SECOND ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR NONLINEAR PARABOLIC CONTROL PROBLEMS WITH STATE CONSTRAINTS *

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Abstract. In this paper, optimal control problems for semilinear parabolic equations with distributed and boundary controls are considered. Pointwise constraints on the control and on the state are given. Main emphasis is laid on the discussion of second order sufficient optimality conditions. Sufficiency for local optimality is verified under different assumptions imposed on the dimension of the domain and on the smoothness of the given data.

Key words. Boundary control, distributed control, semilinear parabolic equations, sufficient optimality conditions, pointwise state constraints

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1. Introduction. This paper is a further contribution to the theory of second order sufficient optimality conditions for optimal control problems governed by nonlinear partial differential equations. We consider the control of semilinear parabolic equations with pointwise constraints on the control and the state. Recently, Casas, Tröltzsch and Unger [6] have discussed second order sufficient conditions for the boundary control of semilinear *elliptic* equations with pointwise state-constraints. It is convenient, to formulate this class of constraints in spaces of continuous functions, hence the associated Lagrange multipliers are Borel measures. The presence of measures in the adjoint equation causes a low regularity of the adjoint state. This fact is crucial in the analysis of second order sufficient optimality conditions. In particular, restrictions on the dimension of the domain had to be imposed in the elliptic case, if pointwise state constraints are given in the whole domain.

In the parabolic case, the situation is even more complicated. If pointwise state-constraints are formulated on the whole domain, then the sufficiency of second order conditions can be proved only for distributed controls appearing linearly in domains of dimension one. Therefore, we also investigate special types of controls, where the regularity of the control-state mapping is better. Moreover, other types of integral state-constraints are discussed. In this way, we are able to deal with problems in domains of higher dimension, although the basic difficulty of low regularity cannot be entirely solved.

The theory for parabolic equations differs from the elliptic case mainly in the regularity of the solutions, while many other aspects are identical. In view of this, we shall heavily rely on the results presented in [6]. Some proofs can be adopted word for word from associated theorems stated therein. Hence we will concentrate on specific features of parabolic problems rather than to repeat lengthy constructions being analogous to [6].

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Second order optimality conditions for control problems governed by semilinear elliptic and parabolic equations have received a good deal of attention in the past years. We refer to Goldberg and Tröltzsch, who deal with the one-dimensional parabolic case without state constraints in [9], and admit in [10] a particular type of state-constraints in higher dimensions. Moreover, we mention Casas, Tröltzsch, and Unger [5] and Bonnans [2], who investigate different aspects of the elliptic case subject to constraints on the control. We also refer to a recent paper by Bonnans and Zidani [3], where elliptic control problems with state-constraints are considered. The extension of sufficient conditions to state-constraints was discussed in [6], while [4] is concerned with the problem of second order *necessary* conditions for state-constrained elliptic problems.

Our paper is organized as follows: After formulating the control problem and stating assumptions in section 2, corresponding first order necessary optimality conditions are recalled, which are known from the literature. Next, the regularity of states and adjoint states is discussed in detail. Then we deal with the important problem of constraint qualifications in connection with certain linearizations of the problem. The main results on second order sufficient conditions are formulated in section 6. In the last part of the paper, we investigate different choices of functionals, state-constraints and dimensions, where we are able to verify the sufficiency of our second order optimality conditions.

2. The optimal control problem. We consider the problem **(P)**

$$\begin{aligned}\min J(y,v,u) = \int_{\overline{Q}} F(x,t,y(x,t))\,d\mu(x,t) + \int_Q f(x,t,y(x,t),v(x,t))\,dxdt \\ + \int_\Sigma g(x,t,y(x,t),u(x,t))\,dS(x)dt\end{aligned}$$

subject to

$$\begin{aligned}(y_t + Ay)(x,t) + d(x,t,y(x,t),v(x,t)) &= 0 \quad \text{in } Q\\ \partial_{\nu_A} y(x,t) + b(x,t,y(x,t),u(x,t)) &= 0 \quad \text{on } \Sigma\\ y(x,0) - y_o(x) &= 0 \quad \text{in } \Omega,\end{aligned} \tag{2.1}$$

$$v \in V_{ad}, \quad u \in U_{ad}, \tag{2.2}$$

$$E(y) \in K. \tag{2.3}$$

In this setting, $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary Γ, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$. The mapping $F : \overline{\Omega} \times [0,T] \times \mathbb{R} \to \mathbb{R}^m$ and the measure $\mu \in \mathcal{M}(\overline{Q};\mathbb{R}^m)$ express different types of observations.

Some cases of interest are specified below. The operator A is a second order elliptic operator, ∂_{ν_A} stands for the conormal derivative with respect to A, $d : Q \times \mathbb{R}^2 \to \mathbb{R}$ and $b : \Sigma \times \mathbb{R}^2 \to \mathbb{R}$ define the nonlinearities of the state-equation, and $y_o \in C(\overline{\Omega})$ is a given fixed initial state.

The control set V_{ad} (respectively U_{ad}) is supposed to be nonempty, convex, closed and bounded in $L^\infty(Q)$ (respectively $L^\infty(\Sigma)$). They will be specified below. E denotes a regular mapping from $C(\overline{Q})$ into a real Banach space Z, while K is a nonempty closed convex subset K in Z.

The following choice of F fits into this setting:

$$F = (F_i)_{1\le i\le m}, \quad \mu = (\mu_i)_{1\le i\le m}$$

where F_i are continuous function on $\overline{Q} \times \mathbb{R}$, $\mu_i = \delta_{(x_i,t_i)}$ for $1 \le i \le k_1$ (with $(x_i, t_i) \in \overline{Q}$), $\mu_i = \delta_{x_i} \otimes dt$ for $k_1 + 1 \le i \le m-1$ (with $x_i \in \bar{\Omega}$), $\mu_m = dx \otimes \delta_T$. This choice corresponds to

$$\int\limits_Q F(x,t,y(x,t))\, d\mu(x,t) = \sum_{i=1}^{k_1} F_i(x_i,t_i,y(x_i,t_i)) + \int\limits_0^T \sum_{k_1+1}^{m-1} F_i(x_i,t,y(x_i,t))\, dt$$
$$+ \int\limits_\Omega F_m(x,T,y(x,T))\, dx.$$

To formulate second order optimality conditions, with the control set V_{ad} (resp. U_{ad}), we associate a space $V \supset L^\infty(Q)$ (resp. $U \supset L^\infty(\Sigma)$) whose structure depends on V_{ad} (resp. U_{ad}). The control sets V_{ad} and U_{ad} are assumed to have one of the following forms.

(i) $V_{ad} = \{v \in L^\infty(Q) \,|\, v_a \le v \le v_b \text{ a.e. on } \mathcal{O}_v,\ \operatorname{supp} v \subset \overline{\mathcal{O}}_v\}$, $V \equiv L^2_o(\mathcal{O}_v)$,

$U_{ad} = \{u \in L^\infty(\Sigma) \,|\, u_a \le u \le u_b \text{ a.e. on } \mathcal{O}_u,\ \operatorname{supp} u \subset \overline{\mathcal{O}}_u\}$, $U \equiv L^2_o(\mathcal{O}_u)$,

where $\mathcal{O}_v$ is an open subset in Q, and $\mathcal{O}_u$ is an open subset in Σ, functions $v_a \le v_b$ are given in $L^\infty(Q)$, functions $u_a \le u_b$ are given in $L^\infty(\Sigma)$. The space $L^2_o(\mathcal{O}_v)$ (resp. $L^2_o(\mathcal{O}_u)$) is the subspace of functions in $L^2(Q)$ (resp. $L^2(\Sigma)$) with support in $\overline{\mathcal{O}}_v$ (resp. $\overline{\mathcal{O}}_u$), endowed with the norm $\|\cdot\|_{L^2(Q)}$ (resp. $\|\cdot\|_{L^2(\Sigma)}$).

An important particular case of (i) is given by

(ii) $V_{ad} = \{v \in L^\infty(Q) \,|\, v_a \le v \le v_b \text{ a.e. on } Q\}$ and $V \equiv L^2(Q)$,

$U_{ad} = \{u \in L^\infty(\Sigma) \,|\, u_a \le v \le u_b \text{ a.e. on } \Sigma\}$ and $U \equiv L^2(\Sigma)$.

Another meaningful control set is

(iii) $V_{ad} = \{v \in L^\infty(Q) \mid v(x,t) = \sum_i v^i(t)\, e_i(x),\ v^i_a \le v^i \le v^i_b \text{ a.e. on } (0,T)\}$, and $V \equiv L^2(0,T; L^\infty(\Omega))$. The functions $e_i \in L^\infty(\Omega)$, and $v^i_a \le v^i_b \in L^\infty(0,T)$, $i = 1,..,\ell_d$, are given. For instance, $\cup_{i=1}^{\ell_d} \overline{\Omega}_i \subset \overline{\Omega}$, $e_i = \chi_{\Omega_i}$, where χ_{Ω_i} is the characteristic function of Ω_i, is meaningful for certain practical applications. Analogous constructions work for U_{ad} and its associated space U.

The following state constraints fit in our setting:

(iv) $e(x,t,y(x,t)) \le 0$ on $\overline{Q}_o$ with $Z = C(\overline{Q}_o)$,

where $Q_o \subset Q$ (in particular, $Q_o = Q$ is possible in some cases).

(v) $e(x_i,t,y(x_i,t)) \le 0$, $i = 1,...,l$, $Z = C([0,T];\mathbb{R}^l)$,

(vi) $e(x_i,t_i,y(x_i,t_i)) \le 0$, $i = 1,...,l$, $Z = \mathbb{R}^l$,

(vii) $\int_\Omega e(x,t,y(x,t))dx \le 0$ on $[0,T]$, $Z = C([0,T])$,

(viii) $\int_Q e(x,t,y(x,t))dx\, dt \le 0$, $Z = \mathbb{R}$.

Combinations of these types are possible as well.

In all what follows, D denotes gradients with respect to (y,v) or (y,u), respectively. For instance, $Dd = (d_y, d_v)$, $Db = (b_y, b_u)$. Hessian matrices w.r. to (y,v) or (y,u) are denoted by D^2. For example, we write

$$D^2 d(x,t,y,v) = \begin{pmatrix} d_{yy}(x,t,y,v) & d_{yv}(x,t,y,v) \\ d_{vy}(x,t,y,v) & d_{vv}(x,t,y,v) \end{pmatrix}.$$

We need the following **General Assumptions.**

(A1) The boundary Γ is of class C^2. The elliptic operator A is defined by

$$Ay(x) = -\sum_{i,j=1}^{N} D_i\,(a_{ij}(x)\,D_j y(x)),$$

where the $a_{ij} \in C^{1,\nu}(\overline{\Omega})$ satisfy, for some positive m_o,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq m_o\,|\xi|^2.$$

(A2) **(i)** The function $d = d(x,t,y,v) : Q \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function. It is supposed to satisfy the following *assumptions of smoothness*:

– For almost all $(x,t) \in Q$, d is of class C^2 with respect to (y,v).

– For all $M > 0$ there is a constant $c_M > 0$ and a continuous monotone nondecreasing function $\eta : [0,\infty) \to \mathbb{R}^+$ with $\eta(0) = 0$ (not depending on M) such that

$$||D^2 d(\cdot, y_1, v_1) - D^2 d(\cdot, y_2, v_2)||_{L^\infty(Q;R^{2\times 2})} \leq c_M\,\eta(|y_1 - y_2| + |v_1 - v_2|)$$

for all $(y_i, v_i) \in \mathbb{R}^2$ with $|y_i| \leq M, |v_i| \leq M$, $i = 1,2$. Moreover,

$$||D^2 d(\cdot,0,0)||_{L^\infty(Q;R^{2\times 2})} + ||Dd(\cdot,0,0)||_{L^\infty(Q;R^2)} + ||d(\cdot,\cdot,0,0)||_{L^q(Q)} \leq c_B$$

holds with some constant c_B and some $q > N/2 + 1$.

(ii) The function $f = f(x,t,y,v)$ is supposed to satisfy the above conditions with the same constants c_M and c_B.

(iii) Analogous conditions are imposed on $b = b(x,t,y,u)$ and $g = g(x,t,y,u)$ on $\Sigma \times \mathbb{R}^2$ with the same constants, where $L^s(\Sigma)$ is substituted for $L^q(Q)$ with some $s > N + 1$.

Let us fix $q > N/2 + 1$ and $s > N + 1$ throughout this paper.

(iv) The mapping F and all entries of DF, D^2F are assumed to be continuous on $\overline{Q} \times \mathbb{R}$. Moreover, F is supposed to satisfy Lipschitz conditions analogous to those imposed on d.

(A3) (Monotonicity) Let c_{max} denote a common L^∞-bound for all controls in V_{ad} and U_{ad}. We assume the existence of a real constant c_o and of functions $d_1 \in L^q(Q)$, $b_1 \in L^s(\Sigma)$ such that

$$\begin{array}{ll} c_o \leq d_y(x,t,y,v) \leq c_M\, d_1(x,t) & \text{a.e. on } Q \\ c_o \leq b_y(x,t,y,u) \leq c_M\, b_1(x,t) & \text{a.e. on } \Sigma \end{array}$$

holds for all real v, u satisfying $max\{|v|,|u|\} \leq c_{max}$.

Before defining the mapping E, let us introduce the spaces

$$W(0,T) = \{y \in L^2(0,T;H^1(\Omega))\,|\,\frac{dy}{dt} \in L^2(0,T;(H^1(\Omega))')\},$$

(see Lions and Magenes [12]) and

$$Y = \{y \in W(0,T)\,|\,y_t + A\,y \in L^q(Q),\ \partial_{\nu_A} y \in L^s(\Sigma),\ y(0) \in C(\overline{\Omega})\}$$

endowed with the norm

$$\|y\|_Y = \|y\|_{W(0,T)} + \|y_t + Ay\|_{L^q(Q)} + \|\partial_{\nu_A} y\|_{L^s(\Sigma)} + \|y(0)\|_{C(\overline{\Omega})}.$$

Since $q > \frac{N}{2} + 1$ and $s > N + 1$, the embedding of Y into $C(\overline{Q})$ is continuous. We refer to [7] and [15].

For each pair of admissible controls the state system (2.1) admits a unique weak solution $y \in C(\overline{Q}) \cap L^2(0,T;H^1(\Omega))$. It belongs to Y [15]. To formulate the next general assumption, we need a special second order space and its corresponding norm:

$$Y_2 = \{y \in W(0,T) \,|\, y_t + A\,y \in V,\ \partial_{\nu_A} y \in U,\ y(0) = 0\}$$

$$\|y\|_{Y_2} = \|y_t + A\,y\|_V + \|\partial_{\nu_A} y\|_U.$$

(A5) The mapping $E : C(\overline{Q}) \to Z$ is of class C^2. For a fixed reference state $\bar{y}$ we assume the existence of a positive constant c_E such that

$$\begin{aligned}
&\|E'(\bar{y})\,y\|_Z \le c_E\,\|y\|_{Y_2}\\
&\|E''(\bar{y})[y,w]\|_Z \le c_E\,\|y\|_{Y_2}\,\|w\|_{Y_2}\\
&\|E'(y_1)\,y - E'(y_2)\,y\|_Z \le c_M\,\|y_1 - y_2\|_{Y_2}\,\|y\|_{Y_2}\\
&\|(E''(y_1) - E''(y_2))[y,w]\|_Z \le c_M\,\eta(\|y_1 - y_2\|_{L^\infty(Q)})\,\|y\|_{Y_2}\,\|w\|_{Y_2}
\end{aligned}$$

for all $y,\ w \in C(\overline{Q}) \cap Y_2$, and all $y_i \in C(\overline{Q}) \cap Y_2$ with $\|y_i\|_{C(\overline{Q})} \le M$, $i = 1,2$.

We should mention at this point that (A5) is a very hard restriction. This point is addressed in section 7.

3. First order necessary optimality conditions. Let us write our control problem as a problem of differentiable optimization in Banach spaces. To do so, we introduce the control-state mapping G: $(v,u) \mapsto y$ from $L^\infty(Q) \times L^\infty(\Sigma)$ to Y. Then problem (P) admits the form

$$\text{(3.1)} \qquad min\, J(G(u,v),u,v) \text{ subject to } (v,u) \in V_{ad} \times U_{ad}, \qquad E(G(v,u)) \in K.$$

Second order sufficient optimality conditions should be applicable to *locally optimal* solutions of the problem that are not necessarily globally optimal. Therefore, we do not discuss the existence of optimal controls by standard techniques, since this problem is concerned with the existence of global optima. We just assume once and for all that a fixed $(\bar{y},\bar{v},\bar{u}) \in Y \times V_{ad} \times U_{ad}$ is a local solution for (P).

The mapping G is Fréchet differentiable from $L^\infty(Q) \times L^\infty(\Sigma)$ into Y. Its derivative $y = G'(\bar{v},\bar{u})(v,u)$ is obtained by solving the *linearized equation*

$$\text{(3.2)} \qquad \begin{aligned}
y_t + Ay + d_y(\bar{y},\bar{v})\ y + d_v(\bar{y},\bar{v})\ v &= 0\\
\partial_{\nu_A} y + b_y(\bar{y},\bar{u})\ y + b_u(\bar{y},\bar{u})\ u &= 0\\
y(0) &= 0.
\end{aligned}$$

The *linearized cone* of $V_{ad} \times U_{ad}$ at $(\bar{v},\bar{u})$ is the set

$$\begin{aligned}
C(\bar{v},\bar{u}) = \{&(v,u) \in L^\infty(Q) \times L^\infty(\Sigma) \,|\, (v,u) = \rho\,(\tilde{v} - \bar{v}, \tilde{u} - \bar{u}),\\
&(\tilde{v},\tilde{u}) \in V_{ad} \times U_{ad},\ \rho \ge 0\} = \textstyle\bigcup_{\rho \ge 0} \rho\,(V_{ad} \times U_{ad} - \{(\bar{v},\bar{u})\}).
\end{aligned}$$

In the same way, the *conical hull* of $K - E(\bar y)$ is introduced by

$$K(E(\bar y)) = \bigcup_{\rho \ge 0, k \in K} \rho\,\{k - E(\bar y)\}.$$

In our abstract setting, the feasible set $\mathcal{M}$ of (P) has the form

$$\mathcal{M} = \{(y, v, u) \,|\, y = G(v, u), E(y) \in K, (v, u) \in V_{ad} \times U_{ad}\}.$$

Its *linearized cone* at $\bar w = (\bar y, \bar v, \bar u)$ is defined by

$$L(\mathcal{M}, \bar w) = \{(y, v, u) \,|\, (v, u) \in C(\bar v, \bar u), E'(\bar y)y \in K(E(\bar y)) \text{ where } y \text{ solves (3.2)}\}.$$

It is well known that a regularity condition is needed to derive first order necessary optimality conditions in a qualified form. We shall work with the following regularity condition (R), adopted from Zowe and Kurcyusz [19],

$$\textbf{(R)} \qquad E'(\bar y)G'(\bar v, \bar u)C(\bar v, \bar u) - K(E(\bar y)) = Z.$$

If $(\bar y, \bar v, \bar u)$ obeys the regularity condition (R), then there exists a *Lagrange multiplier* $\bar\lambda \in Z^*$ fulfilling the *complementary slackness condition*

$$\langle \kappa - E(\bar y)\,,\, \bar\lambda\rangle_{Z\times Z^*} \le 0 \quad \forall \kappa \in K \tag{3.3}$$

and the *variational inequalities*

$$\int_Q (f_v(\bar y, \bar v) - \bar p\, d_v(\bar y, \bar v))(v - \bar v)\, dxdt \ge 0 \quad \forall v \in V_{ad} \tag{3.4}$$

$$\int_\Sigma (g_u(\bar y, \bar u) - \bar p\, b_u(\bar y, \bar u))(u - \bar u)\, dS(x)dt \ge 0 \quad \forall u \in U_{ad}, \tag{3.5}$$

where the associated *adjoint state* $\bar p$ is the weak solution of

$$\begin{aligned} -\bar p_t + A^*\bar p + d_y(\bar y, \bar v)\bar p &= (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_Q + f_y(\bar y, \bar v) && \text{in } Q\\ \partial_{\nu_{A^*}}\bar p + b_y(\bar y, \bar u)\,\bar p &= (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_\Sigma + g_y(\bar y, \bar u) && \text{on } \Sigma\\ \bar p(T) &= (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_{\overline{\Omega}_T} && \text{in } \overline{\Omega}. \end{aligned} \tag{3.6}$$

The terms $(F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_Q$, $(F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_\Sigma$, and $(F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_{\overline{\Omega}_T}$ respectively denote the restriction of the measure $F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda$ to Q, Σ, and to $\overline{\Omega}_T = \overline{\Omega} \times \{T\}$. The measure $F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda$ is defined by $\langle z\,,\, F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda\rangle_{C(\overline{Q})\times\mathcal{M}(\overline{Q})} = \langle F_y(\bar y)z\,,\,\mu\rangle_{C(\overline{Q})\times\mathcal{M}(\overline{Q})} + \langle E'(\bar y)z\,,\,\bar\lambda\rangle_{Z\times Z^*}$ for every $z \in C(\overline{Q})$.

The adjoint state $\bar p$ belongs to $L^{\tilde\delta}(0, T; W^{1,\delta}(\Omega))$ for every $\delta > 1$, $\tilde\delta > 1$ satisfying $\frac{N}{2} + \frac{1}{2} < \frac{N}{2\delta} + \frac{1}{\tilde\delta}$, see Theorem 4.3. Therefore $\bar p$ belongs to $L^{q'}(Q)$ and $\bar p|_\Sigma$ belongs to $L^{s'}(\Sigma)$.

The adjoint equation (3.6) and the variational inequalities (3.4), (3.5) may be expressed by means of a *Lagrange function*

$$\begin{aligned} \mathcal{L}(y, v, u, \bar p, \bar\lambda) = J(y, v, u) &- \int_Q (y_t + Ay + d(y, v))\,\bar p\, dxdt\\ &- \int_\Sigma (\partial_{\nu_A} y + b(y, u))\,\bar p\, dSdt + \langle E(y)\,,\,\bar\lambda\rangle_{Z\times Z^*}. \end{aligned}$$

More precisely, the adjoint equation (3.6) is equivalent to $\mathcal{L}_y(\bar y, \bar v, \bar u, \bar p, \bar\lambda)\; y = 0$ for all $y \in Y$ satisfying $y(0) = 0$, and the variational inequalities (3.4), (3.5) are equivalent to

$$\mathcal{L}_{(v,u)}(\bar y, \bar v, \bar u, \bar p, \bar\lambda)\;((v, u) - (\bar v, \bar u)) \ge 0 \quad \text{for all } (v, u) \in V_{ad} \times U_{ad}. \tag{3.7}$$

4. Regularity results for the state and the adjoint equations. The continuity in $Y_2 \times V \times U$ of the quadratic form $\mathcal{L}''(\bar y, \bar v, \bar u, \bar p, \bar\lambda)[(y_1, v_1, u_1), (y_2, v_2, u_2)]$ plays a crucial role in the second order analysis (see section 6). This continuity property depends on regularity results for the adjoint state $\bar p$ and for solutions of the linearized state equation with source terms belonging to V and U. In order to deal with different choices of state-constraints, state-observation, and control sets (see section 7), here recall some regularity results for the adjoint equation and the linearized state equation. Consider the linear equation

$$\begin{aligned} y_t + Ay + \alpha\, y &= v \\ \partial_{\nu_A} y + \beta\, y &= u \\ y(0) &= 0, \end{aligned} \tag{4.1}$$

where α belongs to $L^\infty(Q)$ and β belongs to $L^\infty(\Sigma)$.

THEOREM 4.1. *Let y be the weak solution of 4.1.* (i) Distributed control. *Suppose that $v \in L^2(Q)$ and $u \equiv 0$. Then the mapping $v \mapsto y$ is continuous from $L^2(Q)$ to $L^{\tilde r}(0,T; L^r(\Omega))$ for every $\tilde r \geq 2$, $r \geq 2$ satisfying $\frac{N}{4} + \frac{1}{2} < \frac{N}{2r} + \frac{1}{\tilde r} + 1$. Moreover, the mapping $v \mapsto y|_\Sigma$ is continuous from $L^2(Q)$ to $L^{\tilde\sigma}(0,T; L^\sigma(\Gamma))$ for every $\tilde\sigma \geq 2$, $\sigma \geq 2$ satisfying $\frac{N}{4} + \frac{1}{2} < \frac{N-1}{2\sigma} + \frac{1}{\tilde\sigma} + 1$.*
If v belongs to $L^2(0,T; L^\infty(\Omega))$ and if $u \equiv 0$, then the mapping $v \mapsto y$ is continuous from $L^2(0,T; L^\infty(\Omega))$ to $C(\overline{Q})$.

(ii) Boundary control. *Suppose that $v \equiv 0$ and u belongs to $L^2(\Sigma)$. Then the mapping $u \mapsto y$ is continuous from $L^2(\Sigma)$ to $L^{\tilde r}(0,T; L^r(\Omega))$ for every $\tilde r \geq 2$, $r \geq 2$ satisfying $\frac{N-1}{4} < \frac{N}{2r} + \frac{1}{\tilde r}$. The mapping $u \mapsto y|_\Sigma$ is continuous from $L^2(\Sigma)$ to $L^{\tilde\sigma}(0,T; L^\sigma(\Gamma))$ for every $\tilde\sigma \geq 2$, $\sigma \geq 2$ satisfying $\frac{N-1}{4} < \frac{N-1}{2\sigma} + \frac{1}{\tilde\sigma}$.*
If $v \equiv 0$ and u belongs to $L^2(0,T; L^\infty(\Sigma))$, then the mapping $u \mapsto y$ is continuous from $L^2(0,T; L^\infty(\Sigma))$ to $L^{\tilde r}(0,T; L^\infty(\Omega))$ for every $\tilde r < \infty$. Moreover, the mapping $u \mapsto y|_\Sigma$ is continuous from $L^2(0,T; L^\infty(\Sigma))$ to $L^{\tilde\sigma}(0,T; L^\infty(\Gamma))$ for every $\tilde\sigma < \infty$.

Proof. The above regularity results for y may be obtained as in Propositions 3.1 and 3.2 of [17] (see also [11]). Using the same method as in [17], regularity results for $y|_\Sigma$ in $L^{\tilde\sigma}(0,T; L^\sigma(\Gamma))$ may be obtained with classical trace theorems, by proving regularity results in $L^{\tilde\sigma}(0,T; W^{k,\sigma}(\Gamma))$, for some $k > \frac{1}{\sigma}$. □

Next, we analyse the regularity of the adjoint state. The adjoint equation is of the form

$$\begin{aligned} -p_t + A^* p + \alpha p &= \bar\mu_Q && \text{in } Q \\ \partial_{\nu_{A^*}} p + \beta\, p &= \bar\mu_\Sigma && \text{on } \Sigma \\ p(T) &= \bar\mu_T && \text{in } \Omega, \end{aligned} \tag{4.2}$$

with $\alpha = d_y(\bar y, \bar v) \in L^\infty(Q)$, $\beta = b_y(\bar y, \bar u) \in L^\infty(\Sigma)$, $\bar\mu_Q = (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_Q + f_y(\bar y, \bar v)$, $\bar\mu_\Sigma = (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_\Sigma + g_y(\bar y, \bar u)$, $\bar\mu_T = (F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_{\Omega_T}$, $\bar\mu_Q \in \mathcal{M}_b(Q)$, $\bar\mu_\Sigma \in \mathcal{M}_b(\Sigma)$, $\bar\mu_T \in \mathcal{M}(\overline{\Omega} \times \{T\})$. The regularity of $\bar p$ clearly depends on the regularity of $\bar\mu_Q$, $\bar\mu_\Sigma$ and $\bar\mu_T$. In the cases we consider here, the term $f_y(\bar y, \bar v)$ is always more regular than $\bar\mu_Q$, and the term $g_y(\bar y, \bar u)$ is always more regular than $\bar\mu_\Sigma$. For the simplicity of the analysis, we suppose that either $[F_y(\bar y)^*\mu]_Q$ and $E'(\bar y)^*\bar\lambda|_Q$ belong to the same Lebesgue space or that both the terms are measures. The same simplifications are assumed for boundary and conditions.

THEOREM 4.2. (Integrable data) **(i)** *Let $\bar\mu_Q$ be in $L^{\tilde r}(0,T; L^r(\Omega))$, $\bar\mu_\Sigma \equiv 0$ and $\bar\mu_T \equiv 0$. Then the weak solution p of 4.2 belongs to $L^{\tilde\alpha}(0,T; L^\alpha(\Omega))$ for every $\tilde\alpha \geq \tilde r$,*

$\alpha \geq r$ satisfying $\frac{N}{2r}+\frac{1}{\tilde r} < \frac{N}{2\alpha}+\frac{1}{\tilde\alpha}+1$. Moreover, the trace $p|_\Sigma$ belongs to $L^{\tilde\beta}(0,T;L^\beta(\Gamma))$ for every $\tilde\beta \geq \tilde r$, $\beta \geq r$ satisfying $\frac{N}{2r}+\frac{1}{\tilde r} < \frac{N-1}{2\beta}+\frac{1}{\tilde\beta}+1$.

(ii) *If $\bar\mu_Q \equiv 0$, $\bar\mu_\Sigma$ belongs to $L^{\tilde\sigma}(0,T;L^\sigma(\Gamma))$ and $\bar\mu_T \equiv 0$, then the weak solution p of 4.2 is in $L^{\tilde\alpha}(0,T;L^\alpha(\Omega))$ for every $\tilde\alpha \geq \tilde\sigma$, $\alpha \geq \sigma$ satisfying $\frac{N-1}{2\sigma}+\frac{1}{\tilde\sigma}+\frac{1}{2} < \frac{N}{2\alpha}+\frac{1}{\tilde\alpha}+1$. Moreover, its trace $p|_\Sigma$ is in $L^{\tilde\beta}(0,T;L^\beta(\Gamma))$ for every $\tilde\beta \geq \tilde\sigma$, $\beta \geq \sigma$ satisfying $\frac{N-1}{2\sigma}+\frac{1}{\tilde\sigma} < \frac{N-1}{2\beta}+\frac{1}{\tilde\beta}+\frac{1}{2}$.*

(iii) *Suppose that $\bar\mu_Q \equiv 0$, $\bar\mu_\Sigma \equiv 0$ and $\bar\mu_T \in L^r(\Omega)$. Then p belongs to $L^{\tilde\alpha}(0,T;L^\alpha(\Omega))$ for every $\tilde\alpha < \infty$, $\alpha \geq r$ satisfying $\frac{N}{2r} < \frac{N}{2\alpha}+\frac{1}{\tilde\alpha}$. Moreover, the trace $p|_\Sigma$ is contained in $L^{\tilde\beta}(0,T;L^\beta(\Gamma))$ for every $\tilde\beta \geq \tilde r$, $\beta \geq r$ satisfying $\frac{N}{2r} < \frac{N-1}{2\beta}+\frac{1}{\tilde\beta}$.*

Proof. The proof is similar to the one of Theorem 4.1. □

THEOREM 4.3. (Measures as data) *Let $\bar\mu_Q+\bar\mu_\Sigma+\bar\mu_T$ be in $\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}\times\{0\})$. Then the weak solution p of 4.2 is contained in $L^{\tilde\alpha}(0,T;L^\alpha(\Omega))$ for every $\tilde\alpha \geq 1$, $\alpha \geq 1$ satisfying $\frac{N}{2} < \frac{N}{2\alpha}+\frac{1}{\tilde\alpha}$. Moreover, the trace $p|_\Sigma$ belongs to $L^{\tilde\beta}(0,T;L^\beta(\Gamma))$ for every $\tilde\beta \geq 1$, $\beta \geq 1$ satisfying $\frac{N}{2} < \frac{N-1}{2\beta}+\frac{1}{\tilde\beta}$.*

Proof. The first part of the theorem is stated in [14]. The regularity result for the trace may be obtained by combining the techniques in [14] and [17]. □

5. Regularity condition and linearization. Since our control problem is written as a problem of differentiable optimization in Banach spaces, we can take advantage of the results stated in [6]. Due to Theorem 4.1, the operator $G'(\bar v,\bar u)$ is continuous from $L^q(Q)\times L^s(\Sigma)$ to Y, and from $V\times U$ to Y_2, that is

$$||G'(\bar v,\bar u)(v,u)||_{Y_2} \leq c\,(||v||_V + ||u||_U).$$

As before, we regard our fixed reference triplet $\bar w = (\bar y,\bar v,\bar u)$ satisfying together with $(\bar p,\bar\lambda)$ the first order necessary optimality system and the regularity condition (R). Further, we define the norms

$$\begin{aligned} ||(v,u)||_{L^\infty} &= ||v||_{L^\infty(Q)} + ||u||_{L^\infty(\Sigma)},\\ ||(v,u)||_{L^2} &= ||v||_V + ||u||_U. \end{aligned}$$

The next result is completely analogous to Theorem 4.2 in [6].

THEOREM 5.1. *If the regularity condition (R) is satisfied, then for all triplets $(\hat y,\hat v,\hat u) \in \mathcal{M}$ there is a triplet $(y,v,u) \in L(\mathcal{M},\bar w)$ such that the difference $r = (r^y,r^v,r^u) = (\hat y,\hat v,\hat u)-(\bar y,\bar v,\bar u)-(y,v,u)$ fulfils the estimates*

$$||r||_{Y\times L^\infty(Q)\times L^\infty(\Sigma)} \leq C_L\,||(\hat v,\hat u)-(\bar v,\bar u)||_{L^\infty}\;(||\hat v-\bar v||_{L^q(Q)}+||\hat u-\bar u||_{L^s(\Sigma)}) \tag{5.1}$$

and

$$||r^y||_2 + ||(r^v,r^u)||_{L^2} \leq C_L\,||(\hat v,\hat u)-(\bar v,\bar u)||_{L^\infty}\;||(\hat v,\hat u)-(\bar v,\bar u)||_{L^2}. \tag{5.2}$$

Proof: The proof is almost identical with that of Theorem 4.2 in [6], which was performed for an elliptic optimal control problem in the abstract form (3.1). The only difference to our setting appears in the concrete meaning of the mapping G. In [6], $G: u \mapsto y$ is the solution operator associated to the elliptic problem

$$-\Delta y + y = 0, \qquad \partial_{\nu_A} y = b(y,u).$$

It is continuous from $L^\infty(\Gamma)$ to $H^1(\Omega)\cap C(\overline{\Omega})$. Moreover, $G'(\bar u)$ is continuous from $L^p(\Gamma)$ to $H^1(\Omega)\cap C(\overline{\Omega})$ for $p > N-1$. Comparing this setting with our problem, we have the following relations: $u \longleftrightarrow (v,u)$, $p > N-1 \longleftrightarrow q > N/2+1 \wedge r > N+1$, $L^\infty(\Gamma) \longleftrightarrow L^\infty(Q)\times L^\infty(\Sigma)$, $L^p(\Gamma) \longleftrightarrow L^q(Q)\times L^s(\Sigma)$, $H^1(\Omega)\cap C(\overline{\Omega}) \longleftrightarrow W(0,T)\times C(\overline{Q})$. By this equivalence, due to regularity results for parabolic equations (see e.g. [7], [15]) the proof in [6] can be transferred to obtain the statement of our theorem. □

6. Second order conditions.

6.1. Space and time dependent controls. In this section, we discuss the second order sufficient conditions for the choices (i) and (ii) for V_{ad} and U_{ad}, defined section 2. The simplest, and at the same time strongest, second order assumption is the *coercivity condition*

$$\mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y,v,u),(y,v,u)] \ge \delta\,(\|y\|_2^2 + \|v\|_V^2 + \|u\|_U^2) \tag{6.1}$$

required for all $y\in Y_2$, $v\in L^\infty(Q)$, $u\in L^\infty(\Sigma)$, where δ is a certain positive constant. Here and below $\mathcal{L}''$ stands for the second order Fréchet derivative of $\mathcal{L}$ with respect to (y,v,u), that is, $\mathcal{L}'' = \mathcal{L}''_{(y,v,u)}$. However, we shall omit the subscript (y,v,u) for convenience. To write $\mathcal{L}''$ in a compact form, let us introduce the "Hamiltonians"

$$\begin{aligned} H^Q(x,t,y,v,p) &= f(x,t,y,v) - p\,d(x,t,y,v)\\ H^\Sigma(x,t,y,u,p) &= g(x,t,y,u) - p\,b(x,t,y,u)\end{aligned}$$

having the following second order derivatives with respect to (y,v) and (y,u), respectively,

$$D^2H^Q = \begin{pmatrix} H^Q_{yy} & H^Q_{yv}\\ H^Q_{yv} & H^Q_{vv}\end{pmatrix},\qquad D^2H^\Sigma = \begin{pmatrix} H^\Sigma_{yy} & H^\Sigma_{yu}\\ H^\Sigma_{yu} & H^\Sigma_{uu}\end{pmatrix}.$$

These Hessian matrices depend on (x,t,y,v,u,p). Then we have

$$\begin{aligned}\mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y,v,u),(y,v,u)] &= \int_Q (y,v)\,D^2\bar H^Q\,(y,v)^\top\,dxdt +\\ +\int_\Sigma (y,u)\,D^2\bar H^\Sigma\,(y,u)^\top\,dSdt &+ \int_{\overline{Q}} \bar F_{yy}\,y^2 d\mu + \langle E''(\bar y)[y,y],\bar\lambda\rangle_{Z\times Z^*},\end{aligned}$$

where the bar in $\bar H^Q$, $\bar H^Q$, $\bar\phi_{yy}$, $\bar F_{yy}$ indicates that these derivatives are taken at the reference point $(\bar y,\bar v,\bar u,\bar p,\bar\lambda)$.

To tighten the gap between necessary and sufficient second order conditions, we shall shrink the subspace of $Y\times L^\infty(Q)\times L^\infty(\Sigma)$, where the coercivity property (6.1) is assumed. A first and most natural step is to assume (6.1) only on the set $L(\mathcal{M},(\bar y,\bar v,\bar u))$, that is on the linearized cone. Then the function y is connected with (v,u) through the linearized equation (5.1). It holds $\|y\|_{Y_2}\le c\,(\|v\|_V + \|u\|_U)$, and therefore (6.1) is equivalent to

$$\mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y,v,u),(y,v,u)] \ge \delta\,(\|v\|_V^2 + \|u\|_U^2), \tag{6.2}$$

for all $(y,v,u)\in L(\mathcal{M},(\bar y,\bar v,\bar u))$. Here, $\delta>0$ is possibly different from that in (6.1). Although being much weaker, this condition is still too far from the associated necessary conditions, since the coercivity of $\mathcal{L}''$ has to hold also for all active constraints, independently on how "positive" the associated Lagrange multiplier is. Therefore, following an idea by Dontchev, Hager, Poore and Yang [8], we consider also *strongly active control constraints* . The control $\bar v(x,t)$ achieves its lower or upper bound in the points (x,t), where

$$|f_v(\bar y,\bar v)(x,t) - \bar p\, d_v(\bar y,\bar v)(x,t)| = |\bar H_v^Q(x,t)| > 0.$$

To make this property stable with respect to perturbations of the reference point in L^∞, for arbitrarily small $\tau > 0$ we introduce the sets of strongly active control constraints by

$$I_\tau^Q = \{(x,t) \in Q \,|\, |\bar H_v^Q(x,t)| \ge \tau\}, \qquad I_\tau^\Sigma = \{(x,t) \in \Sigma \,|\, |\bar H_u^\Sigma(x,t)| \ge \tau\}.$$

Roughly speaking, the coercivity condition (6.2) has to be assumed only for those $(y,v,u) \in L(\mathcal{M},(\bar y,\bar v,\bar u))$ having the additional property $v(x,t) = 0$ on I_τ^Q and $u(x,t) = 0$ on I_τ^Σ. In view of the complicated structure of $L(\mathcal{M},(\bar y,\bar v,\bar u))$ we shall formulate this more precisely below.

Let us first mention that the idea to weaken second order sufficient conditions by strongly active control constraints can be extended to the state constraints as well. This can be done by considering *first order sufficient optimality conditions* introduced by Maurer and Zowe [13]. We refer to Casas, Tröltzsch, and Unger [6] for the elliptic case with state-constraints. Their approach can be directly transferred to our parabolic case. However, it was pointed out in [6] that this further weakening of the second order conditions is only of limited value. Therefore, we concentrate here only on strongly active control constraints.

While the regularity condition (R) is very useful to show the existence of Lagrange multipliers, we need the following stronger constraint qualification to work with a second order condition, which is closer to conditions known from the optimization theory in $\mathbb{R}^n$. Define

$$C_\tau(\bar v,\bar u) = \{(v,u) \in C(\bar v,\bar u) \,|\, v = 0 \text{ a.e. on } I_\tau^Q \text{ and } u = 0 \text{ a.e. on } I_\tau^\Sigma\}.$$

The stronger regularity condition is

$\mathbf{(R)}_\tau$ $\quad E'(\bar y)G'(\bar v,\bar u)C_\tau(\bar v,\bar u) - K(E(\bar y)) = Z.$

Now let us establish the *second order sufficient optimality condition* as follows:

$\mathbf{(SSC)}_\tau$ *There are positive constants δ and τ such that the coercivity condition (6.1) is satisfied for all $(y,v,u) \in L(\mathcal{M},(\bar y,\bar v,\bar u))$ satisfying $(v,u) \in C_\tau(\bar v,\bar u)$.*

To verify that $\mathbf{(SSC)}_\tau$ implies local optimality, we have to approximate differences of the form $(\hat y,\hat v,\hat u) - (\bar y,\bar v,\bar u)$ by associated elements (y,v,u) of $L(\mathcal{M},(\bar y,\bar v,\bar u))$ with controls (v,u) belonging to $C_\tau(\bar v,\bar u)$. First of all, we need the regularity condition $(\mathrm{R})_\tau$ for this purpose. Moreover, and this is crucial in the whole analysis, the quadratic form $\mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y_1,v_1,u_1),(y_2,v_2,u_2)]$ has to depend continuously on (y_i,v_i,u_i) in the L^2-norm (we need this to estimate $\mathcal{L}''$ for remainder terms). Therefore, we must assume the continuity estimate

$$\mathbf{(A6)} \qquad \mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y_1,v_1,u_1),(y_2,v_2,u_2)] \le c_{\mathcal{L}} \prod_{i=1}^2 (\|y_i\|_{Y_2} + \|(v_i,u_i)\|_{L^2})$$

for all (y_i,u_i,v_i) of $Y \cap Y_2 \times L^\infty(Q) \times L^\infty(\Sigma)$. Unfortunately, (A6) is a hard restriction.

To see this, regard the concrete expression for $\mathcal{L}''$

$$
\begin{aligned}
&\mathcal{L}''(\bar y,\bar v,\bar u,\bar p,\bar\lambda)[(y_1,v_1,u_1),(y_2,v_2,u_2)] = \int\limits_Q F_{yy}(\cdot,\bar y)y_1y_2\,d\mu - \langle E''(\bar y)[y_1,y_2]\,,\,\bar\lambda\rangle\\
&\quad+\int\limits_Q (y_1,v_1)D^2f(\cdot,\bar y,\bar v)(y_2,v_2)^\top\,dxdt + \int\limits_\Sigma (y_1,u_1)D^2g(\cdot,\bar y,\bar u)(y_2,u_2)^\top\,dSdt\\
&\quad-\int\limits_Q (y_1,v_1)D^2d(\cdot,\bar y,\bar v)(y_2,v_2)^\top\bar p\,dxdt\\
&\quad-\int\limits_\Sigma (y_1,u_1)D^2b(\cdot,\bar y,\bar u)(y_2,u_2)^\top\bar p\,dSdt.
\end{aligned}\tag{6.3}
$$

The difficulties to estimate this expression arise from the presence of state-constraints. In general we cannot assume that $\bar p$ is a bounded function. Therefore, the last two terms in (6.3) require additional assumptions, while the term containing E'' is handled by (A5). We shall discuss these points in section 7.

Moreover, this assumption allows us to estimate the second order remainder term of $\mathcal{L}$. The remainder term $r_2^{\mathcal{L}}$ is defined by the second order Taylor expansion

$$\mathcal{L}(\hat w,\bar\lambda,\bar p)-\mathcal{L}(\bar w,\bar\lambda,\bar p) = \mathcal{L}'(\bar w,\bar\lambda,\bar p)(\hat w-\bar w)+\frac{1}{2}\mathcal{L}''(\bar w,\bar\lambda,\bar p)[\hat w-\bar w,\hat w-\bar w]+r_2^{\mathcal{L}}(\hat w,\hat w-\bar w).$$

Assumption (A6) applies to derive the estimate

$$
\begin{aligned}
&|r_2^{\mathcal{L}}(\hat w,\hat w-\bar w)| \le\\
&c_{\mathcal{L}}\,\eta(||\hat y-\bar y||_{C(\overline{Q})}+||(\hat v,\hat u)-(\bar v,\bar u)||_{L^\infty})\,(||\hat y-\bar y||^2_{Y_2}+||(\hat v,\hat u)-(\bar v,\bar u)||^2_{L^2}).
\end{aligned}\tag{6.4}
$$

Theorem 6.1. *Let the feasible triplet $\bar w=(\bar y,\bar v,\bar u)$ satisfy together the regularity condition* $(\mathbf{R})_\tau$*, the first order necessary optimality conditions (3.3)–(3.6), and the second order sufficient optimality condition* $(\mathbf{SSC})_\tau$*. Suppose further that the general assumptions* **(A1)–(A6)** *are satisfied. Then there are constants $\varrho>0$ and $\sigma>0$ such that*

$$J(\hat y,\hat v,\hat u)\ge J(\bar y,\bar v,\bar u)+\sigma||(\hat v,\hat u)-(\bar v,\bar u)||^2_{L^2}\tag{6.5}$$

holds for all feasible $\hat w=(\hat y,\hat v,\hat u)$ such that

$$||(\hat v,\hat u)-(\bar v,\bar u)||_{L^\infty}<\varrho.\tag{6.6}$$

Proof. Let an arbitrary feasible triplet $\hat w=(\hat y,\hat v,\hat u)$ be given. We introduce for short the Lagrange multiplier $\bar l=(\bar p,\bar\lambda)$ appearing in the first order necessary optimality conditions. Then

$$J(\hat w)-J(\bar w)=\mathcal{L}(\hat w,\bar l)-\mathcal{L}(\bar w,\bar l)-\langle\bar\lambda,E(\hat y)-E(\bar y)\rangle\tag{6.7}$$

follows from the state equation. The complementary slackness condition implies

$$-\langle\bar\lambda,E(\hat y)-E(\bar y))\rangle\ge 0.$$

After deleating this term, a second order Taylor expansion yields

$$
\begin{aligned}
J(\hat w)-J(\bar w)&\ge\mathcal{L}(\hat w,\bar l)-\mathcal{L}(\bar w,\bar l)\\
&=\int\limits_Q \bar H_v^Q\,(\hat v-\bar v)+\int\limits_\Sigma \bar H_u^\Sigma\,(\hat u-\bar u)+\frac{1}{2}\mathcal{L}''(\bar w,\bar l)[\hat w-\bar w]^2+r_2^{\mathcal{L}}(\bar w,\hat w-\bar w)
\end{aligned}
$$

with $\bar H_v^Q = f_v(\cdot,\bar y,\bar v) - \bar p\, d_v(\cdot,\bar y,\bar v)$, $\bar H_u^\Sigma = g_u(\cdot,\bar y,\bar u) - \bar p\, b_u(\cdot,\bar y,\bar u)$. Here and below we shall omit for short the differentials in integrals. Using the variational inequalities (3.4), (3.5), and the definition of I_τ^Q and I_τ^Σ, we find

$$(6.8)\quad J(\hat w) - J(\bar w) \ge \tau\Big(\int\limits_{I_\tau^Q} |\hat v - \bar v| + \int\limits_{I_\tau^\Sigma} |\hat u - \bar u|\Big) + \frac{1}{2}\mathcal{L}''(\bar w,\bar l)[\hat w - \bar w]^2 + r_2^{\mathcal{L}}(\bar w, \hat w - \bar w).$$

In the following, c will denote a generic constant. Let us introduce for convenience the bilinear form $B = \mathcal{L}''(\bar w,\bar l)$. Next we approximate $\hat w - \bar w$ by $w = (y,v,u) \in L(\mathcal{M},\bar w)$, according to Theorem 5.1. In this way we get the remainder $r = (r^y, r^v, r^u)$ satisfying $\hat w - \bar w = w + r$ and the estimate

$$(6.9)\qquad \|r\| \le C_L \|(\hat v,\hat u) - (\bar v,\bar u)\|_{L^\infty} \|(\hat v,\hat u) - (\bar v,\bar u)\|_{L^2},$$

where

$$\|r\| := \|r^y\|_{Y_2} + \|(r^v, r^u)\|_{L^2}.$$

It follows that $B[\hat w - \bar w]^2 = B[w]^2 + 2B[r,w] + B[r]^2$. We are looking for an estimate from below for $B[\hat w - \bar w]^2$. For this, we separately estimate the three terms.

Step 1. Estimate of $B[w]^2$. To use $\mathbf{(SSC)}_\tau$ we set $(v,u) = (v_1,u_1) + (v_2,u_2)$, where $v_1 = \chi_{I_\tau^Q} v$, $u_1 = \chi_{I_\tau^\Sigma} u$. This yields $v_2 = 0$ on I_τ^Q and $u_2 = 0$ on I_τ^Σ. Let y_i, for $i = 1,2$, be the solutions to the linearized equation (3.2) associated with (v_i,u_i). Observe that (v_2,u_2) belongs to $C_\tau(\bar v,\bar u)$ and

$$E'(\bar y)G'(\bar v,\bar u)(v_2,u_2) \in K(E(\bar y)) - E'(\bar y)G'(\bar v,\bar u)(v_1,u_1).$$

Since in general $E'(\bar y)G'(\bar v,\bar u)(v_1,u_1)$ is non zero, the triplet $w_2 = (y_2,v_2,u_2)$ does not belong to the linearized cone $L(\mathcal{M},\bar w)$. The regularity condition $(\mathrm{R})_\tau$ permits to apply a theorem by Robinson (see Theorem 1 in [18]). Thus, there exist an element (v_H,u_H) in $C_\tau(\bar v,\bar u)$, and a constant c_H such that

$$(6.10)\qquad E'(\bar y)G'(\bar v,\bar u)(v_H,u_H) \in K(E(\bar y))$$

and

$$(6.11)\qquad \|(v_2,u_2) - (v_H,u_H)\|_{L^\infty} \le c_H \|e\|_Z.$$

From assumption (A5) and continuous imbeddings, it follows that

$$(6.12)\qquad \|(v_2,u_2) - (v_H,u_H)\|_{L^2} \le c\|e\|_Z \le c\,\|(v_1,u_1)\|_{L^2}.$$

Define $y_H = G'(\bar v,\bar u)(v_H,u_H)$. By (6.10), $w_H = (y_H,v_H,u_H)$ belongs to $L(\mathcal{M},\bar w)$). With $\mathbf{(SSC)}_\tau$ we obtain $B[w_H]^2 \ge \delta\|(v_H,u_H)\|_{L^2}^2$. If we set $w_I = (y_I,v_I,u_I) = w - w_H$, then, with (6.12), we have

$$\begin{aligned}\|y_I\|_{Y_2} &= \|y_1 + y_2 - y_H\|_{Y_2} \le \|y_1\|_{Y_2} + \|y_2 - y_H\|_{Y_2}\\ &\le c(\|(v_1,u_1)\|_{L^2} + c\,\|(v_2,u_2) - (v_H,u_H)\|_{L^2}) \le c\,\|(v_1,u_1)\|_{L^2}\end{aligned}$$

and

$$(6.13)\quad \|(v_I,u_I)\|_{L^2} \le \|(v_1,u_1)\|_{L^2} + \|(v_2,u_2) - (v_H,u_H)\|_{L^2} \le c\,\|(v_1,u_1)\|_{L^2},$$

$$(6.14)\qquad \|w_I\| = \|y_I\|_{Y_2} + \|(v_I,u_I)\|_{L^2} \le c\,\|(v_1,u_1)\|_{L^2}.$$

With the above estimates and with (A6) we deduce

$$
\begin{aligned}
B[w]^2 &= B[w_H]^2 + 2B[w_H, w_I] + B[w_I]^2 \\
&\geq \delta \|(v_H, u_H)\|_{L^2}^2 - 2c_{\mathcal{L}} \|w_H\| \, \|w_I\| - c_{\mathcal{L}} \, \|w_I\|^2 \\
&\geq \delta \|(v_H, u_H)\|_{L^2}^2 - 2c_{\mathcal{L}} (\|y_I\|_{Y_2} + \|(v_I, u_I)\|_{L^2})(\|y_H\|_{Y_2} + \|(v_H, u_H)\|_{L^2}) \\
&\quad - c_{\mathcal{L}} (\|y_I\|_{Y_2} + \|(v_I, u_I)\|_{L^2})^2 \\
&\geq \delta \|(v_H, u_H)\|_{L^2}^2 - c\|(v_1, u_1)\|_{L^2} \|(v_H, u_H)\|_{L^2} - c\|(v_1, u_1)\|_{L^2}^2 .
\end{aligned}
\tag{6.15}
$$

We apply the Young inequality to obtain

$$
\begin{aligned}
B[w]^2 &\geq \frac{\delta}{2} \|(v_H, u_H)\|_{L^2}^2 - c \, \|(v_1, u_1)\|_{L^2}^2 \\
&= \frac{\delta}{2} \Big(\int_{Q \setminus I_\tau^Q} v_H^2 + \int_{\Sigma \setminus I_\tau^\Sigma} u_H^2 \Big) - c \, \Big(\int_{I_\tau^Q} v_1^2 + \int_{I_\tau^\Sigma} u_1^2 \Big).
\end{aligned}
$$

By definition, it holds $v_H = \hat{v} - \bar{v} - v_I - r^v$ and $v_1 = \hat{v} - \bar{v} - r^v$ on I_τ^Q (notice that $v_2 = 0$ holds on I_τ^Q). Analogous representations are found for u_H and u_1. We substitute these expressions in the integrals above and expand the squares. Moreover, the Young inequality is applied in the form $|\hat{v} - \bar{v}||v_I| \leq \varepsilon |\hat{v} - \bar{v}|^2 + c\,|v_I|^2$, $|\hat{v} - \bar{v}||r^v| \leq \varepsilon |\hat{v} - \bar{v}|^2 + c\,|r^v|^2$, where $\varepsilon > 0$ can be chosen arbitrarily small. Then

$$
\frac{\delta}{2} \int_{Q \setminus I_\tau^Q} v_H^2 - c \int_{I_\tau^Q} v_1^2 \geq \Big(\frac{\delta}{2} - c\,\varepsilon \Big) \int_{Q \setminus I_\tau^Q} (\hat{v} - \bar{v})^2 - c \, \Big\{ \int_{Q \setminus I_\tau^Q} v_I^2 + \int_{I_\tau^Q} (\hat{v} - \bar{v})^2 + \int_Q |r^v|^2 \Big\}
$$

follows for the terms associated with v. According to (6.13), the integral containing v_I^2 can be estimated by the L^2-norm of v_1 on Q, which is handled as follows

$$
\int_Q v_1^2 = \int_{I_\tau^Q} v_1^2 = \int_{I_\tau^Q} (\hat{v} - \bar{v} + r^v)^2 \leq c \, \Big(\int_{I_\tau^Q} (\hat{v} - \bar{v})^2 + \int_Q (r^v)^2 \Big).
$$

Therefore, we find

$$
\frac{\delta}{2} \int_{Q \setminus I_\tau^Q} v_H^2 - c \int_{I_\tau^Q} v_1^2 \geq \Big(\frac{\delta}{2} - c\,\varepsilon \Big) \int_{Q \setminus I_\tau^Q} (\hat{v} - \bar{v})^2 - c \, \Big(\int_{I_\tau^Q} (\hat{v} - \bar{v})^2 + \int_Q (r^v)^2 \Big).
$$

An analogous estimation works for the parts associated with u. Finally, we arrive at

$$
\begin{aligned}
B[w]^2 &\geq \Big(\frac{\delta}{2} - c\,\varepsilon \Big) \Big\{ \int_{Q \setminus I_\tau^Q} (\hat{v} - \bar{v})^2 + \int_{\Sigma \setminus I_\tau^\Sigma} (\hat{u} - \bar{u})^2 \Big\} \\
&\quad - c \, \Big\{ \int_{I_\tau^Q} (\hat{v} - \bar{v})^2 + \int_{I_\tau^\Sigma} (\hat{u} - \bar{u})^2 + \int_Q (r^v)^2 + \int_\Sigma (r^u)^2 \Big\}.
\end{aligned}
\tag{6.16}
$$

Step 2. The treatment of $B[r, w]$ and $B[r]^2$ is simpler. For instance, by (6.9) we find

$$
\begin{aligned}
|B[r, w]| &\leq c\|r\| \|(v, u)\|_{L^2} = c \, \|r\| \|(\hat{v}, \hat{u}) - (\bar{v}, \bar{u}) + (r^v, r^u)\|_{L^2} \\
&\leq c \, \|(\hat{v}, \hat{u}) - (\bar{v}, \bar{u})\|_{L^\infty} \|(\hat{v}, \hat{u}) - (\bar{v}, \bar{u}) + (r^v, r^u)\|_{L^2}^2 \\
&\leq c \, \varrho \|(\hat{v}, \hat{u}) - (\bar{v}, \bar{u})\|_{L^2}^2 .
\end{aligned}
$$

The same type of estimate applies to $B[r]^2$. Altogether, it follows that

$$B[\hat{w}-\bar{w}]^2 \geq (\frac{\delta}{2} - c\,\varepsilon)(\int\limits_{Q\setminus I_\tau^Q} (\hat{v}-\bar{v})^2 + \int\limits_{\Sigma\setminus I_\tau^\Sigma} (\hat{u}-\bar{u})^2)$$
$$-c\,\varrho\,(\int\limits_{I_\tau^Q} |\hat{v}-\bar{v}| + \int\limits_{I_\tau^\Sigma} |\hat{u}-\bar{u}|) - c\,\varrho\,\|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^2}^2$$

(here, terms of the type $\|\hat{v}-\bar{v}\|^2_{L^2(I_\tau^Q)}$ are estimated by $\varrho\|\hat{v}-\bar{v}\|_{L^2(I_\tau^Q)}$). By substituting this result in (6.8), we obtain

$$J(\hat{w}) - J(\bar{w}) \geq (\tau - c\varrho)(\int\limits_{I_\tau^Q} |\hat{v}-\bar{v}| + \int\limits_{I_\tau^\Sigma} |\hat{u}-\bar{u}|) + (\frac{\delta}{2} - c\,\varepsilon)(\int\limits_{Q\setminus I_\tau^Q} (\hat{v}-\bar{v})^2 +$$
$$+ \int\limits_{\Sigma\setminus I_\tau^\Sigma} |\hat{u}-\bar{u}|^2) - c\,\varrho\,\|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^2}^2 - |r_2^{\mathcal{L}}(\bar{w},\hat{w}-\bar{w})|$$
$$\geq \frac{\tau}{2}(\int\limits_{I_\tau^Q} |\hat{v}-\bar{v}| + \int\limits_{I_\tau^\Sigma} |\hat{u}-\bar{u}|) + \frac{\delta}{2}(\int\limits_{Q\setminus I_\tau^Q} (\hat{v}-\bar{v})^2 + \int\limits_{\Sigma\setminus I_\tau^\Sigma} (\hat{u}-\bar{u})^2)$$
$$-c\varrho\|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^2}^2 - |r_2^{\mathcal{L}}(\bar{w},\hat{w}-\bar{w})|,$$

if ε and ϱ are chosen sufficiently small. Let us assume $\varrho = \|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^\infty} \leq 1$. Then it holds $|\hat{u}-\bar{u}| \geq |\hat{u}-\bar{u}|^2$ and $|\hat{v}-\bar{v}| \geq |\hat{v}-\bar{v}|^2$ almost everywhere. Using this in the first integral, setting $\delta' = \min\{\tau/2, \delta/2\}$, and substituting the estimate (6.4) for $r_2^{\mathcal{L}}$, we complete our estimation by

$$J(\hat{w}) - J(\bar{w}) \geq \|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^2}^2(\delta' - c\varrho - c\eta(c\varrho))$$
$$\geq \frac{\delta'}{2}\|(\hat{v},\hat{u})-(\bar{v},\bar{u})\|_{L^2}^2$$

for sufficiently small $\varrho > 0$. □

6.2. Time dependent controls. We now discuss how to adapt the results of the previous section to the case where controls are only depending on the time variable. This corresponds to case (iii) in section 2, for V_{ad} and U_{ad}. Let us recall the structure of the control sets

$$V_{ad} = \{v \in L^\infty(Q) \mid v(x,t) = \textstyle\sum_{i=1}^{\ell_d} v^i(t)\, e_i(x), \quad v_a^i \leq v^i \leq v_b^i \text{ a.e. on } (0,T)\}$$
$$U_{ad} = \{u \in L^\infty(\Sigma) \mid u(x,t) = \textstyle\sum_{i=1}^{\ell_b} u^i(t)\, \eta_i(x), \quad u_a^i \leq u^i \leq u_b^i \text{ a.e. on } (0,T)\},$$

where v_a^i, v_b^i, u_a^i, u_b^i are given constants in $L^\infty(0,T)$. Let us introduce integral forms of Hamiltonian derivatives

$$\tilde{H}_{v,i}^Q(t,\bar{y},\bar{v},\bar{p}) = \textstyle\int_\Omega (f_v(x,t,\bar{y},\bar{v}) - \bar{p}\, d_v(x,t,\bar{y},\bar{v}) e_i(x))dx$$

$$\tilde{H}_{u,i}^\Sigma(t,\bar{y},\bar{u},\bar{p}) = \textstyle\int_\Gamma (g_u(x,t,\bar{y},\bar{u}) - \bar{p}\, b_u(x,t,\bar{y},\bar{u})\eta_i(x))dS(x).$$

For $\tau > 0$, the sets of strongly active control constraints are now defined by

$$I_{\tau,i}^Q = \{t \in [0,T] \mid \ |\bar{H}_{v,i}^Q(t,\bar{y},\bar{v})| \geq \tau\}, \qquad I_{\tau,i}^\Sigma = \{t \in [0,T] \mid \ |\bar{H}_{u,i}^\Sigma(t,\bar{y},\bar{u})| \geq \tau\}.$$

Let us set

$$\tilde{C}_\tau(\bar{v},\bar{u}) = \{(v,u) = (\Sigma_i v^i e_i, \Sigma_i u^i \gamma_i) \in C(\bar{v},\bar{u}) \mid v^i = 0 \text{ a.e. on } I^Q_{\tau,i} \text{ for } 1 \le i \le \ell_d, \text{ and } u^i = 0 \text{ a.e. on } I^\Sigma_{\tau,i} \text{ for } 1 \le i \le \ell_b\}.$$

Now, the stronger regularity condition is

$(\tilde{\mathbf{R}})_\tau$ $\quad E'(\bar{y})G'(\bar{v},\bar{u})\tilde{C}_\tau(\bar{v},\bar{u}) - K(E(\bar{y})) = Z,$

and the second order sufficient optimality condition is formulated as follows:

$(\widetilde{\mathbf{SSC}})_\tau$ *There are positive constants δ and τ such that the coercivity condition (6.1) is satisfied for all $(y,v,u) \in L(\mathcal{M},(\bar{y},\bar{v},\bar{u}))$ satisfying $(v,u) \in \tilde{C}_\tau(\bar{v},\bar{u})$.*

By substituting $(\tilde{\mathbf{R}})_\tau$ and $(\widetilde{\mathbf{SSC}})_\tau$ to $(\mathbf{R})_\tau$ and $(\mathbf{SSC})_\tau$ in the statement of Theorem 6.1, we obtain the version corresponding to time dependent controls.

7. Some applications. In this section we want to exhibit examples for which the regularity condition for $E'(\bar{y})$, stated in Assumption (A5), and the continuity condition for $\mathcal{L}''(\bar{y},\bar{v},\bar{u},\bar{p},\bar{\lambda})$, assumed in (A6), are satisfied. Let us first examine the simplest case in which state constraints and controls are separately supported.

7.1. Constraints and controls with disjoint supports. We here suppose that

$$V_{ad} = \{v \in L^\infty(Q) \mid v_a \le v \le v_b \text{ a.e. on } \mathcal{O}_v, \ \operatorname{supp} v \subset \overline{\mathcal{O}}_v\}, \ V \equiv L^2_o(\mathcal{O}_v),$$

$$U_{ad} = \{u \in L^\infty(\Sigma) \mid u_a \le u \le u_b \text{ a.e. on } \mathcal{O}_u, \ \operatorname{supp} u \subset \overline{\mathcal{O}}_u\}, \ U \equiv L^2_o(\mathcal{O}_u),$$

where $\mathcal{O}_v$ is an open subset in Q, and $\mathcal{O}_u$ is an open subset in Σ (see the definition of spaces L^2_o in section 2). State constraints and state observations are supposed to satisfy $\operatorname{supp}(F_y(\bar{y})^*\mu) \subset \overline{\mathcal{O}}_\mu$, $\operatorname{supp}(E'(\bar{y})^*\lambda) \subset \overline{\mathcal{O}}_\lambda$, and $(\overline{\mathcal{O}}_u \cup \overline{\mathcal{O}}_v) \cap (\overline{\mathcal{O}}_\mu \cup \overline{\mathcal{O}}_\lambda) = \emptyset$, where $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ are open subsets in $\overline{Q}$. Observe that $f_y(\bar{y},\bar{v})$ and $g_y(\bar{y},\bar{u})$ are bounded functions. Thus, by using cut-off functions and a bootstrap argument, we can prove as in Proposition 3.2 of [1], that there exist compact subsets $\overline{Q}_1 \subset \overline{Q}$, $\overline{Q}_2 \subset \overline{Q}$ with the following properties. It holds $\overline{Q}_1 \cup \overline{Q}_2 = \overline{Q}$, $\overline{Q}_1 \supset (\overline{\mathcal{O}}_u \cup \overline{\mathcal{O}}_v)$, $\overline{Q}_2 \supset (\overline{\mathcal{O}}_\mu \cup \overline{\mathcal{O}}_\lambda)$. Moreover, $\bar{p}_{|\overline{Q}_1}$ belongs to $C(\overline{Q}_1)$, $y(u,v)_{|\overline{Q}_2}$ is continuous on $\overline{Q}_2$, and $\|y(u,v)\|_{C(\overline{Q}_2)} \le C\|(u,v)\|_{L^2}$ holds for all $u \in U$ and all $v \in V$. Here and below, $y(u,v)$ denotes the solution to (3.2) associated with (u,v). Due to these regularity results, we can easily check Assumption (A6). Assumption (A5) can be verified for many examples with the estimate $\|y(u,v)\|_{C(\overline{Q}_2)} \le C\|(u,v)\|_{L^2}$. For instance, if e is a real function of class $C^{2,1}$, we can set $E(y)(x,t) = e(y(x,t))$ and $Z = C(\overline{Q}_2)$.

7.2. Distributed controls with $V = L^2(Q)$. We suppose that there is no boundary control, in other words $b \equiv b(x,t,y)$ and $g \equiv g(x,t,y)$. Denote by $y(v)$ the solution of (3.2) corresponding to v (notice that $b_u \equiv 0$ in the boundary condition). In this case, due to Theorem 4.1, the mapping $v \to (y(v), y(v)|_\Sigma)$ is continuous from V to $L^{\tilde{r}}(0,T;L^r(\Omega)) \times L^{\tilde{\sigma}}(0,T;L^\sigma(\Gamma))$, where

$$\frac{N}{4}+\frac{1}{2} < \frac{N}{2r}+\frac{1}{\tilde{r}}+1, \ 2 \le r, \ 2 \le \tilde{r}, \quad \frac{N}{4}+\frac{1}{2} < \frac{N-1}{2\sigma}+\frac{1}{\tilde{\sigma}}+1, \ 2 \le \sigma, \ 2 \le \tilde{\sigma}. \tag{7.1}$$

Suppose that $(F_y(\bar{y})^*\mu + E'(\bar{y})^*\bar{\lambda})|_Q + f_y(\bar{y},\bar{v})$ belongs to $X^{\gamma_d}(Q)$, $(F_y(\bar{y})^*\mu + E'(\bar{y})^*\bar{\lambda})|_\Sigma + g_y(\bar{y})$ belongs to $X^{\gamma_b}(\Sigma)$, $(F_y(\bar{y})^*\mu + E'(\bar{y})^*\bar{\lambda})|_{\Omega_T}$ belongs to $X^{\gamma_T}(\overline{\Omega})$, with the convention that $X^{\gamma_d}(Q) = L^{\gamma_d}(Q)$ if $1 < \gamma_d \le \infty$, and $X^{\gamma_d}(Q) = \mathcal{M}_b(Q)$, if $\gamma_d = 1$. Analogous conventions are adopted for the spaces $X^{\gamma_b}(\Sigma)$ and $X^{\gamma_T}(\overline{\Omega})$. Clearly, the exponents γ_d, γ_b and γ_T depend on the nature of state constraints and of observations (i.e. of the cost functional). Due to Theorem 4.2, the solution $\bar{p}$ to (3.6) belongs to $L^{\tilde{\alpha}}(0,T;L^{\alpha}(\Omega))$, and its trace on Σ belongs to $L^{\tilde{\beta}}(0,T;L^{\beta}(\Gamma))$, if

$$\frac{N+2}{2\gamma_d} < \frac{N}{2\alpha} + \frac{1}{\tilde{\alpha}} + 1,\ \gamma_d \le \alpha,\ \gamma_d \le \tilde{\alpha}, \quad \frac{N+1}{2\gamma_b} < \frac{N}{2\alpha} + \frac{1}{\tilde{\alpha}} + \frac{1}{2},\ \gamma_b \le \alpha,\ \gamma_b \le \tilde{\alpha},$$

$$\frac{N}{2\gamma_T} < \frac{N}{2\alpha} + \frac{1}{\tilde{\alpha}},\ \gamma_T \le \alpha,\ 1 \le \tilde{\alpha},$$

$$\frac{N+2}{2\gamma_d} < \frac{N-1}{2\beta} + \frac{1}{\tilde{\beta}} + 1,\ \gamma_d \le \beta,\ \gamma_d \le \tilde{\beta}, \quad \frac{N+1}{2\gamma_b} < \frac{N-1}{2\beta} + \frac{1}{\tilde{\beta}} + \frac{1}{2},\ \gamma_b \le \beta,\ \gamma_b \le \tilde{\beta},$$

$$\frac{N}{2\gamma_T} < \frac{N-1}{2\beta} + \frac{1}{\tilde{\beta}},\ \gamma_T \le \beta,\ 1 \le \tilde{\beta}.$$

7.2.1. We first examine the case when $d \equiv d(x,t,y)+v$. Here, we have $d_{yv} = d_{vv} = 0$. Therefore, the only terms appearing with $\bar{p}$ in $\mathcal{L}''(\bar{y},\bar{v},\bar{u},\bar{p},\bar{\lambda})$ are $\int_Q \bar{d}_{yy} y_1 y_2 \bar{p}$ and $\int_\Sigma \bar{b}_{yy} y_1 y_2 \bar{p}$. Thus, to verify Assumption (A6) we are looking for conditions on γ_d, γ_b, γ_T ensuring

$$\|\bar{p}y(v_1)y(v_2)\|_{L^1(Q)} + \|\bar{p}y(v_1)y(v_2)\|_{L^1(\Sigma)} \le C\|v_1\|_V\|v_2\|_V. \tag{7.2}$$

Due to the previous estimates on $\bar{p}$ and $y(v_i)$, condition (7.2) holds when

$$\frac{N+2}{5} < \gamma_d, \qquad \frac{N+1}{4} < \gamma_b, \quad \text{and} \qquad \frac{N}{3} < \gamma_T. \tag{7.3}$$

Let us briefly explain how to do such calculations. We know that $y(v_1)|_\Sigma$ and $y(v_2)|_\Sigma$ belongs to $L^{\tilde{\sigma}}(0,T;L^{\sigma}(\Gamma))$ with $\frac{N}{4}+\frac{1}{2} < \frac{N-1}{2\sigma} + \frac{1}{\tilde{\sigma}} + 1$, $2 \le \sigma$, $2 \le \tilde{\sigma}$. Moreover, $\bar{p}|_\Sigma$ belongs to $L^{\tilde{\beta}}(0,T;L^{\beta}(\Gamma))$ with $\frac{N+2}{2\gamma_d} < \frac{N}{2\beta} + \frac{1}{\tilde{\beta}} + 1$, $\gamma_d \le \beta$, $\gamma_d \le \tilde{\beta}$. In view of these inequalities, we find

$$\frac{N}{2} + 1 + \frac{N+2}{2\gamma_d} < \frac{(N-1)}{2}\frac{2}{\sigma} + \frac{2}{\tilde{\sigma}} + 2 + \frac{N}{2\beta} + \frac{1}{\tilde{\beta}} + 1.$$

Due to Hölder inequality's, $(\bar{p}y(v_1)y(v_2))|_\Sigma$ is in $L^1(\Sigma)$ if $\frac{1}{\beta}+\frac{2}{\sigma} \le 1$ and $\frac{1}{\tilde{\beta}}+\frac{2}{\tilde{\sigma}} \le 1$. This leads to $\frac{N}{2}+1+\frac{N+2}{2\gamma_d} < \frac{N-1}{2}+3$, that is to $\frac{N+2}{5} < \gamma_d$. The other calculations are done in the same way.

We have found conditions on γ_d, γ_b, and γ_T for which assumption (A6) holds when $d \equiv d(x,t,y)+v$. Now we are looking for additional conditions on γ_d, γ_b, and γ_T so that (A5) be satisfied. For this, we separately study the case of pointwise state constraints and of integral state constraints. We next analyse the role of the cost functional.

Problems with pointwise state constraints. For a problem with pointwise state constraints on $\overline{Q}$, the associated multiplier is a measure on $\overline{Q}$. In this case we must set $\gamma_d = \gamma_b = \gamma_T = 1$. Therefore, according to (7.3), pointwise state constraints can be considered only for $N < 3$. But this condition is not yet sufficient. Indeed, we must verify Assumption (A5). The mapping $E'(\bar y)$ has to be continuous from $(C(\overline{Q}), \|\cdot\|_{Y_2})$ (i.e. the space $C(\overline{Q})$ endowed with the norm $\|\cdot\|_{Y_2}$) into Z (see (A5)). Thus, in the case of pointwise state constraints on $\overline{Q}$, we require that the identity mapping be continuous from $(C(\overline{Q}), \|\cdot\|_{Y_2})$ into $(C(\overline{Q}), \|\cdot\|_{C(\overline{Q})})$. This continuity condition is satisfied if (7.1) holds for $r = \tilde r = \infty$. The only possible case is $N = 1$. If $N = 1$ and if e is a mapping from $\mathbb{R}$ to $\mathbb{R}$ of class $C^{2,1}$, then assumption $(A5)$ is satisfied for state constraints of the form $E(y) \in K$ defined by $E(y)(x,t) = e(y(x,t))$ (where K is a closed convex subset in $C(\overline{Q})$).

Problems with integral state constraints. Now, consider integral state constraints of the form

$$\int_Q e_1(x,t)\, y(x,t)\, dxdt \le c, \quad \int_\Sigma e_2(x,t)\, y(x,t)\, dS(x)dt \le c, \quad \int_\Omega e_3(x)\, y(x,T)\, dx \le c,$$

with $e_1 \in L^{\ell_1}(Q)$, $e_2 \in L^{\ell_2}(\Sigma)$, $e_3 \in L^{\ell_3}(\Omega)$. Since e_1, e_2, and e_3 appear in the adjoint equation, according to (7.3), assumption (A6) is satisfied for $\frac{N+2}{5} < \gamma_d \le \ell_1$, $\frac{N+1}{4} < \gamma_b \le \ell_2$, $\frac{N}{3} < \gamma_T \le \ell_3$. To check assumption (A5) we are looking for conditions so that the mapping $v \to (y, y_{|\Sigma}, y(T))$ be continuous from V to $L^{\lambda_1}(Q) \times L^{\lambda_2}(\Sigma) \times L^{\lambda_3}(\Omega)$, with $\lambda_1 \ge \ell_1'$, $\lambda_2 \ge \ell_2'$, $\lambda_3 \ge \ell_3'$. Due to Theorem 4.1, we must have $\max(2, \ell_1') \le \lambda_1 < \frac{2N+4}{(N-2)^+}$, $\max(2, \ell_2') \le \lambda_2 < \frac{2N+2}{(N-2)^+}$, $\max(2, \ell_3') \le \lambda_3 < \frac{2N}{(N-2)^+}$. More generally, for state constraints of the form $y \in \mathcal{C} \subset L^{\ell_1'}(Q)$, where $\mathcal{C}$ is a closed convex subset of $L^{\ell_1'}(Q)$, we must choose $\frac{N+2}{5} < \gamma_d \le \ell_1$ and $\max(2, \ell_1') \le \lambda_1 < \frac{2N+4}{(N-2)^+}$.

For simplicity in the analysis we have supposed that $(F_y(\bar y)^*\mu + E'(\bar y)^*\bar\lambda)|_Q + f_y(\bar y, \bar v)$ belongs to $L^{\gamma_d}(Q)$ or to $\mathcal{M}_b(Q)$. But for constraints of the form

$$\int_\Omega e(x) y(x,t) dx \le c \qquad \text{for every } t \in [0,T],$$

the term $E'(\bar y)^*\bar\lambda|_Q$ belongs to the space $\mathcal{M}([0,T]; L^\ell(\Omega))$ if the function e belongs to $L^\ell(\Omega)$. The previous analysis can also be carried out for this kind of constraint.

The role of the cost functional. The exponents γ_d, γ_b and γ_T are also related to the cost functional. A functional of the form $y \to \int_\Omega |y(x,T) - y_d|^\ell dx$ leads to the term $\ell |y(x,T) - y_d|^{\ell-2}(y(x,T) - y_d)$ in the terminal condition of the adjoint equation. If y_d belongs to $L^k(\Omega)$, then $|y(x,T) - y_d|^{\ell-1}$ belongs to $L^{k/(\ell-1)}(\Omega)$. Thus we must choose $\frac{N}{3} < \gamma_T \le \frac{k}{\ell-1}$. The other possibilities can be analysed in the same way. In particular a functional of the form $y \to (y(x_0,t_0) - r_0)^2$, with $(x_0,t_0) \in Q$, corresponds to $\gamma_d = 1$.

7.2.2. The second case corresponds to a function d of the form $d \equiv d(x,t,y)v$. Here the terms $\int_Q \bar d_y y_1 v_2 \bar p$ and $\int_Q \bar d_y y_2 v_1 \bar p$ appear in $\mathcal{L}''(\bar y, \bar v, \bar u, \bar p, \bar\lambda)$. Hence, in addition to condition (7.2) we must have $\|\bar p y(v)\|_{L^2(Q)} \le C\|v\|_V$. It will be realised if

$$\frac{N+2}{7/2} < \gamma_d, \quad \frac{N+1}{5/2} < \gamma_b, \quad \frac{N}{3/2} < \gamma_T.$$

Therefore pointwise state constraints on $\overline{Q}$ may be still considered for $N = 1$.

7.2.3. The last case corresponds to a function d of the form $d \equiv d(x,t,y,v)$. Then the terms $\int_Q \bar d_{vv} v_1 v_2 \bar p$ also appears in $\mathcal{L}''(\bar y, \bar v, \bar u, \bar p, \bar\lambda)$. In this case, Assumption

(A6) is satisfied when $\bar{p}$ belongs to $L^\infty(Q)$. Due to regularity results for $\bar{p}$, we must have

$$\frac{N+2}{2} < \gamma_d, \quad N+1 < \gamma_b, \quad \gamma_T = \infty.$$

Therefore, in this case, we cannot consider pointwise state constraints on $\overline{Q}$. An integral state constraint of the form $\int_\Omega e_3(x)\, y(x,T)\, dx \le c$ may be considered only for $e_3 \in L^\infty(\Omega)$ (since we must have $\gamma_T = \infty$).

7.3. Boundary controls with $U = L^2(\Sigma)$**.** Suppose that there is no distributed control ($d \equiv d(x,t,y)$ and $f \equiv f(x,t,y)$). Denote by $y(u)$ the solution of (3.2) corresponding to u. We perform the same kind of analysis as above. Due to Theorem 4.1, the mapping $u \to (y(u), y(u)|_\Sigma)$ is continuous from U to $L^{\tilde{r}}(0,T;L^r(\Omega)) \times L^{\tilde{\sigma}}(0,T;L^\sigma(\Gamma))$ if

$$\frac{N-1}{4} < \frac{N}{2r} + \frac{1}{\tilde{r}},\ 2 \le r,\ 2 \le \tilde{r}, \quad \frac{N-1}{4} < \frac{N-1}{2\sigma} + \frac{1}{\tilde{\sigma}},\ 2 \le \sigma,\ 2 \le \tilde{\sigma}.$$

From these regularity results, we see that pointwise state constraints (up to the boundary) cannot be considered in this case. Indeed, we cannot set $r = \tilde{r} = \sigma = \tilde{\sigma} = \infty$ in the above inequality. When b is of the form $b \equiv b(x,t,y) + u$, due to the terms $\int_\Sigma \bar{b}_{yu} \bar{p} y_i u_j$ in $\mathcal{L}''$, the estimate

$$\|\bar{p} y(u_1) y(u_2)\|_{L^1(Q)} + \|\bar{p} y(u_1) y(u_2)\|_{L^1(\Sigma)} \le C\|u_1\|_U \|u_2\|_U$$

must be checked. It holds if

$$\frac{N+2}{4} < \gamma_d, \quad \frac{N+1}{3} < \gamma_b, \quad \frac{N}{2} < \gamma_T.$$

The second case corresponds to a function b of the form $b \equiv b(x,t,y)u$. Assumption (A6) is satisfied when the estimate $\|\bar{p} y(u)\|_{L^2(\Sigma)} \le C\|u\|_U$ holds. This leads to

$$\frac{N+2}{3} < \gamma_d, \quad \frac{N+1}{2} < \gamma_b, \quad N < \gamma_T.$$

The last case corresponds to a function b of the form $b \equiv b(x,t,y,u)$. As above $\bar{p}|_\Sigma$ belongs to $L^\infty(\Sigma)$ if $\frac{N+2}{2} < \gamma_d$, $N+1 < \gamma_b$, $\gamma_T = \infty$.

7.4. Distributed controls with $V = L^2(0,T;L^\infty(\Omega))$**.** We suppose that there is no boundary control ($b \equiv b(x,t,y)$ and $g \equiv g(x,t,y)$). We adopt the notation of section 7.2. The mapping $v \to y(v)$ is continuous from V to $C(\overline{Q})$. Therefore assumption (A5) can be easily verified in classical situations, even for pointwise state constraints. Moreover, when d is of the form $d \equiv d(x,t,y) + v$, Assumption (A6) is satisfied. The second case corresponds to a function d of the form $d \equiv d(x,t,y)v$. Due to the structure of the control set, we have to check the estimate $\|\bar{p} y(v)\|_{L^2(0,T;L^1(\Omega))} \le C\|v\|_V$. This holds for every $\gamma_d \ge 1$, every $\gamma_b \ge 1$, and every $\gamma_T \ge 1$. In the case when d of the form $d \equiv d(x,t,y,v)$, Assumption (A6) is fulfilled, if $\bar{p}$ belongs to $L^\infty(0,T;L^1(\Omega))$. This result is true even if $\gamma_d = \gamma_b = \gamma_T = 1$. This regularity property does not follow from Theorem 4.3, but it is proved in Proposition 4.4 of [14].

7.5. Boundary controls with $U = L^2(0,T;L^\infty(\Gamma))$. Suppose that there is no distributed control, in other words $d \equiv d(x,t,y)$ and $f \equiv f(x,t,y)$. We adopt the notation of section 7.3. The mapping $u \to (y(u), y(u)|_\Sigma)$ is continuous from U to $L^{\tilde\sigma}(0,T;L^\infty(\Omega)) \times L^{\tilde\sigma}(0,T;L^\infty(\Gamma))$ for any $\tilde\sigma < \infty$. Therefore assumption (A5) can be verified for integral state constraints. However, we cannot consider neither pointwise state constraints on $\overline{Q}$, nor constraints of the form $\int_\Gamma e(x)y(x,t)dS(x) \le c$, because the mapping $u \to y(u)|_\Sigma$ is not continuous from U to $L^\infty(0,T;L^1(\Gamma))$. When b is of the form $b \equiv b(x,t,y)+u$, Assumption (A6) is satisfied. The second case corresponds to a function b of the form $b \equiv b(x,t,y)u$. The estimate $\|\bar p y(u)\|_{L^2(0,T;L^1(\Gamma))} \le C\|u\|_U$ must be checked. This holds when $1 < \gamma_d$, $1 < \gamma_b$, and $1 < \gamma_T$. In the case when b of the form $b \equiv b(x,t,y,u)$, we have to verify that $\bar p$ belongs to $L^\infty(0,T;L^1(\Gamma))$. It holds if $\frac{3}{2} < \gamma_d$, $2 < \gamma_b$, and $\gamma_T = \infty$.

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