Flow control with regularized state constraints

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Summary

We consider the distributed optimal control of the Navier-Stokes equations in presence of pointwise state constraints. A Lavrentiev regularization of the constraints is proposed and a first order optimality system is derived. The regularity of the mixed constraint multiplier is investigated and second order sufficient optimality conditions are studied. In the last part of the paper, a semi-smooth Newton method is applied for the numerical solution of the control problem and numerical experiments are carried out.

1 Introduction

In the recent past, optimal control of fluid flow has become an attractive multidisciplinary research field with a wide range of ongoing and promising applications. The optimization problems in this context consist in minimizing or maximizing an objective functional (e.g. drag, lift, etc.) subject to the constitutive fluid flow equations and additional control and/or state constraints.

The controls involved are usually considered in distributed form on a sub-domain or as boundary condition acting on some wall sectors. While the design of boundary controls is technically posible, the implementation of a distributed control action presents important difficulties. Lately, an increasing attention has been paid to this kind of controls, mainly within the field of magneto-hydro-dynamics (MHD). Weakly conductive fluids are controlled through the action of Lorentz forces, induced by magnetic fields (see [17, 25]).

Let us briefly comment on the literature. The distributed optimal control problem of the Navier-Stokes equations has been mathematically analyzed and numerically studied in many research papers, see for example [1, 3, 10, 11, 16, 23]. In these articles optimality conditions and/or numerical methods for the solution of the control problem were discussed. The same topics were considered, for the boundary optimal control problem, in [6, 10, 12, 13]. In [6, 12, 13] Dirichlet controls were studied, while in [10] the action of Neumann boundary conditions was investigated. In presence of pointwise control constraints, optimality conditions and numerical methods have been treated in [4, 15, 24]. In particular semi-smooth Newton methods have been applied in this context (see [6, 15, 24]).

In presence of pointwise state constraints the problem has received much less attention. The mathematical analysis of the optimal control problem has been considered in [5] and [9] for the stationary and time dependent problems, respectively. In [7] the numerical solution utilizing a penalized problem together with a semi-smooth Newton method has been studied.

In this paper we consider a bounded two-dimensional domain $\Omega \subset \mathbb{R}^2$ and pointwise state constraints of box type

$$a(x) \leq y(x) \leq b(x), \tag{1.1}$$

where $y = (y_1, y_2)$ stands for the velocity vector field. These constraints are imposed in order to reduce backward flow and, consequently, diminish recirculations. Among other applications, such restrictions can have an important effect in avoiding flow separation or reducing the drag of a body.

For the numerical solution of the control problem we propose a Lavrentiev regularization of the pointwise state constraints, i.e. we consider the modified box constraints

$$a(x) \leq y(x) + \varepsilon u(x) \leq b(x), \qquad \varepsilon > 0. \tag{1.2}$$

Due to the mixed nature of the pointwise constraints (1.2), the corresponding Lagrange multiplier is expected to be more regular than in the state constrained case (cf. [5]). It is also expected that, as ε tends to zero, the solutions converge to the optimal solution of the state constrained problem (see [18]).

Based on the methodology developed in [19] for semilinear elliptic equations, we locally reformulate the mixed problem as a control constrained control problem in a new variable. After that, necessary and sufficient conditions for optimality are studied. Also, thanks to the efficiency of semi-smooth Newton methods for nonlinear control constrained optimal control problems (cf. [6, 14]), we apply a method of this type for the numerical solution of the control problem.

The outline of the paper is as follows. In Section 2, the optimal control problem is stated and existence of a global optimal solution is verified. In Section 3, the problem is reformulated as a control constrained optimal control problem and first order necessary optimality conditions are obtained. Sufficient conditions of second order type are the topic of Section 4. In Section 5, a semi-smooth Newton algorithm for the solution of the problem is stated. Reports on numerical experiments are summarized in Section 6.

2 Problem statement and existence of solution

Consider a bounded regular domain $\Omega \subset \mathbb{R}^2$. Our objective is to find the optimal control u^* and its associated state y^*, solution of the following problem:

$$\begin{cases} \min\ J(y,u) = \frac{1}{2}\int\limits_{\Omega} |y - z_d|^2\, dx + \frac{\alpha}{2}\int\limits_{\Omega} |u|^2\, dx \\ \text{subject to} \\ -\nu \Delta y + (y\cdot\nabla)y + \nabla p = u \\ \text{div}\, y = 0 \\ y|_{\Gamma} = g \\ a \leq \varepsilon u + y \leq b \text{ a.e.,} \end{cases} \tag{2.1}$$

where $\alpha > 0,\ \varepsilon > 0$ is the Lavrentiev regularization parameter, z_d is the desired state, g is a non-homogeneous Dirichlet boundary condition and $a(\cdot),\ b(\cdot)$, with $a(x) \leq b(x)$, are the lower and upper constraint functions, respectively. The constant $\nu > 0$ stands for the viscosity coefficient of the fluid and $Re := 1/\nu$ for its Reynolds number.

It is well known that there exists a solution for the stationary two-dimensional Navier-Stokes system (cf. [21]). Moreover, if ν is sufficiently large or u sufficiently small, an appropriate estimate and uniqueness of the solution are obtained.

Next, we verify the existence of an optimal solution for our control problem. For that purpose let us define the set of admissible solutions

$$\mathcal{T}_{ad} = \{(y,u) \text{ which satisfy the restrictions in (2.1) }\}.$$

Theorem 1 *If $\mathcal{T}_{ad}$ is non-empty, then there exists an optimal solution for* (2.1).

Proof. We refer to [8, p. 3].

In the previous result the existence of a feasible solution was assumed. This hypothesis makes sense, since no pure control constraints are involved. In presence of control constraints the admissible set could possibly be empty.

3 First-order necessary optimality conditions

Once the existence of an optimal solution is verified, it is important to derive conditions that characterize any local solution of the optimization problem. To this aim a necessary condition involving first order derivatives is obtained. This condition takes the form of a system of partial differential equations (Navier-Stokes and adjoint equations) coupled with a nonlinear complementarity problem.

Let us consider the interior of the set of controls for which a unique associated Navier-Stokes solution exists and let us denote this set by U. Introducing the control-to-state operator $G : u \mapsto y(u)$ that assigns to each $u \in U$ the corresponding

Navier-Stokes solution $y(u)$, problem (2.1) can equivalently be expressed in reduced form as

$$\begin{cases} \min_{u\in U} \ J(u) = \frac{1}{2} \int\limits_{\Omega} |G(u) - z_d|^2 \, dx + \frac{\alpha}{2} \int\limits_{\Omega} |u|^2 \, dx \\ \text{subject to: } a \leq \varepsilon u + G(u) \leq b \text{ a.e. in } \Omega. \end{cases} \tag{$\mathcal{P}$}$$

Let us introduce the constant $\mathcal{M}(y) := \sup\limits_{v\in V} \frac{\left|\int_\Omega (v\cdot\nabla)y\cdot v \ dx\right|}{\int_\Omega |\nabla v|^2 \ dx}$, where V is the space of divergence free square integrable functions, with square integrable weak derivatives, that vanish at the boundary Γ. If $\nu > \mathcal{M}(y(u))$, it can be verified that the control to state operator G is twice Fréchet differentiable at u and its derivatives $w := G'(u)h$ and $z := G''(u)[h]^2$ are given by the unique solutions of the systems:

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)y + (y\cdot\nabla)w + \nabla\pi &= h \\ \text{div } w &= 0 \\ w|_\Gamma &= 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\nu\Delta z + (z\cdot\nabla)y + (y\cdot\nabla)z + \nabla\varrho &= -2(w\cdot\nabla)w \\ \text{div } z &= 0 \\ z|_\Gamma &= 0. \end{aligned} \tag{3.2}$$

The idea now consists in reformulating problem ($\mathcal{P}$) in a new variable $v := \varepsilon u + G(u)$ and treat it as a control-constrained optimal control problem. In order to express u as a function of v we consider the operator

$$F : (v, u) \mapsto \varepsilon u + G(u) - v$$

and the solvability of the equation $F(v, u) = 0$.

It can be verified (see [8]), that there are constants $r, r_0 > 0$ such that for each v with $\left(\int_\Omega |v - v^*|^2 \, dx\right)^{1/2} \leq r_0$, there exists a unique $u := K(v)$ with $\left(\int_\Omega |u - u^*|^2 \, dx\right)^{1/2} \leq r$ such that

$$\varepsilon K(v) + G(K(v)) = v. \tag{3.3}$$

Moreover, since F is twice continuously Fréchet differentiable, the implicit function theorem also implies that K is twice continuously Fréchet differentiable. Let us denote by $K''(v)[\xi, \eta]$ the second derivative of K in directions ξ and η and introduce $K''(v)[\xi]^2 := K''(v)[\xi, \xi]$. Taking the first and second derivatives on both sides of (3.3) in direction ξ yields

$$(\varepsilon + G'(K(v)))K'(v)\xi = \xi, \tag{3.4}$$

$$(\varepsilon + G'(K(v)))K''(v)[\xi]^2 = -G''(K(v))[K'(v)\xi]^2, \tag{3.5}$$

which implies that

$$K'(v) = (\varepsilon + G'(K(v)))^{-1}$$

and

$$K''(v)[\xi]^2 = -(\varepsilon + G'(K(v)))^{-1} G''(K(v))[K'(v)\xi]^2.$$

Locally around u^*, our control problem can therefore be formulated as:

$$\begin{cases} \min\ \mathcal{J}(v) =: J(y(K(v)), K(v)) \\ \text{subject to} \quad a \leq v \leq b \text{ a.e.} \\ \qquad\qquad\quad v \in B_{r_0}(v^*). \end{cases} \tag{$\mathcal{P}_r$}$$

Next an optimality system which characterizes the solutions to ($\mathcal{P}$) is stated. The proof is given in the Appendix.

Theorem 2 *Let u^* be a local optimal solution of ($\mathcal{P}$) with $\nu > \mathcal{M}(y(u^*))$. Then there exist adjoint variables $\lambda,\ q$ and Lagrange multipliers $\mu_a,\ \mu_b$ such that*

$$\begin{aligned} -\nu \Delta y^* + (y^* \cdot \nabla) y^* + \nabla p &= u^* \\ \textit{div}\ y^* &= 0 \\ y^*|_\Gamma &= g, \end{aligned} \tag{3.6}$$

$$\begin{aligned} -\nu \Delta \lambda - (y^* \cdot \nabla)\lambda + (\nabla y^*)^T \lambda + \nabla q = z_d - y^* + \mu_a - \mu_b \\ \textit{div}\ \lambda^* = 0 \\ \lambda^*|_\Gamma = 0, \end{aligned} \tag{3.7}$$

$$\lambda - \alpha u^* = \varepsilon(\mu_b - \mu_a), \tag{3.8}$$

$$\begin{aligned} & a \leq \varepsilon u + y^* \leq b, \\ & \mu_a,\ \mu_b \geq 0, \\ \int_\Omega \mu_{a_i}(a_i - \varepsilon u_i^* - y_i^*)\, dx = & \int_\Omega \mu_{b_i}(b_i - \varepsilon u_i^* - y_i^*)\, dx = 0,\ \textit{for}\ i = 1, 2. \end{aligned} \tag{3.9}$$

Optimality systems are important to understand the regularity of the control, state and adjoint variables and to apply a wide variety of numerical methods for the solution of the optimization problem. If no inequality constraints are present, the system can be solved as a system of partial differential equations. In general, however, it is constituted by the state equations, the adjoint equations and a nonlinear complementarity system and, in this case, additional methods for the solution of complementarity problems have to be considered.

4 Second order sufficient condition

Next, we turn to second order sufficient optimality conditions for problem ($\mathcal{P}$). This type of conditions allows the identification of a stationary point (a solution to optimality system (3.6)-(3.9)) as a minimum for the optimal control problem. Additionally, they are of importance in the convergence analysis of Newton type methods applied to the optimization problem.

Following [19], the idea consists in utilizing the second order sufficient optimality properties of the pure control constrained problem ($\mathcal{P}_r$) and translate them to the original setting. By introducing the Lagrangian

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \nu \int_\Omega \nabla\lambda : \nabla y \, dx + \int_\Omega (y\cdot\nabla)y\cdot\lambda \, dx - \int_\Omega \lambda\cdot u \, dx,$$

the equivalence of its second derivative with the one of the reduced functional $\mathcal{J}$ can be verified. The second derivative of the reduced cost functional in direction ξ therefore satisfies $\mathcal{J}''(v^*)[\xi]^2 = \mathcal{L}''(y^*,u^*,\lambda)(w,h)^2$, where $h = K'(v^*)\xi$ and w is the solution to (3.1) with h on the right hand side.

Let us now introduce the set of strongly active constraints $\mathcal{A}_{\tau,i} := \{x\in\Omega : |\mu_i(x)| \geq \tau\}$ and the critical cone

$$\tilde{C}_\tau = \left\{ v\in \mathbf{L}^2(\Omega) : \begin{array}{l} v_i(x) = 0 \text{ if } x\in\mathcal{A}_{\tau,i} \\ v_i(x)\geq 0 \text{ if } v_i^*(x) = a_i, x\notin\mathcal{A}_{\tau,i} \\ v_i(x)\leq 0 \text{ if } v_i^*(x) = b_i, x\notin\mathcal{A}_{\tau,i} \end{array} \right\}.$$

For the investigation of optimality for a given stationary pair (y^*,u^*) let us hereafter assume that for some $\delta>0$ the following second order condition holds: there exist $\tau>0$, $\delta>0$ such that

$$\mathcal{L}''(y^*,u^*,\lambda)(w,h)^2 \geq \delta \int_\Omega |h|^2 \, dx, \tag{SSC}$$

for all $(w,h)\in C_\tau$, where C_τ consists of all pairs (w,h) such that system (3.1) is satisfied and $\varepsilon h + w \in \tilde{C}_\tau$.

Theorem 3 *If u^* is a stationary point of ($\mathcal{P}$) and (SSC) holds for some $\delta>0$, $\tau>0$, then there exist constants $\rho>0$ and $\sigma>0$ such that*

$$J(y,u) \geq J(y^*,u^*) + \sigma\int_\Omega |u-u^*|^2 \, dx \tag{4.1}$$

for all (y,u) such that $y=G(u)$, $a\leq \varepsilon u + y\leq b$ and $\left(\int_\Omega |u-u^*|^2 dx\right)^{1/2}\leq\rho$.

Proof. See [8, p. 10]

Remark 4 *For the analysis of second order numerical methods, a stronger condition is needed: there exist constants $\tau>0,\ \delta>0$ such that*

$$\mathcal{L}''(y^*,u^*,\lambda)(w,h)^2 \geq \delta\int_\Omega |h|^2\, dx \tag{$\overline{SSC}$}$$

for all pairs (w,h) that solve (3.1) and satisfy $\varepsilon h_i + w_i = 0$ on $\mathcal{A}_{\tau,i}$, for $i=1,2$.

5 Semi-smooth Newton method

In this section we propose a semi-smooth Newton method for the numerical solution of ($\mathcal{P}$). These generalized Newton methods for nonsmooth equations are based on the notion of Newton differentiability, which, differently from othe differentiability concepts, allows to prove local superlinear convergence of the method (cf. [14]).

For the application of the method to the optimality system (3.6)-(3.9) we introduce the variable $x = (y, u, \lambda, q, \mu)$, with $\mu := \mu_b - \mu_a$. The system can then be reformulated as an operator equation $T(x) = 0$ and a semi-smooth Newton step is given by $G(x_k)\delta x = -T(x_k)$, where G is the Newton derivative of T.

In our case the difficulty is given by the complementarity system (3.9). Using the max and min functions, this problem can be reformulated as the operator equation

$$\mu = \max(0, \mu + c(v - b)) + \min(0, \mu + c(v - a)) \tag{5.1}$$

for all $c > 0$. The Newton differentiability of the max and min function then imply the Newton differentiability of the whole system. The derivative candidates

$$G_{max}(y)(x) = \begin{cases} 1 \text{ if } y(x) \geq 0 \\ 0 \text{ if } y(x) < 0; \end{cases} \qquad G_{min}(y)(x) = \begin{cases} 1 \text{ if } y(x) \leq 0 \\ 0 \text{ if } y(x) > 0, \end{cases} \tag{5.2}$$

constitute Newton derivatives of $max(0, y)$ and $min(0, y)$, respectively (cf. [14]).

By choosing $c := \alpha/\varepsilon^2$ in (5.1) and considering the derivatives (5.2), the complete algorithm can be formulated as an active set strategy through the following steps.

Algorithm 5

1. Initialize the variables $u_0,\ y_0,\ \mu_0 = 0$ and set $k = 1$.
2. Until a stopping criterion is satisfied, set for $i = 1, 2$

$$\begin{aligned} \mathcal{A}_{b_i}^n &= \{x : \mu_i^{n-1} + \frac{\alpha}{\varepsilon^2} \left(\varepsilon u_i^{n-1} + y_i^{n-1} - b_i\right) > 0\}, \\ \mathcal{A}_{a_i}^n &= \{x : \mu_i^{n-1} + \frac{\alpha}{\varepsilon^2} (\varepsilon u_i^{n-1} + y_i^{n-1} - a_i) < 0\}, \\ \mathcal{I}_i^n &= \Omega \backslash (\mathcal{A}_{b_i}^n \cup \mathcal{A}_{a_i}^n). \end{aligned}$$

and find the solution (y, p, λ, q) of:

$$\begin{aligned} -\nu \Delta y_i + y_1^{n-1} \partial_1 y_i + y_2^{n-1} \partial_2 y_i + y_1 \partial_1 y_i^{n-1} + y_2 \partial_2 y_i^{n-1} & \\ + \partial_i p = y_1^{n-1} \partial_1 y_i^{n-1} + y_2^{n-1} \partial_2 y_i^{n-1} &+ \begin{cases} \frac{1}{\varepsilon}(b_i - y_i) & \text{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i}{\alpha} & \text{on } \mathcal{I}_i^n \\ \frac{1}{\varepsilon}(a_i - y_i) & \text{on } \mathcal{A}_{a_i}^n \end{cases} \\ \operatorname{div}\ y_i &= 0 \\ y_i|_\Gamma &= g \end{aligned}$$

$$
\begin{aligned}
-\nu\Delta\lambda_i + \frac{1}{\varepsilon}\lambda_i - y_1\partial_1\lambda_i^{n-1} - y_2\partial_2\lambda_i^{n-1} - y_1^{n-1}\partial_1\lambda_i - y_2^{n-1}\partial_2\lambda_i + \lambda_1\partial_i y_1^{n-1} \\
+\lambda_2\partial_i y_2^{n-1} + \lambda_1^{n-1}\partial_i y_1 + \lambda_2^{n-1}\partial_i y_2 + \partial_i q = z_{d,i} - y_i - y_1^{n-1}\partial_1\lambda_i^{n-1} \\
-y_2^{n-1}\partial_2\lambda_i^{n-1} + \lambda_1^{n-1}\partial_i y_1^{n-1} + \lambda_2^{n-1}\partial_i y_2^{n-1} + \begin{cases} \frac{\alpha}{\varepsilon^2}(b_i - y_i) & \texttt{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i}{\varepsilon} & \texttt{on } \mathcal{I}_i^n \\ \frac{\alpha}{\varepsilon^2}(a_i - y_i) & \texttt{on } \mathcal{A}_{a_i}^n \end{cases} \\
\texttt{div } \lambda_i = 0 \\
\lambda_i|_\Gamma = 0.
\end{aligned}
$$

`Set` $(y^n, p^n, \lambda^n, q^n) = (y, p, \lambda, q)$, $u_i^n = \begin{cases} \frac{1}{\varepsilon}(b_i - y_i^n) & \texttt{on } \mathcal{A}_{b_i}^n \\ \frac{\lambda_i^n}{\alpha} & \texttt{on } \mathcal{I}_i^n \\ \frac{1}{\varepsilon}(a_i - y_i^n) & \texttt{on } \mathcal{A}_{a_i}^n. \end{cases}$, $\mu^n = \frac{1}{\varepsilon}(\lambda^n - \alpha u^n)$`, and goto step 2.`

Note that the system to be solved in step (2) corresponds to the optimality system of a quadratic control problem with affine constraints. Under the satisfaction of the second order condition ($\overline{SSC}$) and if $\left(\int_\Omega |\nabla(y^{n-1} - y^*)|^2\, dx\right)^{1/2}$ and $\left(\int_\Omega |\nabla(\lambda^{n-1} - \lambda^*)|^2\, dx\right)^{1/2}$ are sufficiently small, convexity of the optimization problem can be argued. Therefore, under these conditions, there exists a unique solution for the system in step (2). Sufficient conditions for local superlinear convergence of the semi-smooth Newton method applied to ($\mathcal{P}$) are investigated in [8].

6 Numerical results

For the numerical tests, a ”forward facing step channel” was utilized (see Figure 1). The fluid flows from left to right with parabolic inflow condition and ”do nothing” output condition. In the remaining boundary parts an homogeneous Dirichlet condition was imposed. The geometry was discretized using a staggered grid and an upwinding finite differences scheme was applied. The behavior of the uncontrolled fluid flow with Reynolds number $Re = 1000$ is depicted in Figure 2. Two main recirculation zones, which increase their size together with the Reynolds number, can be clearly identified from the graphic. These results can be verified experimentally (see [2, 20]).

The target of our control problem is to properly diminish the recirculations of interest by considering, together with the tracking type cost functional, adequate pointwise control-state constraints.

For the solution of the optimality system, Algorithm 5 was utilized. The semi-smooth Newton algorithm stops when the state increment norm is lower than 10^{-4}. Unless otherwise specified, the mesh step $h = 1/240$ was considered. For the solution of the linear systems, Matlab's exact solver was utilized.

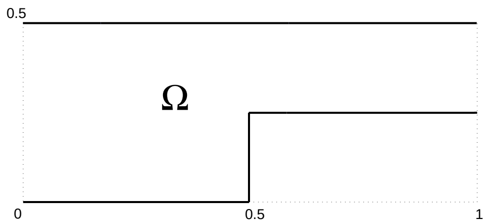

Figure 1 Forward facing step channel.

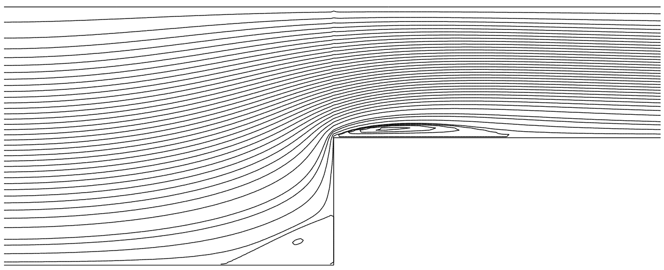

Figure 2 Streamlines of the uncontrolled state.

6.1 Example 1

In this first experiment we consider the elimination of bubbles in the channel by imposing the constraint $y_1 + \varepsilon u_1 \geq -10^{-7}$. For ε sufficiently small, this constraint avoids backward flow in the channel and thus possible recirculations. Additionally, the tracking type component of the cost functional is responsible for a more linear behavior of the flow field. The remaining parameter data utilized are $h = 1/240$, $Re = 1000, \varepsilon = 10^{-4}$ and $\alpha = 0.1$. The semi-smooth Newton method (SSN) stops after 9 iterations, with the final active set containing 28 grid points. The cost functional takes the final value $J(y^*, u^*) = 0.00445224$ and the NCP function residuum the value 2.2737×10^{-9}. The optimal control field is depicted in Figure 3, where the concentration of the control action on the recirculation zones can be observed. The desired recirculation diminishing effect of the control can be visualized from the plot of the reached controlled state streamlines in Figure 4. In Table 1 the number of SSN iterations, the final cost functional value and the size of the active set are registered for different ε values. It can be observed that as ε tends to 0, the problem becomes harder to solve and more SSN iterations are required.

Subsequently we consider the limit case where the tracking type part of the cost functional is dismissed. We aim to find the control of minimum norm that allows the satisfaction of the state constraint $y_1 + \varepsilon u_1 \geq 10^{-7}$ over the domain of interest. As before, the constraint takes care that no important backward flow arises. By considering the constraint on the whole domain, i.e. $\Omega_S = \Omega$, both recirculations before and after the step are diminished (see Figure 5). From Figure 5 it can also be

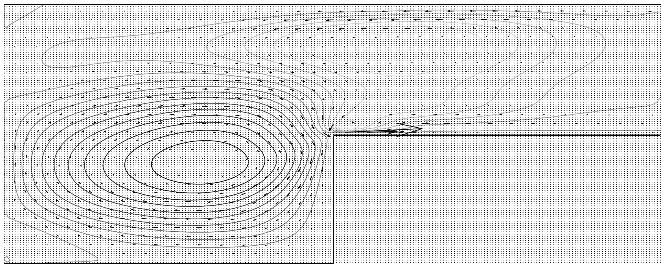

Figure 3 Example 1: control vector field with tracking component.

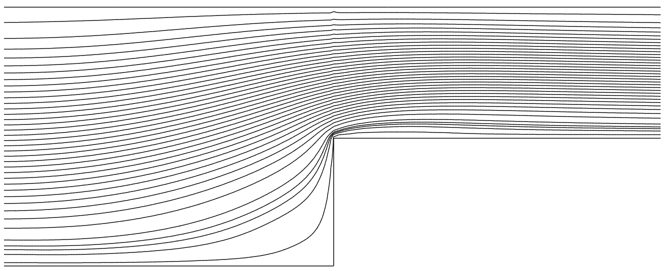

Figure 4 Example 1: streamlines of the controlled state with tracking component.

Table 1 Example 1: h=1/240, $tol = 10^{-4}$.

ε	SSN Iter.	$J(y^*, u^*)$	$\|\mathcal{A}^a \cup \mathcal{A}^b\|$
10^{-1}	5	0.00399972	33
10^{-2}	6	0.00410360	42
10^{-3}	8	0.00438273	29
10^{-4}	9	0.00445224	28
10^{-5}	9	0.00445989	32

observed that the behavior of the fluid flow, mainly before the step, is not as closer to a Stokes flow as in the case where the tracking type component is present (see Figure 4). From the control vector plot (see Figure 6) it can be observed that the control action in this case is even more concentrated on the recirculations zones. The parameter values for this case are $Re = 1000, \varepsilon = 10^{-4}$ and $\alpha = 0.1$. The number of SSN iterations needed is 29 and the cost functional takes the final value 8.99816×10^{-4}.

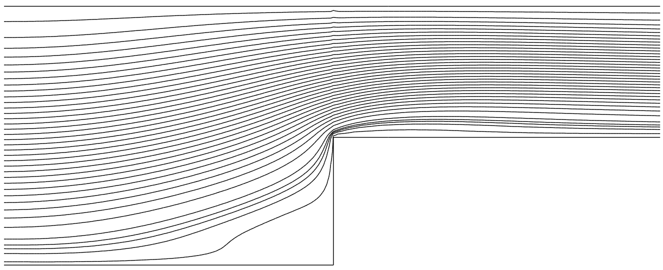

Figure 5 Example 1: streamlines of the controlled state without tracking component.

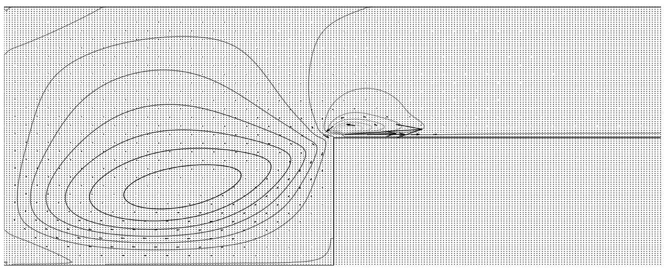

Figure 6 Example 1: control vector field without tracking component.

In many practical cases, the recirculations reduction or elimination on the whole domain is not necessary, if not undesirable. In such cases the state constraint may be imposed in the sectors where the bubble to be diminished is localized. In the case of our geometry the essential recirculation to be diminished is the one after the step. By considering the state constraint on the subdomain $\Omega_S := [0.5, 0.75] \times [0.25, 0.5]$, this elimination is attached with the cost functional value 8.98898×10^{-4} in 6 SSN iterations. The final controlled state is shown in Figure 7, where it can be observed that the recirculation after the step is numerically eliminated, although the one before the step becomes bigger than in the uncontrolled case.

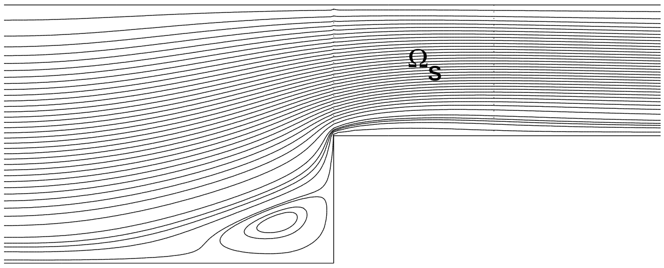

Figure 7 Example 1: streamlines of the controlled state without tracking component; state constraint subdomain.

6.2 Example 2

As an alternative strategy for the reduction of the recirculation after the step, we consider in this example a state constraint that guarantees an homogeneous outgoing velocity. The constraint imposed is $y_1 + \varepsilon u_1 \leq 1.7$ and the remaining parameter values are $Re = 1000$, $\varepsilon = 10^{-3}$ and $\alpha = 0.01$. In this case, the SSN algorithm stops after 15 iterations and the resulting active set contains 2283 grid points. The cost functional takes the final value $J(y^*, u^*) = 0.003470768$. The controlled state is depicted in Figure 8, where an important reduction of the recirculations can be visualized.

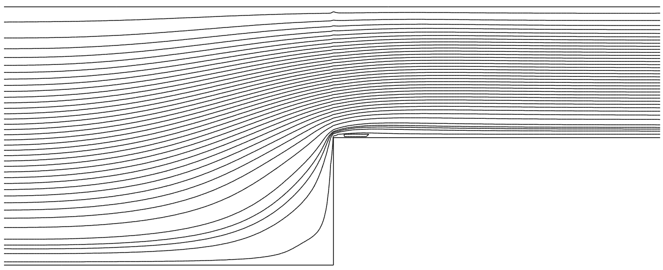

Figure 8 Example 2: streamlines of the controlled state

Since the outgoing velocity is the quantity of interest, it is natural to consider the case where the constraint is imposed only in the last part of the channel. By considering the domain $\Omega_S := [0.5, 0.75] \times [0.25, 0.5]$, the recirculation diminishing effect does also take place (see Figure 9), but with a lower final cost functional value $J(y^*, u^*) = 0.0031112131$. The SSN algorithm stops after 10 iterations with a final active set containing 906 active points. The remaining parameter values are the same as in the case $\Omega_S = \Omega$.

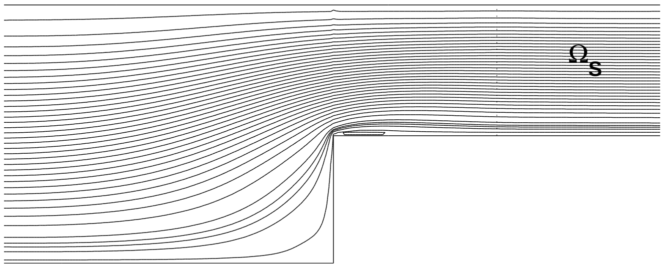

Figure 9 Example 2: streamlines of the controlled state; state constraint subdomain.

7 Conclusion

In this paper the optimal control problem of the Navier-Stokes equations with regularized pointwise state constraints of box type was considered. The problem was mathematically analyzed, yielding optimality conditions of first and second order. A semi-smooth Newton method for the solution of the problem was proposed. For the numerical realization a forward facing step channel was considered and a finite differences scheme was utilized for the partial differential equations involved.

The results show that the state constrained approach succeeded in reducing the recirculations of interest. This happened in the case where the constraint held all over the domain and also in the more realistic case, when it was restricted to a subdomain. Both limiting backward flow and imposing a more homogeneous outgoing velocity profile showed a positive effect with respect to recirculation reduction

Distributed controls are currently applied in magneto-hydro-dynamic problems, where the results obtained here can be used. It seems also possible to extend the analysis to the case of boundary optimal control problems with state constraints.

Appendix: Proof of Theorem 2

We consider the space of square integrable functions on Ω, denoted by $L^2(\Omega)$, and introduce the bold notation for the product of spaces. We denote by $(\cdot,\cdot)$ the inner product in $\mathbf{L}^2(\Omega)$ and by $\|\cdot\|$ the associated norm. Since u^* is a locally optimal solution of ($\mathcal{P}$), we get for some $r > 0$ that $J(y^*, u^*) \leq J(y(u), u)$, for all $u \in B_r(u^*)$ with $a \leq \varepsilon u + y(u) \leq b$. Equivalently, since $u = K(v)$ holds locally, $\mathcal{J}(v^*) \leq \mathcal{J}(v)$, for all $v \in B_{r_0}(v^*)$ with $a \leq v \leq b$, and for an appropriate constant $r_0 > 0$. Therefore, the following first order necessary condition follows

$$\mathcal{J}'(v^*)(v - v^*) \geq 0, \forall a \leq v \leq b \tag{7.1}$$

Applying the chain rule, the derivative of $\mathcal{J}(v^*)$ in direction $\xi \in \mathbf{L}^2(\Omega)$ is given by

$$(\mathcal{J}'(v^*), \xi) = (y^* - z_d, G'(u^*)K'(v^*)\xi) + \alpha(u^*, K'(v^*)\xi), \tag{7.2}$$

which, by $h := K'(v^*)\xi$, yields $(\mathcal{J}'(v^*), \xi) = (y^* - z_d, G'(u^*)h) + \alpha(u^*, h)$. Denoting by $\mu \in \mathbf{L}^2(\Omega)$ the Riesz representative of $-\mathcal{J}'(v^*)$ and using explicitly the derivative of K we obtain $(\mu, \xi) = (\mu, (\varepsilon + G'(u^*))h) = \varepsilon(\mu, h) + (\mu, G'(u^*)h)$. Therefore, equation (7.2) is equivalent to

$$(y^* - z_d + \mu, G'(u^*)h) + (\alpha u^* + \varepsilon\mu, h) = 0. \tag{7.3}$$

We now consider the adjoint equations (3.7). Since, by hypothesis $\nu > \mathcal{M}(y^*)$, the ellipticity of the adjoint operator can be easily verified and, therefore for $z_d - y^* - \mu \in \mathbf{L}^2(\Omega)$, there exists a unique solution $\lambda \in V$ for the adjoint system. Consequently, equation (7.3) can be rewritten as $\lambda - \alpha u^* = \varepsilon\mu$.

Utilizing the decomposition $\mu = \mu_b - \mu_a$, with $\mu_b := \mu_+ = \frac{1}{2}(\mu + |\mu|)$ and $\mu_a := \mu_- = \frac{1}{2}(-\mu + |\mu|)$, where $|\mu| = (|\mu_1|, |\mu_2|)^T$, the optimality condition (7.1) can be rewritten as $(\mathcal{J}'(v^*), v^*) = \min_{a \le v \le b}\{(\mu_{a,1}, v_1) - (\mu_{b,1}, v_1) + (\mu_{a,2}, v_2) - (\mu_{b,2}, v_2)\}$. By fixing the second component of the new control variable $v_2 = v_2^*$ and considering the mutual disjoint sets $\{x : \mu_{a,1}(x) > 0\}$ and $\{x : \mu_{b,1}(x) > 0\}$, we obtain that $(\mathcal{J}'(v^*), v^*) = (\mu_{a,1}, a_1) - (\mu_{b,1}, b_1) + (\mu_{a,2}, v_2^*) - (\mu_{b,2}, v_2^*)$ and, consequently, $(\mu_{a,1}, a_1 - \varepsilon u_1^* - y_1^*) - (\mu_{b,1}, b_1 - \varepsilon u_1^* - y_1^*) = 0$. Fixing now the first component of v and proceeding in a similar manner we get that $(\mu_{a,2}, a_2 - \varepsilon u_2^* - y_2^*) - (\mu_{b,2}, b_2 - \varepsilon u_2^* - y_2^*) = 0$. Taking into account that, by definition, $\mu_a,\ \mu_b \ge 0$ componentwise, the complementarity system (3.9) follows.

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