OPTIMAL CONTROL OF THE STATIONARY NAVIER-STOKES EQUATIONS WITH MIXED CONTROL-STATE CONSTRAINTS*

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Abstract. In this paper we consider the distributed optimal control of the Navier-Stokes equations in presence of pointwise mixed control-state constraints. After deriving a first order necessary condition, the regularity of the mixed constraint multiplier is investigated. Second-order sufficient optimality conditions are studied as well. In the last part of the paper, a semi-smooth Newton method is applied for the numerical solution of the control problem. The convergence of the method is proved and numerical experiments are carried out.

Key words. Optimal control, Navier-Stokes equations, mixed control-state constraints, semi-smooth Newton methods.

AMS subject classifications. 49K20, 76D05, 65J15.

1. Introduction. Continuing our efforts in the investigation of optimal control problems governed by the Navier-Stokes equations in presence of pointwise control and state constraints (cf. [7, 8, 9, 10, 25]), we consider the following mixed control-state constrained problem:

$$\begin{cases} \min \ J(y,u) = \frac{1}{2}\int\limits_{\Omega} |y-z_d|^2 \ dx + \frac{\alpha}{2}\int\limits_{\Omega} |u|^2 \ dx \\ \text{subject to} \\ -\nu\Delta y + (y\cdot\nabla)y + \nabla p = u \\ \operatorname{div} y = 0 \\ y|_\Gamma = g \\ a \le \varepsilon u + y \le b \text{ a.e.,} \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $\varepsilon > 0$. Due to the mixed nature of the pointwise constraints, expressed by the last relation of (1.1), the corresponding Lagrange multiplier is expected to be more regular than in the state constrained case (cf. [8]). In fact, such a constraint can be introduced as a way of regularization of the state constrained case and it is expected that, as ε tends to zero, the solutions converge to the optimal solution of the state constrained problem (see [21]).

Optimal control of partial differential equations in presence of state constraints is a very challenging research field, mainly due to the difficult structure of the Lagrange multiplier associated to the state constraints (see [2, 3, 4]). In the case of Navier-Stokes control, the problem has been investigated in [8], where the measure structure of the multiplier was studied.

This paper is a contribution to the numerical analysis of optimal control problems of the Navier-Stokes equations with pointwise state constraints.

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The unconstrained and control-constrained optimal control problems of the Navier-Stokes equations have been studied in many papers (see [1, 5, 7, 14, 15, 16, 18, 19, 25, 26]), where optimality conditions and/or numerical methods were discussed. Moreover, we refer to the detailed references in [13].

In contrast to this, only few papers consider associated problems with state constraints. To our best knowledge, in flow control, only [8, 10, 11, 27] deal with state constraints. In [8] and [11, 27] necessary optimality conditions are derived for the stationary and time dependent problems, respectively. In [10] the numerical solution utilizing a penalized problem together with a semi-smooth Newton method has been studied.

The novelty of our paper consists of a Lavrentiev type regularization of the state constraints. Here we follow an approach introduced in [21, 22] to approximate the state constraints by mixed control-state constraints. This approach permits to work with regular functions rather than with measures, which are unavoidable for pure pointwise state constraints. In this way, we are able to show regularity of Lagrange multipliers and to derive second order sufficient optimality conditions. An additional novelty is the consideration of semi-smooth Newton methods in this context. We set up a semi-smooth Newton algorithm for the numerical solution of the control problem and prove local superlinear convergence of the method. All this issues have not yet been considered in the literature.

The outline of the paper is as follows. In Section 2, the optimal control problem is stated and existence of a global optimal solution is proved. In Section 3, the problem is reformulated as a control constrained optimal control problem and first order necessary optimality conditions are obtained. Sufficient conditions of second order type are the topic of Section 4. In Section 5, a semi-smooth Newton algorithm is stated and the superlinear convergence of the method is proved. Reports on numerical experiments are summarized in Section 6.

2. Problem statement and existence of solution.

Consider a bounded regular domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$. Our objective is to characterize and find a solution $(u^*, y^*) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega)$ of the following optimal control problem:

$$
\begin{cases}
\min\ J(y,u) = \frac{1}{2} \int\limits_\Omega |y - z_d|^2\, dx + \frac{\alpha}{2} \int\limits_\Omega |u|^2\, dx \\
\text{subject to} \\
-\nu \Delta y + (y \cdot \nabla) y + \nabla p = u \\
\operatorname{div} y = 0 \\
y|_\Gamma = g \\
a \le \varepsilon u + y \le b \text{ a.e.,}
\end{cases}
\tag{2.1}
$$

where $\alpha > 0$, $\varepsilon > 0$, z_d is the desired state, $a \le b \in \mathbf{L}^2(\Omega)$ and $g \in \mathbf{H}_0^{1/2}(\Gamma)$, with $\mathbf{H}_0^{1/2}(\Gamma) := \{v \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma v \cdot \vec{n}\ d\Gamma = 0\}$ are given. The inequalities in the last line of (2.1) have to be understood componentwise. We denote by $(\cdot,\cdot)_X$ the inner product in the Hilbert space X and by $\|\cdot\|_X$ the associated norm. The subindex is suppressed if the L^2-inner product or norm are meant. Hereafter, the bold notation stands for the product of spaces. Additionally, we introduce the solenoidal space $V = \{v \in \mathbf{H}_0^1(\Omega) : \operatorname{div} v = 0\}$, the closed subspace $\mathbf{H} := \{v \in \mathbf{H}^1(\Omega) : \operatorname{div} v = 0\}$ and the trilinear form $c : \mathbf{H} \times \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ defined by

$$
c(u,w,v) = ((u \cdot \nabla) w, v). \tag{2.2}
$$

Considering a force term $f \in V'$, the weak formulation of the Navier-Stokes equations is then given by

$$\nu(\nabla y, \nabla v) + c(y, y, v) = \langle f, v\rangle_{V',V}, \text{ for all } v \in V \tag{2.3}$$

$$\gamma_0 y = g, \tag{2.4}$$

where $\nabla y = \begin{pmatrix} \partial_1 y_1 & \dots & \partial_d y_1 \\ \vdots & \ddots & \vdots \\ \partial_1 y_d & \dots & \partial_d y_d \end{pmatrix}$, $(\nabla y, \nabla v) := \sum_{i=1}^{d}\sum_{j=1}^{d} (\partial_i y_j, \partial_i v_j)_{L^2(\Omega)}$ and $\gamma_0 : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ stands for the trace operator. It is nowadays standard to show existence of a solution for (2.3)-(2.4). Also an appropriate estimate and uniqueness, for ν sufficiently large or f sufficiently small, are obtained. The main results are summarized in the following theorem.

THEOREM 2.1. *Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded regular domain, $f \in \mathbf{H}^{-1}(\Omega)$ and $g \in \mathbf{H}_0^{1/2}(\Gamma)$. Then, there exists at least one solution for the non-homogeneous problem* (2.3)-(2.4)*, that satisfies the estimate*

$$\|y - \hat{y}\|_V \leq \frac{2}{\nu} \|F\|_{V'}, \tag{2.5}$$

where $\hat{y} \in \mathbf{H}$ is a function such that $\gamma_0 \hat{y} = g$ and $F = f + \nu\Delta\hat{y} - (\hat{y}\cdot\nabla)\hat{y}$. Moreover, if $\|\hat{y}\|_{\mathbf{H}}$ is sufficiently small, such that

$$|c(v, \hat{y}, v)| \leq \frac{\nu}{2} \|v\|_V^2 \text{ for all } v \in V$$

and $\nu^2 > 4\mathcal{N}\|F\|_{V'}$, with $\mathcal{N} = \sup_{u,v,w\in V} \frac{|c(u,v,\phi)|}{\|u\|_V \|v\|_V \|w\|_V}$, then the solution is unique.

Proof. For the proof we refer to [23], pp. 178-180. ▯

Next, we verify the existence of an optimal solution for our control problem. For that purpose let us define the set of admissible solutions

$$\mathcal{T}_{ad} = \{(y, u) \in \mathbf{H} \times \mathbf{L}^2(\Omega) : (y, u) \text{ satisfies the restrictions in (2.1) }\}.$$

THEOREM 2.2. *If $\mathcal{T}_{ad}$ is non-empty, then there exists an optimal solution for* (2.1).

Proof. Assuming that there is at least one feasible pair for our problem, we take a minimizing sequence $\{(y_n, u_n)\}$ in $\mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega)$ and, considering the quadratic nature of the cost functional, we get that $\{u_n\}$ is uniformly bounded in $\mathbf{L}^2(\Omega)$.

¿From estimate (2.5) it follows that the sequence $\{y_n\}$ is also uniformly bounded in $\mathbf{H}^1(\Omega)$. Therefore, we may extract a weakly convergent subsequence, also denoted by $\{(y_n, u_n)\}$, such that $u_n \rightharpoonup u^*$ in $\mathbf{L}^2(\Omega)$ and $y_n \rightharpoonup y^*$ in $\mathbf{H}^1(\Omega)$. Due to the weak sequential continuity of the nonlinear form (cf. [12], pg. 286), it follows that $c(y_n, y_n, v) \to c(y^*, y^*, v)$. Consequently, due also to the linearity and continuity of the other terms involved, the limit (y^*, u^*) satisfies the state equations.

Since the set $C := \{v \in \mathbf{L}^2(\Omega) : a \leq v \leq b \text{ a.e.}\}$ is closed and convex, it is weakly closed. Hence, from the linearity and continuity of the mapping $(y, u) \to \varepsilon u + y$, it follows that $\varepsilon u^* + y^* \in C$. Taking into consideration that $J(y, u)$ is weakly lower semicontinuous, the result follows in a standard way. ▯

3. First-order necessary optimality conditions. Let us consider the set

$$U = \{u \in \mathbf{L}^2(\Omega) : \|u\| < \left(\nu^2 - 4\mathcal{N}\,\|\nu\Delta\hat{y} - (\hat{y}\cdot\nabla)\hat{y}\|_{V'}\right)/(4\mathcal{N}\hat{c})\},$$

where $\hat{c}$ denotes the embedding constant of $\mathbf{L}^2(\Omega)$ into V' and $\hat{y}$ is a suitable velocity profile from Theorem 2.1. According to Theorem 2.1 there exists, for each $u \in U$ on the right hand side of (2.3), a unique solution of the Navier-Stokes equations. Introducing the control-to-state operator $G : U \to \mathbf{H}$ that assigns to each $u \in U \subset \mathbf{L}^2(\Omega)$ the corresponding Navier-Stokes solution $y(u)$, and using the composite mapping $\mathcal{G} = \mathcal{I} \circ G$, where $\mathcal{I} : \mathbf{H} \to \mathbf{L}^2(\Omega)$ stands for the continuous compact injection, problem (2.1) can be expressed in a reduced form as

$$(\mathcal{P}) \qquad \begin{cases} \min_{u\in U} \ J(u) = \frac{1}{2}\int\limits_\Omega |\mathcal{G}u - z_d|^2 \, dx + \frac{\alpha}{2}\int\limits_\Omega |u|^2 \, dx \\ \text{subject to} \\ a \le \varepsilon u + \mathcal{G}u \le b \text{ a.e. in } \Omega. \end{cases}$$

Since U is open, we cannot expect in general that $(\mathcal{P})$ admits a global solution. However, in what follows, we concentrate on certain local solutions rather than to consider exclusively global ones. Therefore, we are justified to assume $u^* \in U$ below.

In the sequel we will frequently utilize the condition

$$\nu > \mathcal{M}(y^*), \tag{3.1}$$

with $\mathcal{M}(y) := \sup\limits_{v\in V} \frac{|c(v,y,v)|}{\|v\|_V^2}$, which is responsible for the ellipticity of the linearized equations (see Lemma 3.1 below). Condition (3.1) is immediately satisfied for all pairs $(y(u), u)$ that fulfill the hypotheses of Theorem 2.1 (see [7, Remark 3.1]). In particular, it holds for all pairs $(y(u), u)$ with $u \in U$.

LEMMA 3.1. *Let $u \in U$ and $y := G(u)$. The control-to-state operator G is twice Fréchet differentiable at u and its derivatives $w := G'(u)h$ and $z := G''(u)[h]^2$ are given by the unique solutions of the systems:*

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)y + (y\cdot\nabla)w + \nabla\pi &= h \\ \operatorname{div} w &= 0 \\ w|_\Gamma &= 0 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} -\nu\Delta z + (z\cdot\nabla)y + (y\cdot\nabla)z + \nabla\varrho &= -2(w\cdot\nabla)w \\ \operatorname{div} z &= 0 \\ z|_\Gamma &= 0, \end{aligned} \tag{3.3}$$

respectively.

Proof. Let us begin by considering system (3.2). Its variational formulation is given by

$$a_1(w,\phi) := \nu(\nabla w, \nabla\phi) + c(w,y,\phi) + c(y,w,\phi) = (h,\phi),$$

for all $\phi \in V$. Since for all pairs (y,u) with $u \in U$ condition (3.1) holds (see [7, Remark 3.1]), coercivity of $a_1(\cdot,\cdot)$ and, consequently, the existence and uniqueness of the solution w, follows.

Let us denote the increment by $\bar{y} := y_{u+h} - y_u$, where $y_u := G(u)$. Considering that

$$c(y_{u+h}, y_{u+h}, \phi) - c(y_u, y_u, \phi) = c(\bar{y}, \bar{y}, \phi) + c(y_u, \bar{y}, \phi) + c(\bar{y}, y_u, \phi), \tag{3.4}$$

it can be verified that $\bar{y}$ is solution of

$$\nu(\nabla\bar{y}, \nabla\phi) + c(\bar{y}, \bar{y}, \phi) + c(\bar{y}, y_u, \phi) + c(y_u, \bar{y}, \phi) = (h, \phi), \text{ for all } w \in V. \tag{3.5}$$

Taking $\phi = \bar{y}$ as test function in (3.5) yields

$$(h, \bar{y}) = \nu \left\|\bar{y}\right\|_V^2 + c(\bar{y}, y_u, \bar{y}) \geq \nu \left\|\bar{y}\right\|_V^2 - \mathcal{M}(y_u) \left\|\bar{y}\right\|_V^2$$

and therefore

$$\left\|\bar{y}\right\|_V \leq \kappa\sigma(y) \left\|h\right\|, \tag{3.6}$$

where κ denotes the Poincaré inequality constant and $\sigma(y) := \frac{1}{\nu - \mathcal{M}(y)}$. Considering now $\tilde{y} = y_{u+h} - y_u - w$, we obtain the following equation:

$$\begin{aligned}\nu(\nabla\tilde{y}, \nabla\phi) + c(y_{u+h}, y_{u+h}, \phi) - c(y_u, y_u, \phi)& \\ - c(w, y_u, \phi) - c(y_u, w, \phi) = 0,& \text{ for all } w \in V.\end{aligned} \tag{3.7}$$

Using (3.4) and choosing $\tilde{y}$ as test function in (3.7) we get that

$$\nu \left\|\tilde{y}\right\|_V^2 - c(\tilde{y}, \tilde{y}, y_u) = -c(\bar{y}, \bar{y}, \tilde{y}),$$

which together with (3.6) and condition (3.1) yields

$$\left\|\tilde{y}\right\|_V \leq \mathcal{N}\kappa^2\sigma^3(y) \left\|h\right\|^2. \tag{3.8}$$

Hence, the Fréchet differentiability follows. Moreover, since condition (3.1) holds, existence and uniqueness of solutions for equations (3.2) is verified. Therefore, the inverse operator exists for all $u \in U$ as a linear continuous operator and, from the implicit function theorem, the operator G is of class C^2 from U to $\mathbf{H}$. Taking the derivative on both sides of (3.2) yields (3.3) (see [6], p. 14). ☐

The idea now consists in reformulating problem $(\mathcal{P})$ in a new variable $v := \varepsilon u + \mathcal{G}(u)$ and treat it as a control-constrained optimal control problem. In order to express u as a function of v we consider the operator

$$\begin{aligned}F : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) &\to \mathbf{L}^2(\Omega) \\ (v, u) &\mapsto \varepsilon u + \mathcal{G}(u) - v\end{aligned}$$

and the solvability of the equation

$$F(v, u) = 0.$$

To justify existence and uniqueness of u for each $v \in \mathbf{L}^2(\Omega)$, we will consider a $\mathbf{L}^2$ neighborhood of the optimal control u^* contained in U. From the implicit function theorem (cf. [28]) it suffices, since $F(v, u)$ is clearly defined in a neighborhood of u^* and $v^* = \varepsilon u^* + \mathcal{G}(u^*)$, to verify existence and continuity of the mapping $F_u(v^*, u^*)^{-1}$ from $\mathbf{L}^2(\Omega)$ to $\mathbf{L}^2(\Omega)$.

¿From the open mapping theorem, existence and continuity of $F_u(v^*, u^*)^{-1}$ holds if the operator $F_u(v^*, u^*) = \varepsilon + \mathcal{G}'(u^*)$ is bijective. Let us therefore consider the equation

$$(\varepsilon + \mathcal{G}'(u^*))h = \varphi, \tag{3.9}$$

with $\varphi \in \mathbf{L}^2(\Omega)$. It is easy to see that $\mathcal{G}'(u^*) = \mathcal{I} \circ G'(u^*)$ is compact due to the embedding $\mathcal{I} : \mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Omega)$. Since $\varepsilon > 0$ and $\nu > \mathcal{M}(y^*)$, it can be verified that $\ker(\varepsilon + \mathcal{G}'(u^*)) = \{0\}$ and consequently ε is not an eigenvalue of $-\mathcal{G}'(u^*)$. Applying Fredholm's alternative, we get the existence of a unique solution $h \in \mathbf{L}^2(\Omega)$ for (3.9) and consequently the existence and continuity of $F_u(v^*, u^*)^{-1}$.

Therefore, there are constants $r, r_0 > 0$ such that for each $v \in \mathbf{L}^2(\Omega)$ with $\|v - v^*\| \leq r_0$, there exists a unique $u := K(v)$ with $\|u - u^*\| \leq r$ and

$$\varepsilon K(v) + \mathcal{G}(K(v)) = v. \tag{3.10}$$

Moreover, since F is twice continuously Fréchet differentiable, the implicit function theorem (cf. [28]) also implies that K is twice continuously Fréchet differentiable. Let us denote by $K''(v)[\xi, \eta]$ the second derivative of K in directions ξ and η and introduce $K''(v)[\xi]^2 := K''(v)[\xi, \xi]$. Taking the first and second derivatives on both sides of (3.10) in direction ξ yields

$$(\varepsilon + \mathcal{G}'(K(v)))K'(v)\xi = \xi, \tag{3.11}$$

$$(\varepsilon + \mathcal{G}'(K(v)))K''(v)[\xi]^2 = -\mathcal{G}''(K(v))[K'(v)\xi]^2, \tag{3.12}$$

which implies that

$$K'(v) = (\varepsilon + \mathcal{G}'(K(v)))^{-1}$$

and

$$K''(v)[\xi]^2 = -(\varepsilon + \mathcal{G}'(K(v)))^{-1}\mathcal{G}''(K(v))[K'(v)\xi]^2,$$

respectively.

Locally around u^*, our control problem can therefore be equivalently formulated as:

$$(\mathcal{P}_r) \qquad \begin{cases} \min\ \mathcal{J}(v) =: J(y(K(v)), K(v)) \\ \text{subject to} \\ a \leq v \leq b \text{ a.e.} \\ v \in B_{r_0}(v^*). \end{cases}$$

Theorem 3.2. *Let u^* be a local optimal solution of $(\mathcal{P})$. Then there exist Lagrange multipliers $\lambda \in V$, $q \in L^2_0(\Omega)$ and μ_a, $\mu_b \in \mathbf{L}^2(\Omega)$ such that*

$$\begin{aligned} -\nu\Delta y^* + (y^* \cdot \nabla)y^* + \nabla p &= u^* \\ \operatorname{div} y^* &= 0 \\ y^*|_\Gamma &= g, \end{aligned} \tag{3.13}$$

$$\begin{aligned} -\nu\Delta\lambda - (y^* \cdot \nabla)\lambda + (\nabla y^*)^T\lambda + \nabla q &= z_d - y^* + \mu_a - \mu_b \\ \operatorname{div} \lambda^* &= 0 \\ \lambda^*|_\Gamma &= 0, \end{aligned} \tag{3.14}$$

$$\lambda - \alpha u^* = \varepsilon(\mu_b - \mu_a), \tag{3.15}$$

$$\begin{aligned} a \leq \varepsilon u + y^* \leq b,\\ \mu_a,\ \mu_b \geq 0,\\ (\mu_{a_i}, a_i - \varepsilon u_i^* - y_i^*) = (\mu_{b_i}, b_i - \varepsilon u_i^* - y_i^*) = 0,\ \textit{for } i = 1, 2, \end{aligned} \tag{3.16}$$

hold in variational sense.

Proof. Since u^* is a locally optimal solution of $(\mathcal{P})$, we get for some $r > 0$

$$J(y^*, u^*) \leq J(y(u), u),$$

for all $u \in B_r(u^*)$ with $a \leq \varepsilon u + y(u) \leq b$. Equivalently, since $u = K(v)$ holds locally,

$$\mathcal{J}(v^*) \leq \mathcal{J}(v),$$

for all $v \in B_{r_0}(v^*)$ with $a \leq v \leq b$, and for an appropriate constant $r_0 > 0$.

Therefore, the following first order necessary condition follows

$$\mathcal{J}'(v^*)(v - v^*) \geq 0, \forall a \leq v \leq b \tag{3.17}$$

Applying the chain rule, the derivative of $\mathcal{J}(v^*)$ in any direction $\xi \in \mathbf{L}^2(\Omega)$ is given by

$$(\mathcal{J}'(v^*), \xi) = (y^* - z_d, \mathcal{G}'(u^*)K'(v^*)\xi) + \alpha(u^*, K'(v^*)\xi), \tag{3.18}$$

which, by $h := K'(v^*)\xi$, yields

$$(\mathcal{J}'(v^*), \xi) = (y^* - z_d, \mathcal{G}'(u^*)h) + \alpha(u^*, h).$$

Denoting by $\mu \in \mathbf{L}^2(\Omega)$ the Riesz representative of $-\mathcal{J}'(v^*)$ and using explicitly the derivative of K we obtain

$$(\mu, \xi) = (\mu, (\varepsilon + \mathcal{G}'(u^*))h) = \varepsilon(\mu, h) + (\mu, \mathcal{G}'(u^*)h).$$

Therefore, equation (3.18) is equivalent to

$$(y^* - z_d + \mu, \mathcal{G}'(u^*)h) + (\alpha u^* + \varepsilon\mu, h) = 0. \tag{3.19}$$

We now introduce the adjoint system of equations

$$\begin{aligned} -\nu\Delta\lambda - (y^* \cdot \nabla)\lambda + (\nabla y^*)^T\lambda + \nabla q = z_d - y^* - \mu\\ \operatorname{div} \lambda^* = 0\\ \lambda^*|_\Gamma = 0. \end{aligned} \tag{3.20}$$

Since, by hypothesis $\nu > \mathcal{M}(y^*)$, the ellipticity of the adjoint operator can be easily verified and, therefore for $z_d - y^* - \mu \in \mathbf{L}^2(\Omega)$, there exists a unique solution $\lambda \in V$ for the adjoint system.

Consequently, equation (3.19) can be rewritten as

$$\lambda - \alpha u^* = \varepsilon\mu. \tag{3.21}$$

Utilizing the decomposition $\mu = \mu_b - \mu_a$, with

$$\mu_b := \mu_+ = \frac{1}{2}(\mu + |\mu|)$$
$$\mu_a := \mu_- = \frac{1}{2}(-\mu + |\mu|),$$

where $|\mu| = (|\mu_1|, |\mu_2|)^T$, the optimality condition (3.17) can be rewritten as

$$(\mathcal{J}'(v^*), v^*) = \min_{a\le v\le b}(\mu_a - \mu_b, v) = \min_{a\le v\le b}\{(\mu_{a,1}, v_1) - (\mu_{b,1}, v_1) + (\mu_{a,2}, v_2) - (\mu_{b,2}, v_2)\}.$$

By fixing the second component of the new control variable $v_2 = v_2^*$ and considering the mutual disjoint sets $\{x : \mu_{a,1}(x) > 0\}$ and $\{x : \mu_{b,1}(x) > 0\}$, we obtain that

$$(\mathcal{J}'(v^*), v^*) = (\mu_{a,1}, a_1) - (\mu_{b,1}, b_1) + (\mu_{a,2}, v_2^*) - (\mu_{b,2}, v_2^*)$$

and, consequently,

$$(\mu_{a,1}, a_1 - \varepsilon u_1^* - y_1^*) - (\mu_{b,1}, b_1 - \varepsilon u_1^* - y_1^*) = 0.$$

Fixing now the first component of v and proceeding in a similar manner we get that

$$(\mu_{a,2}, a_2 - \varepsilon u_2^* - y_2^*) - (\mu_{b,2}, b_2 - \varepsilon u_2^* - y_2^*) = 0.$$

Taking into account that, by definition, $\mu_a,\ \mu_b \ge 0$ componentwise, the complementarity system (3.16) follows. ∎

Remark 3.3. *Notice that the existence of μ_a, μ_b cannot be deduced in a standard way from Kuhn-Tucker theorems in Banach spaces, since the cone of non-negative functions in $\mathbf{L}^2(\Omega)$ has an empty interior and we work just in this constraint space.*

4. Second order sufficient condition. Next, we turn to second order sufficient optimality conditions for problem $(\mathcal{P})$. Following [22], the idea consists again in utilizing the second order sufficient optimality properties of the pure control constrained problem $(\mathcal{P}_r)$ and translate them to the original setting.

We begin by verifying the relation between the Lagrangian

$$\mathcal{L}(y, u, \lambda) = \frac{1}{2}\|y^* - z_d\|^2 + \frac{\alpha}{2}\|u\|^2 + \nu(\nabla\lambda, \nabla y) + c(y, y, \lambda) - (\lambda, u)$$

and the second derivative of the reduced functional $\mathcal{J}$.

Lemma 4.1. *The second derivative of the reduced cost functional in direction ξ satisfies*

$$\mathcal{J}''(v^*)[\xi]^2 = \mathcal{L}''(y^*, u^*, \lambda)(w, h)^2 \tag{4.1}$$

where $h = K'(v^*)\xi$ and w is the solution to (3.2) *with h on the right hand side.*

Proof. Considering the reduced cost functional and differentiating it twice in direction ξ we get

$$\begin{aligned}\mathcal{J}''(v^*)[\xi]^2 &= J''(K(v^*))[K'(v^*)\xi]^2 + J'(K(v^*))\ K''(v^*)[\xi]^2\\ &= \|\mathcal{G}'(K(v^*))K'(v^*)\xi\|^2 + (y(K(v^*)) - z_d, \mathcal{G}''(K(v^*))[K'(v^*)\xi]^2)\\ &+ (y(K(v^*)) - z_d, \mathcal{G}'(K(v^*))K''(v^*)\xi^2) + \alpha\|K'(v^*)\xi\|^2 + \alpha(K(v^*), K''(v^*)\xi^2),\end{aligned}$$

which, by the relations $h = K'(v^*)\xi$, $u^* = K(v^*)$, $y^* = y(K(v^*))$, $w = \mathcal{G}'(u^*)h$ and $z = \mathcal{G}''(u^*)[h]^2$, yields

$$\begin{aligned}(4.2)\quad \mathcal{J}''(v^*)[\xi]^2 = \|w\|^2 + (y^* - z_d, z)& \\ + (y^* - z_d, \mathcal{G}'(u^*)K''(v)\xi^2) + \alpha\|h\|^2 + \alpha(u^*, K''(v^*)\xi^2).&\end{aligned}$$

¿From the optimality condition (3.19) we get

$$(y^* - z_d, \mathcal{G}'(u^*)K''(v^*)\xi^2) + \alpha(u^*, K''(v^*)\xi^2) = -(\mu, (\varepsilon + \mathcal{G}'(u^*))K''(v^*)\xi^2),$$

which implies that

$$\mathcal{J}''(v^*)[\xi]^2 = \|w\|^2 + \alpha\|h\|^2 + (y^* - z_d, z) - (\mu, (\varepsilon + \mathcal{G}'(u^*))K''(v^*)\xi^2)$$

Additionally, by (3.12) we find

$$-(\mu, (\varepsilon + \mathcal{G}'(u^*))K''(v^*)\xi^2) = (\mu, z).$$

¿From (3.14) we get, using integration by parts and equation (3.3), that

$$\begin{aligned}(y^* - z_d, z) - (\mu, (\varepsilon + \mathcal{G}'(u^*))K''(v^*)[\xi]^2) &= \nu(\Delta z, \lambda) - c(y^*, z, \lambda) - c(z, y^*, \lambda) \\ &= 2c(w, w, \lambda).\end{aligned}$$

We thus obtain

$$\mathcal{J}''(v^*)[\xi]^2 = \|w\|^2 + \alpha\|h\|^2 + 2((w\cdot\nabla)w, \lambda).$$

On the other hand, computing the first and second derivatives of the Lagrangian yields

$$\begin{aligned}\mathcal{L}'(y^*, u^*, \lambda)(w, h) &= (y^* - z_d, w) + \alpha(u^*, h) + \nu(\nabla\lambda, \nabla w) \\ &\qquad + c(y^*, w, \lambda) + c(w, y^*, \lambda) - (\lambda, h) \\ \mathcal{L}''(y^*, u^*, \lambda)(w, h)^2 &= \|w\|^2 + \alpha\|h\|^2 + 2c(w, w, \lambda),\end{aligned}$$

and consequently

$$(4.3)\qquad \mathcal{J}''(v^*)[\xi]^2 = \mathcal{L}''(y^*, u^*, \lambda)(w, h)^2 = \|w\|^2 + \alpha\|h\|^2 + 2c(w, w, \lambda),$$

where w is solution of (3.2) with h on the right hand side. □

Let us now introduce the set of strongly active constraints

$$\mathcal{A}_{\tau,i} := \{x \in \Omega : |\mu_i(x)| \geq \tau\},\ i = 1, \ldots, d,$$

and the critical cone

$$\tilde{C}_\tau = \left\{ v \in \mathbf{L}^2(\Omega) : \begin{array}{c} v_i(x) = 0 \text{ if } x \in \mathcal{A}_{\tau,i} \\ v_i(x) \geq 0 \text{ if } v_i^*(x) = a_i, x \notin \mathcal{A}_{\tau,i} \\ v_i(x) \leq 0 \text{ if } v_i^*(x) = b_i, x \notin \mathcal{A}_{\tau,i} \end{array} \right\}.$$

For the investigation of optimality for a given stationary pair (y^*, u^*) let us hereafter assume that the following second order condition holds: there exist $\tau > 0$, $\delta > 0$ such that

$$(\mathrm{SSC})\qquad \mathcal{L}''(y^*, u^*, \lambda)(w, h)^2 \geq \delta\|h\|^2$$

for all $(w,h) \in C_\tau$, where C_τ consists of all pairs $(w,h) \in V \times \mathbf{L}^2(\Omega)$ such that $\varepsilon h + w \in \tilde{C}_\tau$ and

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)y^* + (y^*\cdot\nabla)w + \nabla\pi &= h \\ \operatorname{div} w &= 0 \\ w|_\Gamma &= 0. \end{aligned} \tag{4.4}$$

THEOREM 4.2. *If u^* is a stationary point of $(\mathcal{P})$ and (SSC) holds for some $\delta > 0$, $\tau > 0$, then there exist constants $\rho > 0$ and $\sigma > 0$ such that*

$$J(y,u) \geq J(y^*,u^*) + \sigma \|u-u^*\| \tag{4.5}$$

for all pairs (y,u) such that $y = G(u), a \leq \varepsilon u + y \leq b$ and $\|u-u^*\| \leq \rho$.

Proof. Utilizing (3.11), (4.1) and (SSC) it follows that

$$\mathcal{J}''(v^*)[\xi]^2 \geq \delta \left\|(\varepsilon + \mathcal{G}'(u^*))^{-1}\xi\right\|^2,$$

which using the continuity of the mapping $(\varepsilon + \mathcal{G}'(u^*))$ yields

$$\mathcal{J}''(v^*)[\xi]^2 \geq \delta \left(\frac{1}{\|\varepsilon+\mathcal{G}'(u^*)\|}\|\xi\|\right)^2 = \delta\left\|\varepsilon + \mathcal{G}'(u^*)\right\|^{-2}\|\xi\|^2 = \tilde{\delta}\|\xi\|^2.$$

Using the second order sufficient conditions for the reduced problem (cf. [24], pg. 190), we get the existence of constants $\tilde{\rho} > 0,\ \tilde{\sigma} > 0$ such that

$$\mathcal{J}(v) \geq \mathcal{J}(v^*) + \tilde{\sigma}\|v-v^*\|^2,$$

for all $a \leq v \leq b,\ \|v-v^*\| \leq \tilde{\rho}$.

By the implicit function theorem there exist constants $r,\ r_0 > 0$ such that for all $v \in \mathbf{L}^2(\Omega)$ with $\|v-v^*\| \leq r_0$, there is a $u = K(v)$ which satisfies $\|u-u^*\| \leq r$.

Taking $\hat{\rho} = \min(\tilde{\rho}, r_0)$ we have that $\|u-u^*\| \leq r$ and

$$J(u) \geq J(u^*) + \tilde{\sigma}\|v-v^*\|^2 \tag{4.6}$$

$$= J(u^*) + \tilde{\sigma}\|\varepsilon(u-u^*) + \mathcal{G}(u) - \mathcal{G}(u^*)\|^2. \tag{4.7}$$

¿From the quadratic nature of the Navier-Stokes nonlinear term we obtain, using Taylor expansion, that

$$\mathcal{G}(u) - \mathcal{G}(u^*) = \mathcal{G}'(u^*)(u-u^*) + \frac{1}{2}\mathcal{G}''(u^*)[u-u^*]^2,$$

which, considering (4.7) implies that

$$J(u) \geq J(u^*) + \tilde{\sigma}\left\|(\varepsilon + \mathcal{G}'(u^*))(u-u^*) + \frac{1}{2}\mathcal{G}''(u^*)(u-u^*)^2\right\|^2 \tag{4.8}$$

$$\geq J(u^*) + \tilde{\sigma}\left(\|(\varepsilon+\mathcal{G}'(u^*))(u-u^*)\| - \left\|\frac{1}{2}\mathcal{G}''(u^*)(u-u^*)^2\right\|\right)^2. \tag{4.9}$$

Since the operator $(\varepsilon + \mathcal{G}'(u^*))^{-1}$ is linear and continuous we get that

$$\begin{aligned}\|u-u^*\| &= \left\|(\varepsilon+\mathcal{G}'(u^*))^{-1}(\varepsilon+\mathcal{G}'(u^*))(u-u^*)\right\| \\ &\leq \left\|(\varepsilon+\mathcal{G}'(u^*))^{-1}\right\| \|(\varepsilon+\mathcal{G}'(u^*))(u-u^*)\|\end{aligned}$$

which implies that

$$\|(\varepsilon+\mathcal{G}'(u^*))(u-u^*)\| \geq \frac{1}{\|(\varepsilon+\mathcal{G}'(u^*))^{-1}\|}\|u-u^*\| = \bar{C}\|u-u^*\|.$$

Additionally, possibly by reducing r,

$$\left\|\frac{1}{2}\mathcal{G}''(u^*)[u-u^*]^2\right\| \leq \frac{\bar{C}}{2}\|u-u^*\|.$$

Therefore, we get that

$$J(u) \geq J(u^*) + \tilde{\sigma}(\bar{C}\|u-u^*\| - \frac{\bar{C}}{2}\|u-u^*\|)^2 = J(u^*) + \sigma\|u-u^*\|^2$$

with $\sigma := \frac{\tilde{\sigma}\bar{C}^2}{4}$ and, consequently, the local optimality of u^* and the quadratic rate follow. □

Remark 4.3. *For the analysis of second order numerical methods, a stronger condition is needed (see [19, 24]): there exist constants $\tau > 0,\ \delta > 0$ such that*

$$(\overline{SSC}) \qquad \mathcal{L}''(y^*, u^*, \lambda)(w,h)^2 \geq \delta\|h\|^2$$

for all pairs $(w,h) \in V \times \mathbf{L}^2(\Omega)$ that solve (4.4) and satisfy $\varepsilon h_i + w_i = 0$ on $\mathcal{A}_{\tau,i}$, for $i = 1, \dots, d$.

5. Semi-smooth Newton method.

In this section we propose a semi-smooth Newton method for the numerical solution of $(\mathcal{P})$. The infinite dimensional method is applied to the optimality system (3.13)-(3.16) and superlinear convergence is proved. Additionally, the close relationship between semi-smooth Newton and primal-dual active set methods (see [17]) allows a practical formulation of the algorithm in terms of active and inactive sets.

We begin by reformulating the complementarity system (3.16) as the following operator equation

$$\mu = \max(0, \mu + c(v-b)) + \min(0, \mu + c(v-a)) \tag{5.1}$$

for all $c > 0$. Equation (5.1) suggests an update strategy based on active and inactive sets information.

Definition 5.1. *Let X and Z be Banach spaces and $D \subset X$ an open subset. The mapping $F : D \to Z$ is called Newton differentiable in the open subset $U \subset D$ if there exists a mapping $\Psi : U \to L(X,Z)$ such that*

$$\lim_{h\to 0}\frac{1}{\|h\|}\|F(x+h) - F(x) - \Psi(x+h)h\| = 0$$

for every $x \in U$.

Since the $max(0,\cdot)$ and $min(0,\cdot)$ functions are Newton differentiable (see [7, 17]) from $L^p(\Omega) \to L^q(\Omega)$, with $q < p$, the application of the semi-smooth Newton method is justified with the special choice $c := \alpha/\varepsilon^2$ (see Theorem 5.5 below). The complete algorithm is defined through the following steps.

ALGORITHM 5.2.

1. *Initialize the variables* $u_0 \in \mathbf{L}^2(\Omega)$, $y_0 \in V$, $\mu_0 = 0$ *and set* $k = 1$.
2. *Until a stopping criterion is satisfied, set for* $i = 1, \ldots, d$,

$$\begin{aligned}
\mathcal{A}_{b_i}^n &= \{x : \mu_i^{n-1} + \frac{\alpha}{\varepsilon^2}\left(\varepsilon u_i^{n-1} + y_i^{n-1} - b_i\right) > 0\},\\
\mathcal{A}_{a_i}^n &= \{x : \mu_i^{n-1} + \frac{\alpha}{\varepsilon^2}(\varepsilon u_i^{n-1} + y_i^{n-1} - a_i) < 0\},\\
\mathcal{I}_i^n &= \Omega \backslash (\mathcal{A}_{b_i}^n \cup \mathcal{A}_{a_i}^n).
\end{aligned}$$

and find the solution (y, p, λ, q) *of:*

$$\begin{aligned}
&-\nu\Delta y_i + y_1^{n-1}\partial_1 y_i + y_2^{n-1}\partial_2 y_i + y_1\partial_1 y_i^{n-1} + y_2\partial_2 y_i^{n-1}\\
&\qquad +\partial_i p = y_1^{n-1}\partial_1 y_i^{n-1} + y_2^{n-1}\partial_2 y_i^{n-1} + \begin{cases} \frac{1}{\varepsilon}(b_i - y_i) & \text{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i}{\alpha} & \text{on } \mathcal{I}_i^n \\ \frac{1}{\varepsilon}(a_i - y_i) & \text{on } \mathcal{A}_{a_i}^n \end{cases}\\
&\qquad \operatorname{div} y = 0\\
&\qquad y|_\Gamma = g
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&-\nu\Delta\lambda_i + \frac{1}{\varepsilon}\lambda_i - y_1\partial_1\lambda_i^{n-1} - y_2\partial_2\lambda_i^{n-1} - y_1^{n-1}\partial_1\lambda_i - y_2^{n-1}\partial_2\lambda_i + \lambda_1\partial_i y_1^{n-1}\\
&\quad +\lambda_2\partial_i y_2^{n-1} + \lambda_1^{n-1}\partial_i y_1 + \lambda_2^{n-1}\partial_i y_2 + \partial_i q = z_{d,i} - y_i - y_1^{n-1}\partial_1\lambda_i^{n-1}\\
&\quad -y_2^{n-1}\partial_2\lambda_i^{n-1} + \lambda_1^{n-1}\partial_i y_1^{n-1} + \lambda_2^{n-1}\partial_i y_2^{n-1} + \begin{cases} \frac{\alpha}{\varepsilon^2}(b_i - y_i) & \text{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i}{\varepsilon} & \text{on } \mathcal{I}_i^n \\ \frac{\alpha}{\varepsilon^2}(a_i - y_i) & \text{on } \mathcal{A}_{a_i}^n \end{cases}\\
&\qquad \operatorname{div} \lambda = 0\\
&\qquad \lambda|_\Gamma = 0.
\end{aligned} \tag{5.3}$$

Set $(y^n, p^n, \lambda^n, q^n) = (y, p, \lambda, q)$, $u_i^n = \begin{cases} \frac{1}{\varepsilon}(b_i - y_i^n) & \text{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i^n}{\alpha} & \text{on } \mathcal{I}_i^n \\ \frac{1}{\varepsilon}(a_i - y_i^n) & \text{on } \mathcal{A}_{a_i}^n. \end{cases}$, $\mu^n = \frac{1}{\varepsilon}(\lambda^n - \alpha u^n)$,

and goto step 2.

Note that the system to be solved in step (2) corresponds to the optimality system of the following optimal control problem

$$
(5.4)\quad \begin{cases}
\min\limits_{\delta_x \in V\times \tilde{C}^n} \ \frac{1}{2}\langle \mathcal{L}''(x^{n-1},\lambda^{n-1})\delta_x, \delta_x\rangle + \langle \mathcal{L}'(x^{n-1},\lambda^{n-1}), \delta_x\rangle \\
\qquad + \frac{\alpha}{2\varepsilon^2}\sum\limits_{i=1}^{d}\int\limits_{\mathcal{A}_{b_i}^n} \left|b_i - y_i^{n-1} - \delta_{y_i}\right|^2 dx + \frac{\alpha}{2\varepsilon^2}\sum\limits_{i=1}^{d}\int\limits_{\mathcal{A}_{a_i}^n} \left|a_i - y_i^{n-1} - \delta_{y_i}\right|^2 dx \\
\qquad\qquad + \frac{1}{\varepsilon}\sum\limits_{i=1}^{d}\int\limits_{\mathcal{A}_{b_i}^n} \lambda_i^{n-1}\cdot \delta_{y_i}\ dx + \frac{1}{\varepsilon}\sum\limits_{i=1}^{d}\int\limits_{\mathcal{A}_{a_i}^n} \lambda_i^{n-1}\cdot \delta_{y_i}\ dx \\
\text{subject to} \\
-\nu\Delta\delta_{y_i} + y_1^{n-1}\partial_1\delta_{y_i} + y_2^{n-1}\partial_2\delta_{y_i} + \delta_{y_1}\partial_1 y_i^{n-1} + \delta_{y_2}\partial_2 y_i^{n-1} + \partial_i p^n \\
\qquad = \nu\Delta y_i^{n-1} - y_1^{n-1}\partial_1 y_i^{n-1} - y_2^{n-1}\partial_2 y_i^{n-1} + \begin{cases} \frac{1}{\varepsilon}(b_i - y_i^{n-1} - \delta_{y_i}) & \text{on } \mathcal{A}_{b_i}^n \\ u_i^{n-1} + \delta_{u_i} & \text{on } \mathcal{I}_i^n \\ \frac{1}{\varepsilon}(a_i - y_i^{n-1} - \delta_{y_i}) & \text{on } \mathcal{A}_{a_i}^n \end{cases} \\
\operatorname{div} \delta_y = 0 \\
\delta_y|_\Gamma = -y^{n-1}|_\Gamma + g,
\end{cases}
$$

where $x^n = (y^n, u^n)$, $\delta_x = x^n - x^{n-1}$ and

$$
\tilde{C}^n := \left\{h \in \mathbf{L}^2(\Omega) : h_i(x) = 0 \text{ for } x \in \mathcal{A}_{b_i}^n \cup \mathcal{A}_{a_i}^n,\ i = 1,\dots,d\right\}.
$$

Problem (5.4) corresponds to a quadratic control problem with affine constraints. Existence and uniqueness of a solution, as well as existence of Lagrange multipliers will be verified next.

Theorem 5.3. *Let $u^* \in U$ be a stationary point of $(\mathcal{P})$ that satisfies the second order condition $(\overline{SSC})$. If $\mathcal{I}_i^n \subset \mathcal{I}_{\tau,i}$, with $\mathcal{I}_{\tau,i} := \Omega\backslash\mathcal{A}_{\tau,i}$, $i = 1,\dots,d$, and $\left\|y^{n-1} - y^*\right\|_V$, $\left\|\lambda^{n-1} - \lambda^*\right\|_V$ are sufficiently small, then there exists a unique solution for system (5.2)-(5.3).*

Proof. Existence of Lagrange multipliers for (5.4) follows from the satisfaction of the regular point condition (see [20]), which in the present case is fulfilled if there exists a unique weak solution $w \in V$ of

$$
(5.5)\quad \begin{aligned}
-\nu\Delta w + (w\cdot\nabla)y^{n-1} + (y^{n-1}\cdot\nabla)w + \nabla\pi &= h \\
\operatorname{div} w &= 0 \\
w|_\Gamma &= 0
\end{aligned}
$$

with $\varepsilon h + w \in \tilde{C}^n$. Multiplying both sides of (5.5) by w, existence and uniqueness follow from the Lax-Milgram theorem if the coercivity condition $\nu > \mathcal{M}(y^{n-1})$ holds. From the definition of $\mathcal{M}(\cdot)$ we get that

$$
\begin{aligned}
\nu - \mathcal{M}(y^{n-1}) &= \nu - \sup_{w\in V}\frac{|c(w, y^{n-1}, w)|}{\|w\|_V^2} \\
&\geq \nu - \sup_{w\in V}\frac{|c(w, y^{n-1} - y^*, w)|}{\|w\|_V^2} - \mathcal{M}(y^*) \\
&\geq \nu - \mathcal{M}(y^*) - \mathcal{N}\left\|y^{n-1} - y^*\right\|_V.
\end{aligned}
$$

Choosing $\left\|y^{n-1} - y^*\right\|_V \leq \frac{\nu - \mathcal{M}(y^*)}{2\mathcal{N}}$, yields

$$
\nu - \mathcal{M}(y^{n-1}) \geq \frac{\nu - \mathcal{M}(y^*)}{2} > 0
$$

and, thus, each solution of (5.4) satisfies the optimality system (5.2)-(5.3).

On the other hand, to see that a solution to (5.2)-(5.3) corresponds to the solution of (5.4) a second order condition has to hold. Denoting by $L(\delta_y, \delta_u)$ the Lagrangian of (5.4), the second order condition can be stated as follows: there exists a constant $\rho > 0$ such that

$$L''(\delta_y, \delta_u)(w,h)^2 \geq \rho \|h\|^2, \tag{5.6}$$

for all $(w,h) \in V \times \mathbf{L}^2(\Omega)$ that solve (5.5) and satisfy $\varepsilon h + w \in \tilde{C}^n$. Taking such a (w,h) arbitrary but fix, we introduce the decomposition $(w,h) = (\xi, \bar{h}) + (\psi, \underline{h})$ with $\xi \in V$ weak solution of

$$\begin{aligned} -\nu\Delta\xi + (\xi \cdot \nabla)y^* + (y^* \cdot \nabla)\xi + \nabla\pi_1 &= \bar{h} \\ \operatorname{div} \xi &= 0 \\ \xi|_\Gamma &= 0, \end{aligned} \tag{5.7}$$

with $\bar{h}_i := \begin{cases} -\frac{1}{\varepsilon}\xi_i & \text{on } \mathcal{A}_{b_i}^n \\ h_i & \text{on } \mathcal{I}_i^n \\ -\frac{1}{\varepsilon}\xi_i & \text{on } \mathcal{A}_{a_i}^n \end{cases}$, for $i = 1, \ldots, d$, and $\psi \in V$ weak solution of

$$\begin{aligned} -\nu\Delta\psi + (\psi \cdot \nabla)y^{n-1} + (y^{n-1} \cdot \nabla)\psi + \nabla\pi_2 & \\ = -((y^{n-1} - y^*) \cdot \nabla)\xi - (\xi \cdot \nabla)(y^{n-1} - y^*) + \underline{h}& \\ \operatorname{div} \psi = 0& \\ \psi|_\Gamma = 0& \end{aligned} \tag{5.8}$$

with $\underline{h}_i = \begin{cases} -\frac{1}{\varepsilon}\psi_i & \text{on } \mathcal{A}_{b_i}^n \\ 0 & \text{on } \mathcal{I}_i^n \\ -\frac{1}{\varepsilon}\psi_i & \text{on } \mathcal{A}_{a_i}^n \end{cases}$, for $i = 1, \ldots, d$. We therefore get that $(\xi, \bar{h})$ solves (5.7) and satisfies $\varepsilon\bar{h} + \xi \in \tilde{C}^n$. From (4.3) and using Cauchy-Schwarz we thus obtain

$$L''(\delta_y, \delta_u)(w,h)^2 \geq \|\xi\|^2 + \alpha \|\bar{h}\|^2 - 2\|\xi\|\|\psi\| - 2\alpha\|\bar{h}\|\|\underline{h}\| + 2c(w, w, \lambda^{n-1}),$$

which implies, using the properties of the trilinear form, that

$$\begin{aligned} L''(\delta_y, \delta_u)(w,h)^2 \geq\ & \mathcal{L}''(y^*, u^*, \lambda^*)(\xi, \bar{h})^2 - 2\|\xi\|\|\psi\| - 2\alpha\|\bar{h}\|\|\underline{h}\| \\ & - 2\mathcal{N}\|\xi\|_V^2 \|\lambda^{n-1} - \lambda^*\|_V - 4\mathcal{N}\|\xi\|_V\|\psi\|_V\|\lambda^{n-1}\|_V - 2\mathcal{N}\|\psi\|_V^2\|\lambda^{n-1}\|_V. \end{aligned}$$

¿From equations (5.7) and (5.8) it can be verified that the following estimates hold

$$\|\xi\|_V \leq \kappa\sigma\|\bar{h}\| \tag{5.9}$$

$$\|\psi\|_V \leq 4\mathcal{N}\kappa\sigma^2 \|y^{n-1} - y^*\|_V \|\bar{h}\|, \tag{5.10}$$

with $\sigma := (\nu - \mathcal{M}(y^*))^{-1}$.

Since by hypothesis u^* satisfies $(\overline{SSC})$ and $\mathcal{I}_i^n \subset \mathcal{I}_{\tau,i}$, it follows that $\varepsilon\bar{h} + \xi = 0$ on $\mathcal{A}_{\tau,i}$ and, using estimates (5.9), (5.10),

$$\begin{aligned} L''(\delta_y, \delta_u)(w,h)^2 \geq \delta\|\bar{h}\|^2 - 8\mathcal{N}\kappa^4\sigma^3\|y^{n-1} - y^*\|_V\|\bar{h}\|^2 - \frac{8}{\varepsilon}\alpha\mathcal{N}\kappa^2\sigma^2\|y^{n-1} - y^*\|_V\|\bar{h}\|^2 \\ -2\mathcal{N}\kappa^2\sigma^2\|\lambda^{n-1} - \lambda^*\|_V\|\bar{h}\|^2 - 16\mathcal{N}^2\kappa^2\sigma^3\|\lambda^{n-1}\|_V\|y^{n-1} - y^*\|_V\|\bar{h}\|^2 \\ -32\mathcal{N}^3\kappa^2\sigma^4\|\lambda^{n-1}\|_V\|y^{n-1} - y^*\|_V^2\|\bar{h}\|^2. \end{aligned}$$

Choosing $\left\|y^{n-1}-y^*\right\|_V$ and $\left\|\lambda^{n-1}-\lambda^*\right\|_V$ sufficiently small such that

$$\rho := \delta - 2\mathcal{N}\kappa^2\sigma^2\left\|\lambda^{n-1}-\lambda^*\right\|_V - 8\mathcal{N}\kappa^2\sigma^2\left\|y^{n-1}-y^*\right\|_V\left[\kappa^2\sigma+\alpha/\varepsilon + 2\mathcal{N}\sigma\left\|\lambda^{n-1}\right\|_V + 4\mathcal{N}^2\sigma^2\left\|\lambda^{n-1}\right\|_V\left\|y^{n-1}-y^*\right\|_V\right] > 0,$$

condition (5.6) is satisfied.

Therefore, system (5.2)-(5.3) is uniquely solvable since it corresponds to the solution of a linear quadratic control problem with convex objective. □

Remark 5.4. *¿From the definition of the inactive sets, it can be verified that the condition* $\mathcal{I}_i^n \subset \mathcal{I}_{\tau,i}$ *holds for* $\left\|y^{n-1}-y^*\right\|_V$ *and* $\left\|\lambda^{n-1}-\lambda^*\right\|_V$ *sufficiently small.*

By considering the state variable y and the newly defined control variable v, the optimal control problem $(\mathcal{P})$ can locally also be expressed as the following control constrained optimal control problem

$$(5.11)\quad \begin{cases} \min\ J(y,v) = \frac{1}{2}\int_\Omega |y-z_d|^2\,dx + \frac{\alpha}{2\varepsilon^2}\int_\Omega |v|^2\,dx - \frac{\alpha}{\varepsilon^2}\int_\Omega v\ y\,dx + \frac{\alpha}{2\varepsilon^2}\int_\Omega |y|^2\,dx \\ \text{subject to} \\ -\nu\Delta y + \frac{1}{\varepsilon}y + (y\cdot\nabla)y + \nabla p = \frac{1}{\varepsilon}v \\ \operatorname{div} y = 0 \\ y|_\Gamma = g \\ a \le v \le b \text{ a.e.,} \end{cases}$$

The presence of the mixed term $\frac{\alpha}{\varepsilon^2}\int_\Omega v\ y\,dx$ in the cost functional is responsible for a different problem structure, which does not allow the application of already known results about convergence of the semi-smooth Newton method for control constrained optimal control problems (see [9, 17]).

In the next theorem sufficient conditions for the local superlinear convergence of the semi-smooth Newton method are stated.

Theorem 5.5. *Let* $u^* \in U$ *be a stationary point of* $(\mathcal{P})$ *that satisfies* $(\overline{SSC})$. *If* $\|\lambda^*\|_V < \frac{\alpha^{1/2}}{4\varepsilon}(\nu - \mathcal{M}(y^*))\left(\frac{\varepsilon(\alpha+\varepsilon^2)-\alpha}{\varepsilon^{1/2}(\alpha+\varepsilon^2)^{1/2}+\alpha^{1/2}}\right)$ *and* $\left\|y^0-y^*\right\|_V$, $\left\|\lambda^0-\lambda^*\right\|_V$ *are sufficiently small, then the sequence* $\{(y^n,v^n,\lambda^n,\mu^n)\}$ *generated by the algorithm converges superlinearly in* $\mathbf{H}^1(\Omega)\times\mathbf{L}^2(\Omega)\times V\times\mathbf{L}^2(\Omega)$ *to* $(y^*,u^*,\lambda^*,\mu^*)$. *Moreover, there exists a constant* $C>0$ *such that*

$$(5.12)\quad \left\|(v^{n+1}-v^*, y^{n+1}-y^*, \lambda^{n+1}-\lambda^*)\right\|_{\mathbf{L}^2\times V\times V} \le C\left(\|y^n-y^*\|_V^2 + \|\lambda^n-\lambda^*\|_V^2\right) + o(\|(y^n-y^*,\lambda^n-\lambda^*)\|_{\mathbf{L}^p\times\mathbf{L}^p}).$$

Proof. By considering system (3.13)-(3.16) and the system in step (2) of Algorithm 5.2, it can be verified that the increments $\delta_y = y^{n+1}-y^*$, $\delta_\lambda = \lambda^{n+1}-\lambda^*, \delta_u, \delta_\mu$ and δ_ϕ satisfy the system

$$(5.13)\quad \begin{aligned} &\nu(\nabla\delta_y,\nabla\phi) + \frac{1}{\varepsilon}(\delta_y,\phi) + c(y^n,\delta_y,\phi) + c(\delta_y,y^n,\phi) \\ &\qquad = \frac{1}{\varepsilon}(\delta_v,\phi) + (((y^n-y^*)\cdot\nabla)(y^n-y^*),\phi), \text{ for all } \phi\in V, \end{aligned}$$

(5.14)
$$\nu(\nabla\delta_\lambda, \nabla\phi) - c(y^n, \delta_\lambda, \phi) - c(\delta_y, \lambda^n, \phi) + c(w, y^n, \delta_\lambda) + c(w, \delta_y, \lambda^n)$$
$$= ((\nabla(y^n - y^*))^T(\lambda^n - \lambda^*) - ((y^n - y^*)\cdot\nabla)(\lambda^n - \lambda^*), \phi) - (\delta_y + \delta_\mu, \phi), \forall\phi \in V.$$

Introducing the auxiliar variable $\varphi := \varepsilon\mu + \frac{\alpha}{\varepsilon}v$, and considering (3.15) and (5.1) together with the semi-smooth Newton update for u^n and μ^n we also obtain that

$$\delta_\lambda - \frac{\alpha}{\varepsilon}\delta_v + \frac{\alpha}{\varepsilon}\delta_y = \varepsilon\delta_\mu \tag{5.15}$$

$$\delta_\varphi = \varepsilon\delta_\mu + \frac{\alpha}{\varepsilon}\delta_v \tag{5.16}$$

$$\delta_\varphi - \frac{\alpha}{\varepsilon}\delta_v = G^n_{\max}(\delta_\varphi) + G^n_{\min}(\delta_\varphi) + R \tag{5.17}$$

where

$$G^n_{\max,i}(\phi) = \begin{cases}\phi \text{ on } \mathcal{A}^{n+1}_{b_i}\\ 0 \text{ on } \Omega\backslash\mathcal{A}^{n+1}_{b_i}\end{cases} \quad \text{and} \quad G^n_{\min,i}(\phi) = \begin{cases}\phi \text{ on } \mathcal{A}^{n+1}_{a_i}\\ 0 \text{ on } \Omega\backslash\mathcal{A}^{n+1}_{a_i}\end{cases},$$

and

$$R = \max(0, \varphi^* + (\varphi^n - \varphi^*) - \frac{\alpha}{\varepsilon}b) - \max(0, \varphi^* - \frac{\alpha}{\varepsilon}b) - G^n_{\max}(\varphi^n - \varphi^*)$$
$$+ \min(0, \varphi^* + (\varphi^n - \varphi^*) - \frac{\alpha}{\varepsilon}a) - \min(0, \varphi^* - \frac{\alpha}{\varepsilon}a) - G^n_{\min}(\varphi^n - \varphi^*).$$

Due to Newton differentiability of the $max(0,\cdot)$ and $min(0,\cdot)$ functions (cf. [17]) from $L^p(\Omega) \to L^q(\Omega)$, with $q < p$, we therefore obtain that

$$\|R\|_{\mathbf{L}^2} = o(\|\varphi^n - \varphi^*\|_{\mathbf{L}^p}), \tag{5.18}$$

with $p > 2$.

Multiplying equation (5.17) by δ_v we get that

$$-(R, \delta_v) = (G^n_{\max}(\delta_\varphi) + G^n_{\min}(\delta_\varphi), \delta_v) - (\delta_\varphi, \delta_v) + \frac{\alpha}{\varepsilon}\|\delta_v\|^2. \tag{5.19}$$

Additionally, from the definition of $G^n_{\max}$ and $G^n_{\min}$,

$$(G^n_{\max}(\delta_\varphi) + G^n_{\min}(\delta_\varphi), \delta_v) - (\delta_\varphi, \delta_v) = (\delta_\varphi, \delta_v)_{\mathcal{I}^n}, \tag{5.20}$$

where $(v, w)_{\mathcal{I}^n} := \int_{\mathcal{I}^n} v \cdot w \, dx$.

On the other hand, substituting (5.15) in (5.14) and multiplying by δ_y we get that

(5.21)
$$\nu(\nabla\delta_\lambda, \nabla\delta_y) + \frac{1}{\varepsilon}(\delta_\lambda, \delta_y) - c(y^n, \delta_\lambda, \delta_y) - c(\delta_y, \lambda^n, \delta_y)$$
$$+ c(\delta_y, y^n, \delta_\lambda) + c(\delta_y, \delta_y, \lambda^n) = ((\nabla(y^n - y^*))^T(\lambda^n - \lambda^*)$$
$$- ((y^n - y^*)\cdot\nabla)(\lambda^n - \lambda^*), \delta_y) - \|\delta_y\|^2 + \frac{\alpha}{\varepsilon^2}(\delta_v, \delta_y) - \frac{\alpha}{\varepsilon^2}\|\delta_y\|^2,$$

which, utilizing (5.13) multiplied by δ_λ yields

$$\frac{1}{\varepsilon}(\delta_v, \delta_\lambda) + (((y^n - y^*)\cdot\nabla)(y^n - y^*), \delta_\lambda) - c(y^n, \delta_y, \delta_\lambda) - c(\delta_y, y^n, \delta_\lambda) \tag{5.22}$$
$$= ((\nabla(y^n - y^*))^T(\lambda^n - \lambda^*) - ((y^n - y^*)\cdot\nabla)(\lambda^n - \lambda^*), \delta_y) - \frac{\alpha}{\varepsilon^2}\|\delta_y\|^2$$
$$+ \frac{\alpha}{\varepsilon^2}(\delta_v, \delta_y) - \|\delta_y\|^2 + c(y^n, \delta_\lambda, \delta_y) + c(\delta_y, \lambda^n, \delta_y) - c(\delta_y, y^n, \delta_\lambda) - c(\delta_y, \delta_y, \lambda^n).$$

Consequently, utilizing the properties of the trilinear form,

(5.23)
$$\left(\frac{\alpha+\varepsilon^2}{\varepsilon^2}\right)\|\delta_y\|^2+\frac{1}{\varepsilon}(\delta_v,\delta_\lambda)+\frac{\alpha}{\varepsilon^2}(\delta_v,\delta_y)-2c(\delta_y,\lambda^n,\delta_y)=((\nabla(y^n-y^*))^T(\lambda^n-\lambda^*)$$
$$-((y^n-y^*)\cdot\nabla)(\lambda^n-\lambda^*),\delta_y)-(((y^n-y^*)\cdot\nabla)(y^n-y^*),\delta_\lambda)+\frac{2\alpha}{\varepsilon^2}(\delta_v,\delta_y).$$

and therefore

$$\frac{1}{\varepsilon}(\delta_v,\delta_\varphi)\le\frac{2\alpha}{\varepsilon^2}(\delta_v,\delta_y)+2\mathcal{N}\|\lambda^n\|_V\|\delta_y\|_V^2-\left(\frac{\alpha+\varepsilon^2}{\varepsilon^2}\right)\|\delta_y\|^2$$
$$+((\nabla(y^n-y^*))^T(\lambda^n-\lambda^*)-((y^n-y^*)\cdot\nabla)(\lambda^n-\lambda^*),\delta_y)$$
$$-(((y^n-y^*)\cdot\nabla)(y^n-y^*),\delta_\lambda). \tag{5.24}$$

Let us now consider the increment equation (5.13) and multiply it by δ_y. We get the estimate

$$\nu\|\delta_y\|_V^2+\frac{1}{\varepsilon}\|\delta_y\|^2-\mathcal{M}(y^n)\|\delta_y\|_V^2\le\frac{1}{\varepsilon}(\delta_v,\delta_y)+\mathcal{N}\|y^n-y^*\|_V^2\|\delta_y\|_V, \tag{5.25}$$

which, by considering a y^* neighborhood such that

$$\nu-\mathcal{M}(y^n)\ge\frac{1}{2}(\nu-\mathcal{M}(y^*))>0 \tag{5.26}$$

and Poincare inequality, implies that

$$\frac{1}{2}(\nu-\mathcal{M}(y^*))\|\delta_y\|_V^2+\frac{1}{\varepsilon}\|\delta_y\|^2\le\frac{1}{\varepsilon}(\delta_v,\delta_y)+\mathcal{N}\|y^n-y^*\|_V^2\|\delta_y\|_V. \tag{5.27}$$

Consequently, we obtain the estimate

$$\|\delta_y\|_V\le 2\sigma(\frac{\kappa}{\varepsilon}\|\delta_v\|+\mathcal{N}\|y^n-y^*\|_V^2), \tag{5.28}$$

with $\sigma:=(\nu-\mathcal{M}(y^*))^{-1}$.

Using (5.27) in (5.24) and grouping terms yields

$$\frac{1}{\varepsilon}(\delta_v,\delta_\varphi)\le\frac{2\alpha+4\sigma\varepsilon\|\lambda^n\|_V}{\varepsilon^2}(\delta_v,\delta_y)-\left(\frac{\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V}{\varepsilon^2}\right)\|\delta_y\|^2$$
$$+4\mathcal{N}\sigma\|\lambda^n\|_V\|y^n-y^*\|_V^2\|\delta_y\|_V-(((y^n-y^*)\cdot\nabla)(y^n-y^*),\delta_\lambda)$$
$$+((\nabla(y^n-y^*))^T(\lambda^n-\lambda^*)-((y^n-y^*)\cdot\nabla)(\lambda^n-\lambda^*),\delta_y). \tag{5.29}$$

Since

$$\left\|c\delta_v-\left(\frac{\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V}{\varepsilon^2}\right)^{1/2}\delta_y\right\|^2=c^2\|\delta_v\|^2$$
$$-\frac{2c}{\varepsilon}\left(\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V\right)^{1/2}(\delta_v,\delta_y)+\left(\frac{\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V}{\varepsilon^2}\right)\|\delta_y\|^2, \tag{5.30}$$

we obtain, by choosing $c = \frac{\alpha+2\sigma\varepsilon\|\lambda^n\|_V}{\varepsilon\sqrt{\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V}}$, that

$$\frac{1}{\varepsilon}(\delta_v,\delta_\varphi) \leq \frac{(\alpha+2\sigma\varepsilon\|\lambda^n\|_V)^2}{\varepsilon^2(\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V)}\|\delta_v\|^2 + 4\mathcal{N}\sigma\|\lambda^n\|_V\|y^n-y^*\|_V^2\|\delta_y\|_V \tag{5.31}$$
$$+ ((\nabla(y^n-y^*))^T(\lambda^n-\lambda^*) - ((y^n-y^*)\cdot\nabla)(\lambda^n-\lambda^*),\delta_y) - (((y^n-y^*)\cdot\nabla)(y^n-y^*),\delta_\lambda).$$

Consequently, from (5.19)-(5.20) and (5.31) we therefore obtain

$$\begin{aligned}|(R,\delta_v)| \geq \frac{\alpha}{\varepsilon}\|\delta_v\|^2 - \Bigg| &\frac{(\alpha+2\sigma\varepsilon\|\lambda^n\|_V)^2}{\varepsilon^2(\alpha+\varepsilon^2+4\sigma\varepsilon\|\lambda^n\|_V)}\|\delta_v\|^2_{\mathcal{I}^n}\\ &+((\nabla(y^n-y^*))^T(\lambda^n-\lambda^*) - ((y^n-y^*)\cdot\nabla)(\lambda^n-\lambda^*),\delta_y)_{\mathcal{I}^n}\\ &-(((y^n-y^*)\cdot\nabla)(y^n-y^*),\delta_\lambda)_{\mathcal{I}^n} + 4\mathcal{N}\sigma\|\lambda^n\|_V\|y^n-y^*\|_V^2\|\delta_y\|_V\Bigg|,\end{aligned}$$

which by considering a (y^*,λ^*) neighborhood such that

$$\|\lambda^n\|_V \leq 2\|\lambda^*\|_V \tag{5.32}$$

implies that

$$\begin{aligned}|(R,\delta_v)| \geq \frac{\alpha}{\varepsilon}\|\delta_v\|^2 &- \frac{(\alpha+4\sigma\varepsilon\|\lambda^*\|_V)^2}{\varepsilon^2(\alpha+\varepsilon^2)}\|\delta_v\|^2\\ &-\mathcal{N}\|y^n-y^*\|_V^2(8\sigma\|\lambda^*\|_V\|\delta_y\|_V+\|\delta_\lambda\|_V) - 2\mathcal{N}\|y^n-y^*\|_V\|\lambda^n-\lambda^*\|_V\|\delta_y\|_V.\end{aligned}$$

Since, by hypothesis, $\|\lambda^*\|_V < \frac{\alpha^{1/2}}{4\varepsilon}(\nu-\mathcal{M}(y^*))\frac{\varepsilon(\alpha+\varepsilon^2)-\alpha}{\varepsilon^{1/2}(\alpha+\varepsilon^2)^{1/2}+\alpha^{1/2}}$ we get that

$$\beta := \frac{\alpha\varepsilon(\alpha+\varepsilon^2)-(\alpha+4\sigma\varepsilon\|\lambda^*\|_V)^2}{\varepsilon^2(\alpha+\varepsilon^2)} > 0 \tag{5.33}$$

and therefore

$$\begin{aligned}\beta\|\delta_v\|^2 \leq \|R\|\|\delta_v\| + \mathcal{N}\|y^n-y^*\|_V^2(8\sigma\|\lambda^*\|_V\|\delta_y\|_V+\|\delta_\lambda\|_V)&\\ +2\mathcal{N}\|y^n-y^*\|_V\|\lambda^n-\lambda^*\|_V\|\delta_y\|_V&.\end{aligned} \tag{5.34}$$

On the other hand, by multiplying (5.14) by δ_λ we get that

$$\begin{aligned}\nu\|\delta_\lambda\|_V^2 + \frac{1}{\varepsilon}\|\delta_\lambda\|^2 - \mathcal{M}(y^n)\|\delta_\lambda\|_V^2 \leq 2\mathcal{N}\|\delta_y\|_V\|\lambda^n\|_V\|\delta_\lambda\|_V&\\ +2\mathcal{N}\|y^n-y^*\|_V\|\lambda^n-\lambda^*\|_V\|\delta_\lambda\|_V + \left(\frac{\alpha}{\varepsilon^2}+1\right)\|\delta_y\|\|\delta_\lambda\| + \frac{\alpha}{\varepsilon^2}\|\delta_v\|\|\delta_\lambda\|&,\end{aligned} \tag{5.35}$$

which, considering (5.32) and (5.26), implies that

$$\begin{aligned}\frac{1}{2}(\nu-\mathcal{M}(y^*))\|\delta_\lambda\|_V \leq \left(\frac{\kappa^2\alpha}{\varepsilon^2}+\kappa^2+4\mathcal{N}\|\lambda^*\|_V\right)\left(\frac{\alpha+\varepsilon^2}{\varepsilon^2}\right)\|\delta_y\|_V&\\ +\frac{\alpha}{\varepsilon^2}\|\delta_v\| + 2\mathcal{N}\|\lambda^n-\lambda^*\|_V\|y^n-y^*\|_V&.\end{aligned} \tag{5.36}$$

Therefore, utilizing estimate (5.28), there exists a constant $\bar{C}$ such that

$$\|\delta_\lambda\|_V \leq \bar{C}(\|\delta_v\| + \|\lambda^n-\lambda^*\|_V^2 + \|y^n-y^*\|_V^2). \tag{5.37}$$

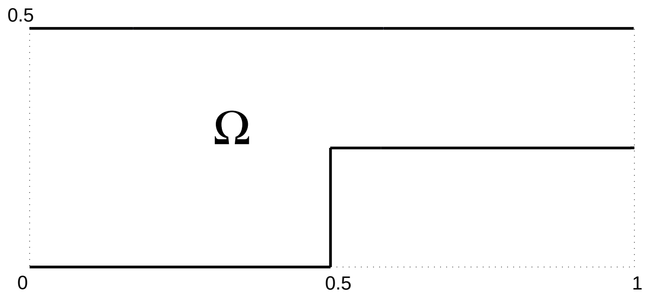

FIG. 6.1. *Forward facing step channel.*

¿From the definition of φ and (5.18) we therefore obtain that

$$\|R\| = o(\|(y^n - y^*, \lambda^n - \lambda^*)\|_{\mathbf{L}^p \times \mathbf{L}^p}),$$

which, considering estimates (5.28) and (5.37) in (5.34), implies the existence of a constant C such that

$$\begin{aligned} \|(\delta_v, \delta_y, \delta_\lambda)\|_{\mathbf{L}^2 \times V \times V} \leq C \left(\|y^n - y^*\|_V^2 + \|\lambda^n - \lambda^*\|_V^2 \right) \\ + o(\|(y^n - y^*, \lambda^n - \lambda^*)\|_{\mathbf{L}^p \times \mathbf{L}^p}). \end{aligned} \tag{5.38}$$

Consequently, the superlinear convergence is verified. □

6. Numerical results. For the numerical tests, a "forward facing step channel" was utilized (see Figure 6.1). The fluid flows from left to right with parabolic inflow condition with maximum value equal to one and "do nothing" outflow condition. In the remaining boundary parts an homogeneous Dirichlet condition was imposed. The geometry was discretized using a staggered grid and an upwinding finite differences scheme was applied. The behavior of the uncontrolled fluid flow with Reynolds number $Re = 1000$ is depicted in Figure 6.2. Two main recirculation zones, which increase their size together with the Reynolds number, can be clearly identified from the graphics.

The target of our control problem is to properly diminish the recirculations of interest by considering, together with the tracking type cost functional, adequate pointwise control-state constraints.

For the solution of the optimality system, Algorithm 5.2 was utilized. The semi-smooth Newton algorithm stops when the $\mathbf{L}^2$-norm of the state increment is lower than 10^{-4}. Unless otherwise specified, the mesh step size $h = 1/240$ was considered. For the solution of the linear systems, MATLAB's exact solver was utilized.

6.1. Example 1. In this first experiment, we consider the elimination of bubbles in the channel by imposing the constraint $y_1 + \varepsilon u_1 \geq -10^{-7}$. For ε sufficiently small, this constraint avoids backward flow in the channel and thus possible recirculations. Additionally, the tracking type component of the cost functional is responsible for a more linear behavior of the flow field. The remaining parameter data utilized are $h = 1/240$, $Re = 1000, \varepsilon = 10^{-4}$ and $\alpha = 0.1$. The semi-smooth Newton method (SSN) stops after 9 iterations, with the final active set containing 28 grid points. The cost functional takes the final value $J(y^*, u^*) = 0.00445224$ and the NCP function residuum the value 2.2737×10^{-9}. The optimal control field

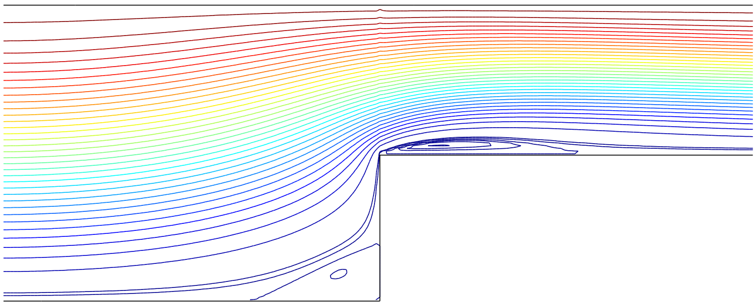

FIG. 6.2. *Streamlines of the uncontrolled state.*

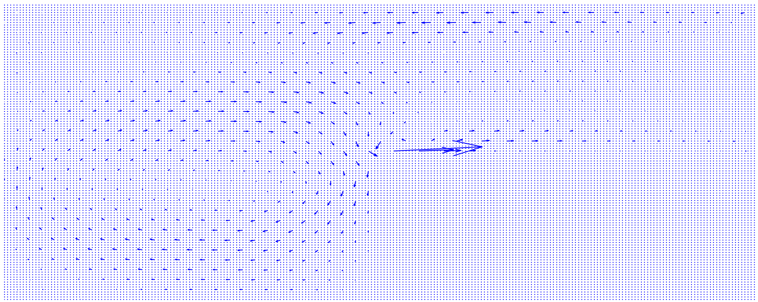

FIG. 6.3. *Example 1: control vector field with tracking component.*

is depicted in Figure 6.3, where the concentration of the control action on the recirculation zones can be observed. The desired recirculation diminishing effect of the control can be visualized from the plot of the reached controlled state streamlines in Figure 6.4. In Table 6.1 the number of SSN iterations, the final cost functional value and the size of the active set are registered for different ε values. It can be observed that as ε tends to 0, the problem becomes harder to solve and more SSN iterations are required.

Subsequently we consider the limit case where the tracking type part of the cost functional is dismissed. We aim to find the control of minimum norm that allows the satisfaction of the state constraint $y_1 + \varepsilon u_1 \geq 10^{-7}$ over the domain of interest. As before, the constraint takes care that no important backward flow arises. By considering the constraint on the whole domain, i.e. $\Omega_S = \Omega$, both recirculations before and after the step are diminished (see Figure 6.5). From Figure 6.5 it can also be observed that the behavior of the fluid flow, mainly before the step, is not close to the Stokes flow, as is the case when the tracking type component is present. From the control vector plot (see Figure 6.6) it can be observed that the control action in this case is even more concentrated on the recirculations zones. The parameter values for this case are $Re = 1000, \varepsilon = 10^{-4}$ and $\alpha = 0.1$. The number of SSN iterations needed is 29 and the cost functional takes the final value 8.99816×10^{-4}.

In many practical cases, the recirculations reduction or elimination on the whole domain is not necessary, if not undesirable. In such cases the state constraint may be imposed in the sectors where the bubble to be diminished is localized. In the case of our geometry the essential recirculation to be diminished is the one after the step. By considering the state constraint on the subdomain $\Omega_S := [0.5, 0.75] \times [0.25, 0.5]$, this elimination is attached with the cost functional value 8.98898×10^{-4} in 6 SSN iterations. The final controlled

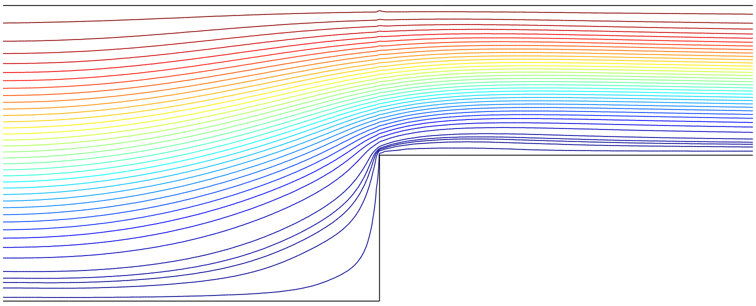

FIG. 6.4. *Example 1: streamlines of the controlled state with tracking component.*

ε	SSN Iter.	$J(y^*, u^*)$	$\lvert\mathcal{A}^a \cup \mathcal{A}^b\rvert$
10^{-1}	5	0.00399972	33
10^{-2}	6	0.00410360	42
10^{-3}	8	0.00438273	29
10^{-4}	9	0.00445224	28
10^{-5}	9	0.00445989	32

TABLE 6.1
Example 1: h=1/240, tol $= 10^{-4}$.

state is shown in Figure 6.7, where it can be clearly seen that the recirculation after the step is numerically eliminated, although the one before the step becomes bigger than in the uncontrolled case.

6.2. Example 2. As an alternative strategy for the reduction of the recirculation after the step, we consider in this example a state constraint that guarantees an homogeneous outflow velocity. The constraint imposed is $y_1 + \varepsilon u_1 \leq 1.7$ and the remaining parameter values are $Re = 1000$, $\varepsilon = 10^{-3}$ and $\alpha = 0.01$. In this case, the SSN algorithm stops after 15 iterations and the resulting active set contains 2283 grid points. The cost functional takes the final value $J(y^*, u^*) = 0.003470768$. The controlled state is depicted in Figure 6.8, where an important reduction of the recirculations can be visualized.

Since the outgoing velocity is the quantity of interest, it is natural to consider the case where the constraint is imposed only in the last part of the channel. By considering the domain $\Omega_S := [0.5, 0.75] \times [0.25, 0.5]$, the recirculation diminishing effect does also take place (see Figure 6.9), but with a lower final cost functional value $J(y^*, u^*) = 0.0031112131$. The SSN algorithm stops after 10 iterations with a final active set containing 906 active points. The remaining parameter values are the same as in the case $\Omega_S = \Omega$.

Finally, in order to visualize the structure of the control-state constraint multiplier, we modify the Reynolds number to 500 and impose the homogeneous outgoing velocity constraint $y_1 + \varepsilon u_1 \leq 1.7$. The evolution of the multiplier as ε decreases can be observed in Figure 6.10. In Table 6.2 the evolution of the SSN is registered. The algorithm stops after 7 iterations with the final active set containing 2465 grid points. As expected from the theoretical results, local superlinear convergence can be observed from the data. Let us point out that, although no monotonic behavior of the cost functional along the iterations occurs, a monotonic decrease of the nonlinear complementarity function and of the size of the active set can be observed.

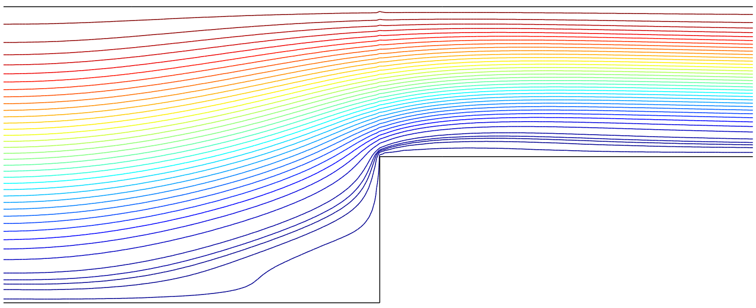

FIG. 6.5. *Example 1: streamlines of the controlled state without tracking component.*

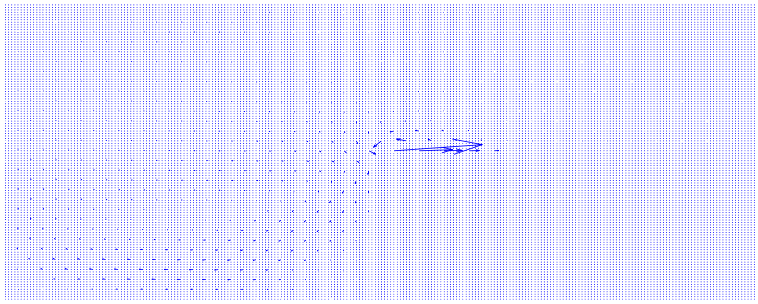

FIG. 6.6. *Example 1: control vector field without tracking component.*

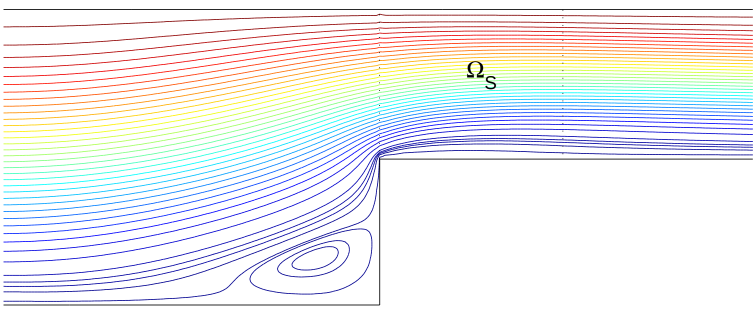

FIG. 6.7. *Example 1: streamlines of the controlled state without tracking component; state constraint subdomain.*

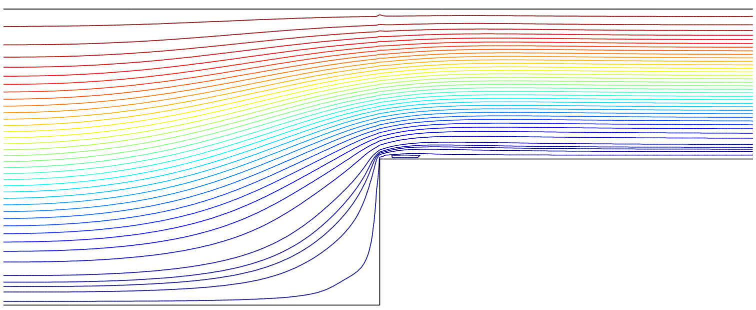

FIG. 6.8. *Example 2: streamlines of the controlled state*

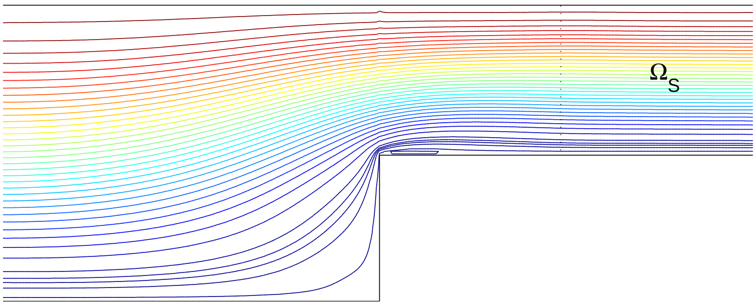

FIG. 6.9. *Example 2: streamlines of the controlled state; state constraint subdomain.*

Iteration	$\mid \mathcal{A}_n \mid$	$J(y,u)$	$\|y_n - y_{n-1}\|$	$\frac{\|y_n - y_{n-1}\|}{\|y_{n-1} - y_{n-2}\|}$	NCP
1	0	0.00156432	9.4321	-	29.43065
2	2743	0.00349897	12.40964	-	4.531425
3	2571	0.003355	1.05301	0.0077	1.663621
4	2494	0.0033477	0.2134005	0.201	0.397106
5	2469	0.00334765	0.0151623	0.07079	0.052505
6	2465	0.00334765	$5.55 \cdot 10^{-4}$	0.03634	$2.22 \cdot 10^{-14}$
7	2465	0.00334765	$2.048 \cdot 10^{-8}$	$3.86 \cdot 10^{-5}$	$2.22 \cdot 10^{-14}$

TABLE 6.2
Example 2: $h = \frac{1}{240}$, $\varepsilon = 10^{-3}$, $Re = 500$.

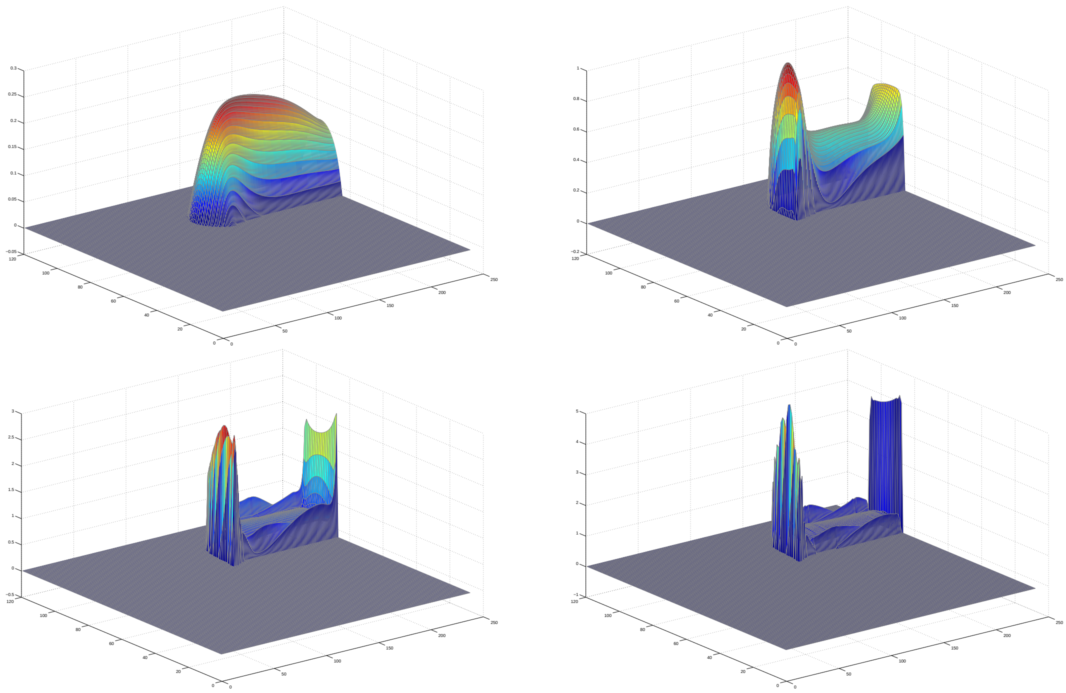

FIG. 6.10. *State constraint multiplier;* $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-4}$

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