begin{aligned} \mathcal{L}(y,u,\varphi,\lambda) &= F(y,u) - \int_\Omega (-\Delta y + y)\,\varphi\,dx \\ &\quad - \int_\Gamma (\partial_\nu y + b(x,y,u))\,\varphi\,dS + \sum_{j=1}^m \lambda_j\,F_j(y,u), \end{aligned}$$

where $\varphi \in W^{1,s}(\Omega)$ for all $s < \frac{N}{N-1}$ is the *adjoint state.* The adjoint state φ together with its trace $\varphi_{|\Gamma}$ forms a Lagrange multiplier of $Z^* = L^{q'}(\Omega) \times L^{p'}(\Gamma)$ having higher regularity. Here (4.5) reduces to the adjoint equation

$$\begin{aligned}-\Delta\varphi+\varphi &= \frac{\partial\gamma_0}{\partial y}+\frac{\partial\psi_0}{\partial y}\mu_0|_\Omega+\sum_{j=1}^m\bar\lambda_j\,(\frac{\partial\gamma_j}{\partial y}+\frac{\partial\psi_j}{\partial y}\mu_j|_\Omega)\\ \partial_\nu\varphi+\frac{\partial b}{\partial y}\varphi &= \frac{\partial f_0}{\partial y}+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial f_j}{\partial y}+\frac{\partial\psi_0}{\partial y}\mu_0|_\Gamma+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial\psi_j}{\partial y}\mu_j|_\Gamma\end{aligned}$$

(all partial derivatives taken at $(x,\bar y(x),\bar u(x))$). This equation has a unique solution $\bar\varphi\in W^{1,s}(\Omega)$ associated with $(\bar y,\bar u,\bar\lambda)$. Notice that for $N=2$ the Sobolev imbedding theorem yields $\varphi\in L^\sigma(\Omega)$ for all $\sigma<\infty$, but not in general $\varphi\in L^\infty(\Omega)$. For $N\ge 3$ the regularity of φ is even lower. This indicates that we have to discuss the assumptions (A1) – (A3) with more care. We shall do this in the last section.

The situation is easier in the case $\Psi_j=0$, $j=0,..,m$. Then all data given in the adjoint equation are bounded and measurable, and the regularity theory of elliptic equations yields $\bar\varphi\in C(\bar\Omega)$ (see [2]).

Let us establish the first and second order derivatives of $\mathcal{L}$. We get

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)\,h=\\ &=\int_\Gamma(\frac{\partial f_0}{\partial u}(x,\bar y,\bar u)+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial f_j}{\partial u}(x,\bar y,\bar u)-\bar\varphi\,\frac{\partial b}{\partial u}(x,\bar y,\bar u))\,h\,dS\end{aligned}$$

and

$$\begin{aligned}&\mathcal{L}''_{(y,u)}(\bar y,\bar u,\bar\varphi,\bar\lambda)[(y_1,h_1),(y_2,h_2)]=\\ &=\int_\Gamma(y_1,h_1)(f_0''(x,\bar y,\bar u)+\sum_{j=1}^m\bar\lambda_j f_j''(x,\bar y,\bar u)-\bar\varphi\,b''(x,\bar y,\bar u))(y_2,h_2)^\top\,dS\\ &+\int_\Omega(\frac{\partial^2\gamma_0}{\partial y^2}(x,\bar y)+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial^2\gamma_j}{\partial y^2}(x,\bar y))\,y_1y_2\,dx\\ &+\int_\Omega\frac{\partial^2\psi_0}{\partial y^2}(x,\bar y)\,y_1y_2\,d\mu_0+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial^2\psi_j}{\partial y^2}(x,\bar y)\,y_1y_2\,d\mu_j.\end{aligned}$$

We observe that, due to our notation, there is almost no difference to the expressions derived for the case of (ODE) in (4.21), (4.23). The first and second order conditions for our elliptic problem (ELL) admit the following form: Put

$$d(x)=\frac{\partial f_0}{\partial u}(x,\bar y(x),\bar u(x))+\sum_{j=1}^m\bar\lambda_j\,\frac{\partial f_j}{\partial u}(x,\bar y(x),\bar u(x))-\bar\varphi\,\frac{\partial b}{\partial u}(x,\bar y(x),\bar u(x)).$$

Then d has the same form as in (4.22). The first and second order optimality conditions are given by the Corollaries 4.1–4.3. We put there $X=\Gamma$ to obtain all first and second order conditions for (ELL). Now the directions (y,h) are coupled through the *linearized boundary value problem*

$$\begin{aligned}&-\Delta y+y=0\\ &\partial_\nu y+\frac{\partial b}{\partial y}(x,\bar y,\bar u)y+\frac{\partial b}{\partial u}(x,\bar y,\bar u)h=0.\end{aligned}\tag{4.27}$$

The derivatives in (4.14), (4.15) admit the form

$$
(4.28)\quad \begin{aligned}
\frac{\partial F_j}{\partial y}(\bar y,\bar u)\,y+\frac{\partial F_j}{\partial u}(\bar y,\bar u)\,h \;=\;& \int_\Omega \frac{\partial \gamma_j}{\partial y}(t,\bar y)\,y\,dx+\int_\Omega \frac{\partial \psi_j}{\partial y}(t,\bar y)\,y\,d\mu_j\\
&+\int_\Gamma\{\frac{\partial f_j}{\partial y}(t,\bar y,\bar u)\,y+\frac{\partial f_j}{\partial u}(t,\bar y,\bar u)\,h\}\,dS.
\end{aligned}
$$

In this way, we have obtained the second order sufficient condition for a simplified elliptic control problem. For the discussion of more general problems we refer to [3], [4]. We should underline again that so far we have stated the optimality condition in a formal way. It remains to verify (A1)–(A3) to make our theory work. Low regularity of the adjoint state φ can be an essential obstacle for this. We refer to section 5.

4.4. Optimal distributed control of a parabolic equation. We confine ourselfes to a distributed parabolic control problem. A more general class, including also boundary control and boundary observation, is considered in a forthcoming paper by Raymond and Tröltzsch [10]. Let Ω be defined as in the last section and put $Q=\Omega\times(0,T)$, $\Sigma=\Gamma\times(0,T)$. Define

$$
\begin{aligned}
F(y,u) \;=\;& \int_\Omega \gamma_0(x,y(x,T))dx+\int_\Omega \psi_0(x,y(x,T))d\mu_0(x)+\\
&+\int_Q f_0(x,t,y(x,t),u(x,t))dxdt\\
F_j(y,u) \;=\;& \int_Q \psi_j(x,t,y(x,t))d\mu_j(x,t)+\int_Q f_j(x,t,y(x,t),u(x,t))dxdt,
\end{aligned}
$$

$j=1,..,m$. We assume again that the functions ψ_j, f_j, and γ_j are of class C^2 on $\bar Q\times\mathbb{R}$ and $\bar Q\times\mathbb{R}^2$, respectively. Moreover, real Borel measures μ_j, $j=0,..,m$ are given on Ω and Q, respectively. Now μ is the Lebesgue measure on Q, $d\mu=dxdt$. Regard the optimal control problem

$$
(\mathrm{PAR})\left\{\begin{array}{ll}
\text{Minimize } F(y,u) & \\
\dfrac{\partial y}{\partial t}-\Delta y+b(x,t,y,u)=0 & \text{in } Q,\\
\partial_\nu y=0 & \text{on } \Sigma,\\
y(x,0)=0 & \text{in } \Omega,\\
 & \\
u_a(x,t)\le u(x,t)\le u_b(x,t) & \text{a.e. on } Q,\\
F_j(y,u)=0 & 1\le j\le m_1,\\
F_j(y,u)\le 0 & m_1+1\le j\le m.
\end{array}\right.
$$

In this setting, the *distributed control* u is looked upon in the space $L^\infty(Q)$, hence we put $X=Q$. The *state* y belongs to $Y=\{y\in W(0,T)\,|\,y(0)=0,\,y_t-\Delta y\in L^q(Q),\,\partial_\nu y\in L^p(\Sigma)\}$, where $q>N/2+1$ and $p>N+1$ are given fixed. It is known that $Y\subset C(\bar Q)$, the embedding being continuous for the graph norm. Assume that $b=b(x,t,y,u)$ satisfies the same conditions as the f_j. Additionally, we require that $\partial b/\partial y(x,t,y,u)\ge 0$ on $Q\times\mathbb{R}\times[\min u_a,\max u_b]$. Define

$$
A:Y\to L^q(Q)\times L^p(\Sigma)\quad\text{and}\quad B:Y\times L^\infty(Q)\to L^q(Q)\times L^p(\Sigma)
$$

by

$$
Ay=\begin{pmatrix}\dfrac{\partial y}{\partial t}-\Delta y\\ \partial_\nu y\end{pmatrix}\quad\text{and}\quad B(y,u)(x,t)=\begin{pmatrix} b(x,t,y(x,t),u(x,t))\\ 0\end{pmatrix}.
$$

The equation $Ay + B(y,u) = 0$, which is equivalent to our parabolic initial-boundary value problem, admits for each $u \in L^\infty(Q)$ exactly one solution $y \in Y$. We refer to [2]. The mapping $u \mapsto y$ is of class C^2 from $L^\infty(Q)$ to Y. Here, the Lagrange function is

$$\begin{aligned}\mathcal{L}(y,u,\varphi,\lambda) &= F(y,u) - \int_Q (y_t - \Delta y - b(x,t,y,u))\,\varphi\,dxdt \\ &\quad - \int_\Sigma \partial_\nu y\,\varphi\,dSdt + \sum_{j=1}^m \lambda_j\,F_j(y,u),\end{aligned}$$

where φ is the *adjoint state* and dS denotes again the Lebesgue surface measure induced on Γ. Equation (4.5) turns out to be the adjoint equation

$$\begin{aligned}-\frac{\partial\varphi}{\partial t} - \Delta\varphi + \frac{\partial b}{\partial y}\varphi &= \frac{\partial f_0}{\partial y} + \sum_{j=1}^m \bar\lambda_j\,(\frac{\partial f_j}{\partial y} + \frac{\partial\psi_j}{\partial y}\mu_j) && \text{in Q,}\\ \partial_\nu\varphi &= 0 && \text{in } \Sigma,\\ \varphi(x,T) &= \frac{\partial\gamma_0}{\partial y}(x,\bar y(x,T)) + \frac{\partial\psi_0}{\partial y}(x,\bar y(x,T))\,\mu_0 && \text{in } \Omega\end{aligned}$$

(all partial derivatives taken at $(x,\bar y,\bar u)$). This equation has a unique solution $\bar\varphi \in W^{1,s}(\Omega)$ associated with $(\bar y,\bar u,\bar\varphi,\bar\lambda)$. If, however, $\psi_j = 0$, $j = 1,..,m$, then $\bar\varphi$ is more regular, $\bar\varphi \in W(0,T) \cap C(\bar Q)$.
The relevant derivatives of $\mathcal{L}$ are

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial u}(\bar y,\bar u,\bar\varphi,\bar\lambda)\,h = \\ &= \int_Q (\frac{\partial f_0}{\partial u}(x,\bar y,\bar u) + \sum_{j=1}^m \bar\lambda_j\,\frac{\partial f_j}{\partial u}(x,\bar y,\bar u) - \bar\varphi\,\frac{\partial b}{\partial u}(x,\bar y,\bar u)\,)h\,dxdt \\ &= \int_Q d(x,t)h(x,t)\,dxdt,\end{aligned}$$

$$\begin{aligned}&\mathcal{L}''_{(y,u)}(\bar y,\bar u,\bar\varphi,\bar\lambda)[(y_1,h_1),(y_2,h_2)] = \\ &= \int_Q (y_1,h_1)(f_0''(x,\bar y,\bar u) + \sum_{j=1}^m \bar\lambda_j f_j''(x,\bar y,\bar u) - \bar\varphi\,b''(x,\bar y,\bar u))(y_2,h_2)^\top\,dxdt \\ &+ \int_\Omega \frac{\partial^2\psi_0}{\partial y^2}(x,\bar y(T))\,y_1(T)y_2(T)\,d\mu_0 + \int_Q \sum_{j=1}^m \bar\lambda_j\,\frac{\partial^2\psi_j}{\partial y^2}(x,\bar y)\;y_1y_2\,d\mu_j \\ &+ \int_\Omega \frac{\partial^2\gamma_0}{\partial y^2}(x,\bar y(T))\,y_1(T)y_2(T)\,dx.\end{aligned}$$

The first and second order conditions for the parabolic case are covered by the Corollaries 4.1–4.3. We have to substitute there Q for X and to replace the variable x by (x,t). Moreover, in the second order conditions y and h are coupled through the *linearized initial-boundary value problem*

$$\begin{aligned}&y_t - \Delta y + \frac{\partial b}{\partial y}(x,t,\bar y,\bar u)\,y + \frac{\partial b}{\partial u}(x,t,\bar y,\bar u)\,h = 0\\ &\partial_\nu y = 0\\ &y(x,0) = 0.\end{aligned} \tag{4.29}$$

We leave the calculations of the derivatives in (4.14) to the reader. They are obtained by an obvious modification of (4.28). We should mention again that these optimality conditions are only meaningful, if the assumptions (A1)–(A3) are satisfied.

5. Verification of the assumptions. Our theory relies on the general assumptions (A1)–(A3). We shall see that (A1)–(A3) are naturally satisfied for the problem (ODE), while the situation is more complicated in the case of the elliptic or parabolic PDE.

(i) Problem (ODE)

(A1): It is obviously sufficient to regard one of the functionals $G_j(u) = F_j(G(u), u)$ to assess the situation. We have

$$G_j'(\bar u)h = \int_0^T \frac{\partial f_j}{\partial y}(t, \bar y, \bar u)\, y\, dt + \int_0^T \frac{\partial f_j}{\partial u}(t, \bar y, \bar u)\, h\, dt, \tag{5.1}$$

where $y = G'(\bar u)h$. Here, $\partial f_j/\partial y$, $\partial f_j/\partial u$ are bounded and measurable functions. Moreover, the estimate

$$\|y\|_{C[0,T]} = \|G'(\bar u)\, h\|_{C[0,T]} \le c\, \|h\|_{L^2(0,T)} \tag{5.2}$$

holds, since $\|y\|_{C[0,T]} \le c\, \|y\|_{H^1(0,T)} \le c\, \|h\|_{L^2(0,T)}$. Thus the mapping $h \mapsto G_j'(\bar u)h$ defines a linear and continuous functional on $L^2(0,T)$. By the Riesz representation theorem,

$$G_j'(\bar u)\, h = \int_0^T g_j(t)\, h(t)\, dt \tag{5.3}$$

must hold with some $g_j \in L^2(0,T)$, hence (A1) is fulfilled.

(A2): Here, the derivative

$$G_j''(\bar u)[h_1, h_2] = \int_0^T (y_1, h_1)\, f_j''(t, \bar y, \bar u)\, (y_2, h_2)^\top\, dt$$

is characteristic for the discussion. All entries of f_j'' are bounded and measurable. If $h_i^k \to h_i$ in $L^2(0,T)$, $k \to \infty$, $i = 1, 2$, then $y_i^k \to y_i$ in $C[0,T]$, hence $G_j''(\bar u)[h_1^k, h_2^k] \to G_j''(\bar u)[h_1, h_2]$. This shows (A2).

(A3): First, we must estimate differences of the type $G_j''(\tilde u) - G_j''(\bar u)$ for $\tilde u$ in a L^∞-neighbourhood of $\bar u$. We get

$$|(G_j''(\tilde u) - G_j''(\bar u))\, h^2| \le \int_0^T |f_j''(t, \tilde y, \tilde u) - f_j''(t, \bar y, \bar u)|\, |(y, h)|^2\, dt,$$

where $\tilde y = G(\tilde u)$, $\bar y = G(\bar u)$, $y = G'(\bar u)\, h$. Due to our assumptions, we find that

$$|(G_j''(\tilde u) - G_j''(\bar u))\, h^2| \le \delta(\|y\|^2_{C[0,T]} + \|h\|^2_{L^2(0,T)}) \le c\, \delta \|h\|^2_{L^2(0,T)}, \tag{5.4}$$

where $\delta \to 0$ as $\|\tilde u - \bar u\|_{L^\infty} \to 0$. Another characteristic part in $\partial^2 L/\partial u^2$ is the coupling of the nonlinearity b with $\bar\varphi$. It is the essential advantage of our simplified case (ODE) that $\bar\varphi \in L^\infty(0,T)$. Therefore, we are justified to estimate

$$\begin{aligned} |\textstyle\int_0^T (y, h)\, b''(t, \bar y, \bar u)\, (y, h)^\top \bar\varphi\, dt| &\le c\, \|\bar\varphi\|_{L^\infty(0,T)}\, (\|y\|^2_{C[0,T]} + \|h\|^2_{L^2(0,T)}) \\ &\le c\, \|h\|^2_{L^2(0,T)}. \end{aligned} \tag{5.5}$$

Discussing all second order terms in this way, we easily verify that (A3) is satisfied, too.

(ii) Elliptic Problem (ELL)

We repeat the discussion of (A1)–(A3) along the lines of (i) but concentrating on the essential differences to the case of (ODE). Here, it holds

$$\begin{aligned} G_j'(\bar u)h \;&=\; \int_\Omega \frac{\partial \gamma_j}{\partial y}(x,\bar y)\, y\, dx + \int_\Omega \frac{\partial \psi_j}{\partial y}(x,\bar y)\, y\, d\mu_j + \\ &\quad + \int_\Gamma \frac{\partial f_j}{\partial y}(x,\bar y,\bar u)\, y\, dS + \int_\Gamma \frac{\partial f_j}{\partial u}(x,\bar y,\bar u)\, h\, dS, \end{aligned}$$

where $y = G'(\bar u)\,h$. In contrast to (5.2), now the mapping $G'(\bar u)$ is not in general continuous from $L^2(\Gamma)$ to $C(\bar\Omega)$. This property only holds for $N = \dim\Omega = 2$ (see ([4]). For $N > 2$ we assume that Ω_j, the support of μ_j, satisfies $\bar\Omega_j \subset \Omega$. Then the mapping $h \mapsto G'(\bar u)\,h$ is continuous from $L^2(\Gamma)$ to $C(\bar\Omega_j)$, hence $h \mapsto G_j'(\bar u)h$ is a linear and continuous functional on $L^2(\Gamma)$. The Riesz theorem yields a representation analogous to (5.3). Hence (A1) is shown under additional assumptions on the subdomains Ω_j. (A2) then holds true in the same way. Notice that the restriction to Ω_j is not needed, if all ψ_j vanish.

To verify (A3) we need even more restrictions on the data. The situation is easy, if $\psi_j = 0$, $j = 1,..,m$. Then all given data in the adjoint equation are bounded and measurable, and the regularity theory of elliptic equations yields $\bar\varphi \in C(\bar\Omega)$. In this case, (A3) is obviously satisfied.

Let us now assume that at least one of the ψ_j is not zero. Then the best regularity of the trace $\bar\varphi_{|\Gamma}$ is $\bar\varphi_{|\Gamma} \in L^r(\Gamma)$ for all $r < (N-1)/(N-2)$. For instance, $\varphi \in L^r(\Gamma)$ for all $r < \infty$ is obtained in the case $N = 2$. We therefore cannot assume that $\bar\varphi \in L^\infty(\Omega)$. Regard the elliptic counterpart to (5.5),

$$(5.6)\quad \begin{aligned} \left|\int_\Gamma (y,h)\, b''(x,\bar y,\bar u)\,(y,h)^\top \bar\varphi\, dS\right| \;&=\; \left|\int_\Gamma \bar\varphi\,\Big(\frac{\partial^2 b}{\partial y^2}y^2 + 2\frac{\partial^2 b}{\partial y\partial u}yh + \frac{\partial^2 b}{\partial u^2}h^2\Big)\, dS\right| \\ &\le\; c \int_\Gamma (|\bar\varphi| y^2 + |\bar\varphi| yh + |\bar\varphi| h^2)\, dS. \end{aligned}$$

This expression has to be estimated for $h \in L^2(\Gamma)$. If $\bar\varphi_{|\Gamma} \notin L^\infty(\Gamma)$, which is the normal case, then we must exclude the third term from (5.6). This means that $\partial^2 b/\partial u^2$ has to disappear – u must appear linearly. Next we consider the second term, where $\|\bar\varphi_{|\Gamma}\, y\|_{L^2(\Gamma)}$ is to estimate against $\|h\|_{L^2(\Gamma)}$. The mapping $h \mapsto y$ is continuous from $L^2(\Gamma)$ to $C(\Gamma)$ $(N = 2)$, to $L^r(\Gamma)$ for all $r < \infty$ $(N = 3)$, and to $L^r(\Gamma)$ for all $r < 2(N-1)/(N-3)$ $(N > 3)$. Therefore, the second term can be estimated iff $N = 2$, while it must be cancelled for $N > 2$. The latter means $\partial^2 b/\partial u\partial y = 0$ – here $b = b_1(x,y) + b_2(x)u$ must hold. In the same way we arrive at the surprising fact that for $N > 3$ the first term in (5.6) must vanish, too. In other words: In the case of elliptic boundary control with *pointwise* functionals F_j we cannot admit nonlinear equations for $N > 3$.

Remark: *We should underline again that these restrictions are not needed, if the functionals F_j are sufficiently regular ($\psi_j = 0$, $j = 1,..,m$). Moreover, the case of distributed controls permits to slightly relax the restrictions on the dimension N.*

(iii) Parabolic Problem (PAR)

Once again, (A1)–(A3) are satisfied, if $\psi_j = 0$, $j = 1,..,m$. This is due to the high regularity $\bar\varphi \in W(0,T) \cap C(\bar Q)$ in this case.

In the opposite case, the problem of regularity is even more delicate than in the elliptic problem. We cannot discuss the general case in detail and refer to the forthcoming paper [10]. Instead of this, let us explain the point for a very particular constraint: Suppose that only one (pointwise) state-constraint of the form

$$g_1(y,u) = \int_0^T y(x_1,t)\,dt = 0$$

is given, where $x_1 \in \Omega$ is a fixed position of observation. To make the theory work, we need some strong restrictions: We assume $N = \dim \Omega = 1$, i.e. $\Omega = (a,b)$ and require that $\partial^2 b/\partial u^2 = 0$ (the control appears linearly). Then the mapping $h \mapsto y = G'(\bar{u})h$ is continuous from $L^2(Q)$ to $C(\bar{Q})$, and the functional $h \mapsto g_1(y,h)$ is continuous on $L^2(Q)$. We know that $\bar{\varphi} \in L^s(Q)$ for all $s < 3$ (this follows from Thm. 4.3 in [10] for $N = 1$ and $\alpha = \tilde{\alpha}$). Hence $\bar{\varphi} \notin L^\infty(Q)$, and that is the reason why we cannot admit a control appearing nonlinearly. The estimate of the parabolic counterpart of (5.6) is

$$\begin{aligned} &|\textstyle\int_Q (\frac{\partial^2 b}{\partial y^2}\bar{\varphi}\, y^2 + 2\frac{\partial^2 b}{\partial y \partial u}\bar{\varphi}\, y\, h)\, dxdt| \le \\ &\le c\,\|\bar{\varphi}\|_{L^1(Q)}\|y\|^2_{L^\infty(Q)} + c\|\bar{\varphi}\|_{L^2(Q)}\|y\|_{L^\infty(Q)}\|h\|_{L^2(Q)} \le c\,\|h\|^2_{L^2(Q)}. \end{aligned}$$

Discussions of this type reveal that (A1)–(A3) are satisfied. However, we needed very strong assumptions, in particular $N = 1$. The case $N = 2$ can be handled under additional restrictions concerning the appearance of control and observations ("control and observations have disjoint supports", see [10]).

If there are no pointwise state-constraints, the situation is easier, as the reader can check.

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