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OPTIMAL CONTROL OF SOME QUASILINEAR MAXWELL EQUATIONS OF PARABOLIC TYPE

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Abstract. An optimal control problem is studied for a quasilinear Maxwell equation of nondegenerate parabolic type. Well-posedness of the quasilinear state equation, existence of an optimal control, and weak Gâteaux-differentiability of the control-to-state mapping are proved. Based on these results, first-order necessary optimality conditions and an associated adjoint calculus are derived.

1. **Introduction.** We consider the optimal control of a system of quasilinear evolution Maxwell equations that models the behavior of magnetic fields in a vector potential formulation. The state equation is the non-degenerate parabolic equation

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \operatorname{curl}\left(\nu(x, |\operatorname{curl}\mathbf{y}|) \operatorname{curl}\mathbf{y} \right) &= \mathbf{f} \quad \text{in} \quad Q := \Omega \times (0,T) \\ \mathbf{y} \times \mathbf{n} &= \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0,T) \\ \mathbf{y}(0) &= \mathbf{y}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with a connected Lipschitz boundary. We further assume that the electric conductivity $\sigma : \Omega \to \mathbb{R}_+$ is a positive constant,

$$\sigma(x) = \sigma > 0 \quad \forall x \in \Omega. \tag{1.2}$$

By $\mathbf{n}(x)$, we denote the outward normal direction in the point $x \in \partial\Omega$. If the magnetic reluctivity $\nu : \Omega \times \mathbb{R}^3 \to \mathbb{R}_+$ is also a positive constant, then (1.1) is a standard linear evolution equation. However, as in Bachinger et al. [2], we allow ν to be a nonlinear function so that equation (1.1) becomes quasilinear. The mapping $s \mapsto \nu(x, s)s$ expresses the so-called $|\mathbf{B}|$-$|\mathbf{H}|$ curve.

In the application to the magnetization processes we have in mind, the real quantity of interest is the magnetic induction $\mathbf{B} : \Omega \times (0,T) \to \mathbb{R}^3$ that will be

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represented here by a vector potential $\mathbf{y} : \Omega \times (0,T) \to \mathbb{R}^3$ obtained from equation (1.1). For the given right-hand side $\mathbf{f} : \Omega \times (0,T) \to \mathbb{R}^3$ we shall require some regularity properties. For instance, we assume $\operatorname{div} \mathbf{f} = 0$ in Ω. We will briefly sketch some special types of associated (control) functions $\mathbf{f}$ at the end of this paper.

Our paper contributes to the fast developing theory of optimal control problems that include, as part of the control system, Maxwell's equations. For instance, Maxwell's equations appear in the control of processes of magneto-hydrodynamics (MHD). We mention examplarily [3], [6], [7], [8].

In these papers, the Maxwell equations are considered in a steady state or time-harmonic setting. The time harmonic approach is also used in [2], [5], [9], [10], [17], [18]. For the linear transient case, we mention the paper [4] on the optimal control of the full (linear) time dependent Maxwell system, where the control function is composed as a product of two functions depending on the time and on the space variable, respectively.

Concentrating on a parabolic vector potential formulation, we continue our investigations in [11], [12], [13], where we considered linear degenerate parabolic Maxwell systems.

The main novelty of this paper is the consideration of quasilinear Maxwell equations of parabolic type in the context of optimal control. Quasilinear Maxwell equations have already been considered in [2]. The extension of optimality conditions to the quasilinear case causes specific difficulties related to the existence and uniqueness of solutions to the state equation and the differentiability of the control-to-state mapping. It turns out that we only have weak Gâteaux-differentiability. However, since the objective functional is quadratic, this is sufficient for deriving first-order necessary optimality conditions.

Our approach extends results of the seminal paper [19] on the optimal control of certain quasilinear elliptic Maxwell equations. In this paper, main ideas were introduced that we were able to adopt in the context of non-stationary quasilinear systems. For proving existence and uniqueness of the solutions to our quasilinear parabolic Maxwell equations, we rely on results of [15]. In this context, we also mention the monography [14], where different important mathematical principles for nonlinear partial differential equations are discussed.

2. Well-posedness of the non-degenerate quasilinear Maxwell evolution system.

Assumption 2.1 (cf. [2, 19])**.** Let $\nu_0 > 0$ denote the magnetic reluctivity in a vacuum. We assume that there exist constants $\underline{\nu} \in (0, \nu_0)$ and $\overline{\nu} \geq \nu_0$ such that there holds

$$\underline{\nu} \leq \nu(x,s) \leq \overline{\nu} \quad \text{for a.a. } x \in \Omega \text{ and all } s > 0, \tag{2.1}$$

$$\lim_{s\to\infty} \nu(x,s) = \nu_0 \text{ for a.a. } x \in \Omega. \tag{2.2}$$

Moreover, for a.e. $x \in \Omega$, the mapping $s \mapsto \nu(x,s)s$ is assumed to satisfy

$$(\nu(x,s)s - \nu(x,\sigma)\sigma)(s-\sigma) \;\geq\; \underline{\nu}\,(s-\sigma)^2 \quad \forall s > 0,\, \sigma > 0, \tag{2.3}$$

$$|\nu(x,s)s - \nu(x,\sigma)\sigma| \;\leq\; \overline{\nu}\,|s-\sigma| \quad \forall s > 0,\, \sigma > 0. \tag{2.4}$$

Notice that, by this assumption, the real function $s \mapsto \nu(x,s)s$ is monotone for a.a. $x \in \Omega$.

Definition 2.2. Following [19], we define a function $\mathcal{F} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathcal{F}(x,\mathbf{s}) = \nu(x,|\mathbf{s}|)\,\mathbf{s}.$$

Lemma 2.3 ([19], Appendix)**.** *Under Assumption 2.1, we have for a.a.* $x \in \Omega$ *that*

$$(\mathcal{F}(x,\mathbf{u}) - \mathcal{F}(x,\mathbf{v}))\cdot(\mathbf{u}-\mathbf{v}) \;\geq\; \underline{\nu}\,|\mathbf{u}-\mathbf{v}|^2 \qquad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^3, \tag{2.5}$$

$$|\mathcal{F}(x,\mathbf{u}) - \mathcal{F}(x,\mathbf{v})| \;\leq\; L\,|\mathbf{u}-\mathbf{v}| \qquad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^3, \tag{2.6}$$

where $L = 2\nu_0 + \overline{\nu}$.

For convenience, we also introduce a mapping $\mathcal{N} : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\mathcal{N}(x,\mathbf{s}) = \nu(x, |\mathbf{s}|).$$

By $\mathcal{N}$, we have $\mathcal{F}(x,\mathbf{s}) = \mathcal{N}(x,\mathbf{s})\,\mathbf{s}$.

Throughout this paper, vector valued functions will be written in boldface and we write $\mathbf{L}^2(E) := L^2(E)^3$ for suitable measurable sets E. The associated inner product and norm will be denoted by $(\cdot,\cdot)_E$ and $\|\cdot\|_E$, respectively; if E is equal to Ω, we will drop the index. In particular, we have

$$(\cdot\,,\,\cdot) := (\cdot\,,\,\cdot)_\Omega \quad \text{and} \quad \|\cdot\| := \|\cdot\|_\Omega.$$

Moreover, we will write $a \lesssim b$, if a generic positive constant C exists such that $a \leq C\,b$ holds.

The divergence constraint is defined in distributional sense, i.e., for $\mathbf{f} \in \mathbf{L}^2(\Omega)$, we say that

$$\operatorname{div}\mathbf{f} = 0 \;\text{ iff }\; \int_\Omega \mathbf{f}(x)\cdot\nabla\varphi(x)\,dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

For later use, we introduce the following spaces:

$$\begin{aligned}
&\mathcal{H} := \mathbf{H}(\text{div=}0,\Omega) = \{\mathbf{y} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{y} = \mathbf{0}\},\\
&\mathbf{H}(\text{curl},\Omega) = \{\mathbf{y} \in \mathbf{L}^2(\Omega) : \operatorname{curl}\mathbf{y} \in \mathbf{L}^2(\Omega)\},\\
&\mathbf{H}_0(\text{curl},\Omega) = \{\mathbf{y} \in \mathbf{H}(\text{curl},\Omega) : \mathbf{y}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\},\\
&\mathbf{V} = \mathbf{H}_0(\text{curl},\Omega) \cap \mathbf{H}(\text{div=}0,\Omega),\\
&\mathbf{W}(0,T) = \{\mathbf{y} \in \mathbf{L}^2(0,T;\mathbf{V}) : \partial\mathbf{y}/\partial t \in \mathbf{L}^2(0,T;\mathbf{V}')\},\\
&\mathbf{L}^2(\text{div=}0,Q) = \mathbf{L}^2(0,T;\mathbf{H}(\text{div=}0,\Omega)) = \mathbf{L}^2(0,T;\mathcal{H}).
\end{aligned}$$

The space $\mathbf{H}(\text{div=}0,\Omega)$ is equipped with the norm of $\mathbf{L}^2(\Omega)$, while $\mathbf{V}$ and $\mathbf{W}(0,T)$ are equipped with their known natural norms. Note that $\mathbf{W}(0,T)$ is continuously embedded into $C([0,T];\mathcal{H})$ (see for instance Theorem 3.10 from [16]). Further by Theorem 2.8 of [1], $\mathbf{V}$ is compactly embedded into $\mathcal{H}$, hence by the Aubin-Lions Lemma [15, Prop. III.1.3], $\mathbf{W}(0,T)$ is compactly embedded into $\mathbf{L}^2(0,T;\mathcal{H})$.

In order to prove the existence and uniqueness of a strong solution to (1.1), we introduce the (nonlinear) operator A in $\mathcal{H}$ as follows:

$$D(A) := \{\mathbf{y} \in \mathbf{V} : \operatorname{curl}(\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y})) \in \mathbf{L}^2(\Omega)\},$$

and

$$A(\mathbf{y}) = \operatorname{curl}(\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y})), \quad \forall \mathbf{y} \in D(A).$$

We recall [15, p. 158] that an operator $A : H \supset D(A) \to H$ in a Hilbert space H is said to be *accretive* (or *monotone*) , if

$$(A(x) - A(y)\,,\,x - y)_H \geq 0 \quad \forall x \in D(A)$$

and *maximal accretive*, if in addition *Range* $(A + I) = H$ holds.

Lemma 2.4. *The operator* A *is maximal accretive in* $\mathcal{H}$.

Proof. Let us start with the accretiveness. Indeed for $\mathbf{y}, \mathbf{z} \in D(A)$, we have $\mathbf{y}-\mathbf{z}$ is in $\mathbf{H}_0(\mathrm{curl}, \Omega)$ and $\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{y}) - \mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{z})$ belongs to $\mathbf{H}(\mathrm{curl}, \Omega)$, hence by Green's formula we can write

$$\begin{aligned}(A(\mathbf{y}) - A(\mathbf{z}), \mathbf{y}-\mathbf{z}) &= (\mathrm{curl}(\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{y}) - \mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{z})), \mathbf{y}-\mathbf{z}) \\ &= (\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{y}) - \mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{z})), \mathrm{curl}(\mathbf{y}-\mathbf{z})).\end{aligned}$$

By the property (2.5), we deduce that

$$(A(\mathbf{y}) - A(\mathbf{z}), \mathbf{y}-\mathbf{z}) \geq \underline{\nu}\, \|\,\mathrm{curl}(\mathbf{y}-\mathbf{z})\|^2 \geq 0, \quad \forall \mathbf{y},\, \mathbf{z} \in D(A) \tag{2.7}$$

and hence A is accretive.

Let us proceed with the maximality. Namely, we have to show that $I + A$ is surjective. In other words, for all $\mathbf{f} \in \mathcal{H}$, the equation

$$\mathbf{u} + A(\mathbf{u}) = \mathbf{f},$$

or equivalently

$$\mathbf{u} + \mathrm{curl}\,\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{u}) = \mathbf{f}$$

must have a solution $\mathbf{u} \in D(A)$.

If such a solution exists, multiplying by $\mathbf{v} \in V$ and integrating by parts as before, we find that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.8}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + (\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{u}), \mathrm{curl}\,\mathbf{v}).$$

The right-hand side of (2.8) defines an element of $\mathbf{V}'$ while its left-hand side defines a monotone, hemicontinuous map T from $\mathbf{V}$ into $\mathbf{V}'$. Hence by [15, Corollary II.2.2], problem (2.8) has a unique solution $\mathbf{u} \in \mathbf{V}$, if T is coercive, that is

$$\frac{\langle T(\mathbf{u}), \mathbf{u}\rangle}{\|\mathbf{u}\|_{\mathbf{V}}} = \frac{a(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|_{\mathbf{V}}} \to +\infty, \text{ as } \|\mathbf{u}\|_{\mathbf{V}} \to +\infty.$$

Here and in all what follows, we denote by $\langle \cdot\,, \cdot\rangle$ the pairing between $\mathbf{V}$ and $\mathbf{V}'$. But this property directly follows from (2.5), since it yields

$$a(\mathbf{u}, \mathbf{u}) \geq \|\mathbf{u}\|^2 + \underline{\nu}\, \|\,\mathrm{curl}\,\mathbf{u}\|^2 \geq \min\{1, \underline{\nu}\}\, \|\mathbf{u}\|^2_{\mathbf{V}}.$$

Now, as $\mathbf{u}, \mathbf{f}$ are divergence free, the identity (2.8) extends to any $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, i.e.,

$$(\mathbf{u}, \mathbf{v}) + (\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{u}), \mathrm{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega). \tag{2.9}$$

Indeed, any $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ admits the Helmholtz decomposition

$$\mathbf{v} = \mathbf{v}_0 + \nabla\varphi,$$

with $\varphi \in H^1_0(\Omega)$ and $\mathbf{v}_0 \in \mathbf{V}$. Hence $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,\mathbf{v}_0$ and $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}_0)$ as well as $(\mathbf{f}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0)$.

As $C_0^\infty(\Omega)^3$ is included (and dense) in $\mathbf{H}_0(\mathrm{curl}, \Omega)$, we conclude that

$$\mathbf{u} + \mathrm{curl}(\mathcal{F}(\cdot, \mathrm{curl}\,\mathbf{u})) = \mathbf{f}$$

in the distributional sense. This means that $\mathbf{u}$ belongs to $D(A)$ and satisfies $\mathbf{u} + A(\mathbf{u}) = \mathbf{f}$. □

Corollary 2.5. *$D(A)$ is dense in $\mathcal{H}$.*

Proof. As the boundary of Ω is assumed to be connected, Corollary 3.19 in [1] guarantees that

$$\|\mathbf{u}\|^2 \lesssim \|\operatorname{curl}\mathbf{u}\| + \|\operatorname{div}\mathbf{u}\|,$$

for all $\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$ such that $\operatorname{div}\mathbf{u} \in L^2(\Omega)$. As a consequence, we deduce that

$$\|\operatorname{curl}\mathbf{u}\|^2 \gtrsim \|\mathbf{u}\|^2, \quad \forall \mathbf{u} \in \mathbf{V}. \tag{2.10}$$

Therefore the map

$$S : \mathbf{V} \longrightarrow \mathbf{V}' : \mathbf{u} \mapsto S\mathbf{u},$$

with $\langle S\mathbf{u}, \mathbf{v}\rangle = (\mathcal{F}(\cdot, \operatorname{curl}\mathbf{u}), \operatorname{curl}\mathbf{v})$, is monotone, hemicontinuous and coercive. This implies that, for all $\mathbf{f} \in \mathcal{H}$, there exists a unique solution $\mathbf{u} \in \mathbf{V}$ of

$$(\mathcal{F}(\cdot, \operatorname{curl}\mathbf{u}), \operatorname{curl}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.11}$$

As before, $\mathbf{f}$ being divergence free, $\mathbf{u}$ also satisfies

$$(\mathcal{F}(\cdot, \operatorname{curl}\mathbf{u}), \operatorname{curl}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \Omega),$$

and therefore

$$\operatorname{curl}(\mathcal{F}(\cdot, \operatorname{curl}\mathbf{u})) = \mathbf{f}$$

holds in the distributional sense; hence $\mathbf{u}$ belongs to $D(A)$ and satisfies $A(\mathbf{u}) = \mathbf{f}$.

Now let us choose any $\mathbf{f}$ in $\mathcal{H} \cap D(A)^\perp$, namely $\mathbf{f} \in \mathcal{H}$ such that

$$(\mathbf{f}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in D(A).$$

Then by taking $\mathbf{u} \in D(A)$ such that $A(\mathbf{u}) = \mathbf{f}$, we deduce that

$$(A(\mathbf{u}), \mathbf{u}) = 0.$$

From (2.7) (applied with $\mathbf{z} = \mathbf{0}$ and the fact that $A(\mathbf{0}) = \mathbf{0}$), we obtain that

$$\operatorname{curl}\mathbf{u} = 0.$$

By (2.10), it follows that $\mathbf{u} = \mathbf{0}$, hence $\mathbf{f} = \mathbf{0}$. □

Let us quote Theorem [15, Thm. IV.4.1] that goes back to Kato:

Theorem 2.6. *Let A be maximal accretive in the Hilbert space H and $\omega \geq 0$. For each $y_0 \in D(A)$ and each absolutely continuous $f : [0, T] \to H$, there is a unique absolutely continuous $y : [0, T] \to H$ such that*

$$y'(t) + A(y(t)) = \omega\, y(t) + f(t) \quad \text{and} \quad y(0) = y_0 \tag{2.12}$$

holds at a.e. $t > 0$. Moreover, y is Lipschitz, right differentiable, $y(t) \in D(A)$ for all $t \geq 0$ and $y', Ay \in L^\infty(0, T; H)$.

The regularity $y' \in L^\infty(0, T; H)$ is stated in the proof of Theorem IV.4.1 of [15]. Furthermore by (2.12) (as A is single-valued), $Ay = -y' + \omega y + f$ that also belongs to $L^\infty(0, T; H)$ since each term has this regularity.

Lemma 2.4 combined with Theorem 2.6 allows to deduce the existence and uniqueness of a strong solution to (1.1).

Theorem 2.7. *Let $\mathbf{y}_0 \in D(A)$ and $\mathbf{f} : [0, T] \longrightarrow \mathcal{H}$ be absolutely continuous. Then there exists a unique (strong) solution $\mathbf{y}$ of (1.1) with the regularity $\mathbf{y} \in C([0, T]; \mathcal{H}) \cap W^{1,\infty}(0, T; \mathcal{H})$ and $A(\mathbf{y}) \in L^\infty(0, T; \mathcal{H})$.*

Weak solutions come from the following energy estimates.

Lemma 2.8. *Let $\mathbf{y}_0, \mathbf{z}_0 \in D(A)$ and $\mathbf{f}, \mathbf{g} : [0,T] \longrightarrow \mathcal{H}$ be absolutely continuous. Let $\mathbf{y}$ (resp. $\mathbf{z}$) be the strong solutions of (1.1) corresponding to an initial datum $\mathbf{y}_0$ (resp. $\mathbf{z}_0$) and right-hand sides $\mathbf{f}$ (resp. $\mathbf{g}$). Then these solutions satisfy*

$$\|\mathbf{y} - \mathbf{z}\|_{\mathbf{L}^2(0,T;\mathbf{V})} \lesssim \|\mathbf{f} - \mathbf{g}\|_Q + \|\mathbf{y}_0 - \mathbf{z}_0\|. \tag{2.13}$$

Proof. Multiplying the equation

$$\sigma \frac{\partial \mathbf{y}}{\partial t} + \operatorname{curl}\left(\, \nu(x, |\operatorname{curl} \mathbf{y}|) \operatorname{curl} \mathbf{y}\,\right) = \mathbf{f} \text{ in } Q,$$

by $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{V})$ and integrating with respect to $(x,t) \in Q = \Omega \times (0,T)$, we find

$$\int_Q (\sigma \frac{\partial \mathbf{y}}{\partial t} \cdot \mathbf{v} + \operatorname{curl}(\mathcal{F}(x, \operatorname{curl} \mathbf{y})) \cdot \mathbf{v})\, dxdt = \int_Q \mathbf{f} \cdot \mathbf{v}\, dxdt.$$

Integrating by parts in space, we get

$$\int_Q (\sigma \frac{\partial \mathbf{y}}{\partial t} \cdot \mathbf{v} + \mathcal{F}(x, \operatorname{curl} \mathbf{y}) \cdot \operatorname{curl} \mathbf{v})\, dxdt = \int_Q \mathbf{f} \cdot \mathbf{v}\, dxdt. \tag{2.14}$$

Note that the same identity holds for $\mathbf{z}$ with $\mathbf{g}$ instead of $\mathbf{f}$. Hence, making the difference between (2.14) and the same identity with $\mathbf{z}$ instead of $\mathbf{y}$, we find that

$$\begin{aligned} \int_Q \Big(\sigma \frac{\partial (\mathbf{y} - \mathbf{z})}{\partial t} \cdot \mathbf{v} + (\mathcal{F}(x, \operatorname{curl} \mathbf{y}) - \mathcal{F}(x, \operatorname{curl} \mathbf{z})) \cdot \operatorname{curl} \mathbf{v}\Big)\, dxdt \\ = \int_Q (\mathbf{f} - \mathbf{g}) \cdot \mathbf{v}\, dxdt, \quad \forall \mathbf{v} \in \mathbf{L}^2(0,T,\mathbf{V}). \end{aligned} \tag{2.15}$$

Taking $\mathbf{v} = \mathbf{y} - \mathbf{z}$, we obtain in particular

$$\begin{aligned} \int_Q \Big(\sigma \frac{\partial (\mathbf{y} - \mathbf{z})}{\partial t} \cdot (\mathbf{y} - \mathbf{z}) + (\mathcal{F}(x, \operatorname{curl} \mathbf{y}) - \mathcal{F}(x, \operatorname{curl} \mathbf{z})) \cdot \operatorname{curl}(\mathbf{y} - \mathbf{z})\Big)\, dxdt \\ = \int_Q (\mathbf{f} - \mathbf{g}) \cdot (\mathbf{y} - \mathbf{z})\, dxdt. \end{aligned}$$

Integrating by parts in time, we finally obtain

$$\begin{aligned} \sigma \int_\Omega |(\mathbf{y} - \mathbf{z})(T)|^2\, dx + \int_Q (\mathcal{F}(x, \operatorname{curl} \mathbf{y}) - \mathcal{F}(x, \operatorname{curl} \mathbf{z})) \cdot \operatorname{curl}(\mathbf{y} - \mathbf{z})\, dxdt \\ = \int_Q (\mathbf{f} - \mathbf{g}) \cdot (\mathbf{y} - \mathbf{z})\, dxdt + \sigma \int_\Omega |\mathbf{y}_0 - \mathbf{z}_0|^2\, dx. \end{aligned}$$

By (2.5), we deduce that

$$\begin{aligned} \frac{\sigma}{2} \int_\Omega |(\mathbf{y} - \mathbf{z})(T)|^2\, dx + \underline{\nu} \int_Q |\operatorname{curl}(\mathbf{y} - \mathbf{z})|^2\, dxdt \\ \leq \int_Q (\mathbf{f} - \mathbf{g}) \cdot (\mathbf{y} - \mathbf{z})\, dxdt + \frac{\sigma}{2} \int_\Omega |\mathbf{y}_0 - \mathbf{z}_0|^2\, dx. \end{aligned}$$

Dropping the first term of this left-hand side, we get

$$\begin{aligned} \underline{\nu}\, \|\operatorname{curl}(\mathbf{y} - \mathbf{z})\|_Q^2 &\leq \|\mathbf{f} - \mathbf{g}\|_Q\, \|\mathbf{y} - \mathbf{z}\|_Q + \frac{\sigma}{2}\, \|\mathbf{y}_0 - \mathbf{z}_0\|^2 \\ &\lesssim \|\mathbf{f} - \mathbf{g}\|_Q\, \|\operatorname{curl}(\mathbf{y} - \mathbf{z})\|_Q + \|\mathbf{y}_0 - \mathbf{z}_0\|^2, \end{aligned}$$

where (2.10) was used in this last estimate. Applying Young's inequality, we find

$$\|\operatorname{curl}(\mathbf{y} - \mathbf{z})\|_Q^2 \lesssim \|\mathbf{f} - \mathbf{g}\|_Q^2 + \|\mathbf{y}_0 - \mathbf{z}_0\|^2. \tag{2.16}$$

Again by (2.10), the estimate (2.16) yields

$$\|\mathbf{y}-\mathbf{z}\|_Q^2 \lesssim \|\mathbf{f}-\mathbf{g}\|_Q^2 + \|\mathbf{y}_0-\mathbf{z}_0\|^2. \tag{2.17}$$

These two estimates prove (2.13). □

Corollary 2.9. *Under the assumptions of Lemma 2.8, we have*

$$\|\mathbf{y}-\mathbf{z}\|_{\mathbf{W}(0,T)} \lesssim \|\mathbf{f}-\mathbf{g}\|_Q + \|\mathbf{y}_0-\mathbf{z}_0\|. \tag{2.18}$$

Proof. In (2.15), using (2.6) and Cauchy-Schwarz's inequality we find that

$$\left|\int_Q \sigma \frac{\partial(\mathbf{y}-\mathbf{z})}{\partial t}\cdot \mathbf{v}\,dxdt\right| \le (L\|\operatorname{curl}(\mathbf{y}-\mathbf{z})\|_Q + \|\mathbf{f}-\mathbf{g}\|_Q)\|\operatorname{curl}\mathbf{v}\|_Q.$$

Hence by the estimate (2.13), we conclude that

$$\left|\int_Q \sigma \frac{\partial(\mathbf{y}-\mathbf{z})}{\partial t}\cdot \mathbf{v}\,dxdt\right| \lesssim (\|\mathbf{f}-\mathbf{g}\|_Q + \|\mathbf{y}_0-\mathbf{z}_0\|_Q)\|\mathbf{v}\|_{\mathbf{L}^2(0,T;\mathbf{V})}.$$

The conclusion directly follows. □

At this stage, we are able to define the notion of weak solutions.

Definition 2.10. For $T>0$, $\mathbf{y}_0\in\mathcal{H}$ and $f\in\mathbf{L}^2(0,T;\mathcal{H})$, we say that $\mathbf{y}:[0,T]\to\mathbf{L}^2(\Omega)^3$ is a weak solution of problem (1.1) if $\mathbf{y}\in\mathbf{W}(0,T)$ satisfies

$$\int_0^T \left\langle \sigma\frac{\partial\mathbf{y}}{\partial t}(\cdot,t)\,,\,\mathbf{z}\right\rangle dt + (\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y})\,,\,\operatorname{curl}\mathbf{z})_Q = (\mathbf{f}\,,\,\mathbf{z})_Q, \quad \forall z\in\mathbf{L}^2(0,T;\mathbf{V})$$

as well as

$$\mathbf{y}(\cdot,0)=\mathbf{y}_0 \text{ in } \Omega.$$

Theorem 2.11. *Let $T>0$ be fixed and assume that $\mathbf{y}_0\in\mathcal{H}$ and $\mathbf{f}\in\mathbf{L}^2(0,T;\mathcal{H})$. Then problem (1.1) has a unique weak solution $\mathbf{y}$ that satisfies*

$$\|\mathbf{y}\|_{\mathbf{W}(0,T)} \lesssim \|\mathbf{f}\|_Q + \|\mathbf{y}_0\|. \tag{2.19}$$

Proof. By Cororally 2.5, there exists a sequence $\{\mathbf{y}_{0,n}\}$ in $D(A)$ such that

$$\mathbf{y}_{0,n}\to\mathbf{y}_0 \text{ in } \mathcal{H}, \text{ as } n\to\infty.$$

Fix another sequence $\{\mathbf{f}_n\}\subset C_0^\infty((0,T),\mathbf{V}), n\in\mathbb{N}$, such that

$$\mathbf{f}_n\to\mathbf{f} \text{ in } \mathbf{L}^2(0,T;\mathbf{V}) \text{ as } n\to\infty.$$

Then by Theorem 2.7, for all $n\in\mathbb{N}$, problem (1.1) with right-hand side $\mathbf{f}_n$ and an initial datum $\mathbf{y}_{0,n}$ has a unique strong solution $\mathbf{y}_n$. Further, owing to Lemma 2.8 and its Corollary 2.9 applied to $\mathbf{y}_n$ and $\mathbf{y}_m$, we have

$$\|\mathbf{y}_n-\mathbf{y}_m\|_{\mathbf{W}(0,T)} \lesssim \|\mathbf{f}_n-\mathbf{f}_m\|_Q + \|\mathbf{y}_{0,n}-\mathbf{y}_{0,m}\|. \tag{2.20}$$

Hence there exists $\mathbf{y}\in\mathbf{W}(0,T)$ such that

$$\mathbf{y}_n\to\mathbf{y} \text{ in } \mathbf{W}(0,T), \text{ as } n\to\infty. \tag{2.21}$$

Finally by (2.6), we have

$$|\mathcal{F}(x,\operatorname{curl}\mathbf{y}_n)-\mathcal{F}(x,\operatorname{curl}\mathbf{y})| \le L|\operatorname{curl}\mathbf{y}_n-\operatorname{curl}\mathbf{y}|, \tag{2.22}$$

and therefore $\mathcal{F}(x,\operatorname{curl}\mathbf{y}_n)$ converges to $\mathcal{F}(x,\operatorname{curl}\mathbf{y})$ in $\mathbf{L}^2(Q)$.

Starting from (2.14) satisfied by $\mathbf{y}_n$ and passing to the limit, we find that $\mathbf{y}$ satisfies (2.19). Finally the estimate (2.19) follows from (2.18) with $\mathbf{y}=\mathbf{y}_n$ and $\mathbf{z}=\mathbf{0}$, $\mathbf{g}=0$, $\mathbf{z}_0=0$. □

3. **Optimal control.** We will discuss the following optimal control problem that is defined upon the state equation (1.1). We consider the objective functional

$$J(\mathbf{y},\mathbf{f}) := \frac{\lambda_T}{2}\int_\Omega |\mathbf{y}(x,T)-\mathbf{y}_T(x)|^2\,dx + \frac{\lambda_Q}{2}\int_Q |\mathbf{y}(x,t)-\mathbf{y}_Q(x,t)|^2\,dxdt + \frac{\lambda_f}{2}\int_Q |\mathbf{f}(x,t)|^2\,dxdt, \tag{3.1}$$

where λ_T, λ_Q, and λ_f are nonnegative constants with $\lambda_T+\lambda_Q>0$, while $\mathbf{y}_Q \in \mathbf{L}^2(\mathrm{div}{=}0,Q)$ and $\mathbf{y}_T \in \mathbf{H}(\mathrm{div}{=}0,\Omega)$ are given functions. Here, the control function $\mathbf{f}$ stands for a distributed current density $\mathbf{f}:\Omega\times[0,T]\to\mathbb{R}^3$.

The optimal control problem is

$$\text{(OCP)} \qquad\qquad \min_{\mathbf{f}\in\mathbf{F}_{ad}} J(\mathbf{y}_\mathbf{f},\mathbf{f}),$$

where $\mathbf{y}_\mathbf{f}$ denotes the solution of the equation (1.1) associated with the control $\mathbf{f}$, and the set of admissible controls $\mathbf{F}_{ad}\subset\mathbf{L}^2(0,T;\mathcal{H})$ is assumed to be non-empty, convex and closed.

Lemma 3.1. *The control-to-state mapping $G:\mathbf{f}\mapsto\mathbf{y}_\mathbf{f}$ for the equation (1.1) is weakly-strongly continuous from $\mathbf{L}^2(\mathrm{div}=0,Q)$ to $\mathbf{L}^2(0,T;\mathbf{V})$. This means that $\mathbf{f}_n \rightharpoonup \mathbf{f}$ (weakly) in $\mathbf{L}^2(\mathrm{div}{=}0,Q)$ implies $\mathbf{y}_{\mathbf{f}_n}\to\mathbf{y}_\mathbf{f}$ (strongly) in $\mathbf{L}^2(0,T;\mathbf{V})$.*

Proof. We begin with the variational formulation for weak solutions of (1.1) and write for short $\mathbf{y}_n := \mathbf{y}_{\mathbf{f}_n}$,

$$\int_0^T \left\langle \sigma\frac{\partial \mathbf{y}_n}{\partial t}\,,\,\mathbf{v}\right\rangle dt + \int_Q \mathcal{F}(x,\mathrm{curl}\,\mathbf{y}_n)\cdot \mathrm{curl}\,\mathbf{v}\,dxdt = \int_Q \mathbf{f}_n\cdot\mathbf{v}\,dxdt \qquad \forall \mathbf{v}\in\mathbf{L}^2(0,T;\mathbf{V}). \tag{3.2}$$

The sequence $\{\mathbf{f}_n\}$ is bounded in $\mathbf{L}^2(0,T;\mathcal{H})$, hence, by Theorem 2.11, the sequence $\{\mathbf{y}_n\}$ is bounded in $\mathbf{W}(0,T)$ so that we can assume $\mathbf{y}_n\rightharpoonup\mathbf{y}$ in $\mathbf{W}(0,T)$ with some $\mathbf{y}\in\mathbf{W}(0,T)$. The embedding $\mathbf{W}(0,T)\subset\mathbf{L}^2(0,T;\mathcal{H})$ is compact so that, after extracting a subsequence again, we can also assume the strong convergence $\mathbf{y}_n\to\mathbf{y}$ in $\mathbf{L}^2(0,T;\mathcal{H})$.

Inserting $\mathbf{y}_n-\mathbf{y}$ as test function in the weak formulation for $\mathbf{y}_n$, we get

$$\int_0^T \left\langle \sigma\frac{\partial \mathbf{y}_n}{\partial t}\,,\,\mathbf{y}_n-\mathbf{y}\right\rangle dt + \int_Q \mathcal{F}(x,\mathrm{curl}\,\mathbf{y}_n)\cdot \mathrm{curl}(\mathbf{y}_n-\mathbf{y})\,dxdt = \int_Q \mathbf{f}_n\cdot(\mathbf{y}_n-\mathbf{y})\,dxdt \quad \forall n\in\mathbb{N}.$$

Adding suitable terms to both sides, we proceed by

$$\begin{aligned}\int_0^T \left\langle \sigma\frac{\partial (\mathbf{y}_n-\mathbf{y})}{\partial t}\,,\,\mathbf{y}_n-\mathbf{y}\right\rangle dt &+ \int_Q (\mathcal{F}(x,\mathrm{curl}\,\mathbf{y}_n)-\mathcal{F}(\cdot,\mathrm{curl}\,\mathbf{y}))\cdot \mathrm{curl}(\mathbf{y}_n-\mathbf{y})\,dxdt \\ &= \int_Q \mathbf{f}_n\cdot(\mathbf{y}_n-\mathbf{y})\,dxdt - \int_0^T \left\langle \sigma\frac{\partial \mathbf{y}}{\partial t},\mathbf{y}_n-\mathbf{y}\right\rangle dt \\ &\quad - \int_Q \mathcal{F}(\cdot,\mathrm{curl}\,\mathbf{y})\cdot\mathrm{curl}(\mathbf{y}_n-\mathbf{y})\,dxdt.\end{aligned} \tag{3.3}$$

The first integral of the right-hand side of (3.3) tends to zero, since $\{\mathbf{y}_n\}$ converges strongly to $\mathbf{y}$ in $\mathbf{L}^2(0,T;\mathcal{H})$ and $\{\mathbf{f}_n\}$ is bounded. The second integral converges to zero, because $\{\mathbf{y}_n\}$ converges weakly in $\mathbf{W}(0,T)$, hence also weakly in $\mathbf{L}^2(0,T;\mathbf{V})$. Also the third integral tends to zero, since $\{\operatorname{curl}\mathbf{y}_n\}$ converges weakly to $\operatorname{curl}\mathbf{y}$ in $\mathbf{L}^2(Q)$. In view of all this, the right-hand side of (3.3) converges to zero, hence this holds also for the left-hand side.

We have already proved before that

$$\int_0^T \left\langle \sigma\frac{\partial(\mathbf{y}_n-\mathbf{y})}{\partial t}\,,\,\mathbf{y}_n-\mathbf{y}\right\rangle dt + \int_Q (\mathcal{F}(x,\operatorname{curl}\mathbf{y}_n)-\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y}))\cdot\operatorname{curl}(\mathbf{y}_n-\mathbf{y})\,dxdt \gtrsim \int_Q |\operatorname{curl}(\mathbf{y}_n-\mathbf{y})|^2\,dxdt,$$

hence we deduce that $\{\mathbf{y}_n\}$ converges to $\mathbf{y}$ in $\mathbf{L}^2(0,T;\mathbf{V})$.

Reminding (2.22) we can pass to the limit in (3.2) to find

$$\int_0^T \left\langle \sigma\frac{\partial\mathbf{y}}{\partial t}\,,\,\mathbf{v}\right\rangle dt + \int_Q \mathcal{F}(x,\operatorname{curl}\mathbf{y}_n)\cdot\operatorname{curl}\mathbf{v}\,dxdt = \int_Q \mathbf{f}\cdot\mathbf{v}\,dxdt \quad \forall\mathbf{v}\in\mathbf{L}^2(0,T;\mathbf{V}),$$

so that $\mathbf{y}$ is a weak solution associated with $\mathbf{f}$. By uniqueness, we have $\mathbf{y}=\mathbf{y}_f$, and thus $\mathbf{y}_n=\mathbf{y}_{\mathbf{f}_n}\to\mathbf{y}_f$ (strongly) in $\mathbf{L}^2(0,T;\mathbf{V})$ as $n\to\infty$. □

Let us next prove that our optimal control problem is well-posed, i.e. that there exists at least one optimal control. Thanks to Lemma 2.8, we know that the control-to-state mapping $G:\mathbf{f}\to\mathbf{y_f}$ is continuous from $\mathbf{L}^2(\text{div=}0,Q)\supset\mathbf{F}_{ad}$ to $\mathbf{W}(0,T)$.

Theorem 3.2 (Existence of an optimal control)**.** *If, in addition to the former assumptions, $\mathbf{F}_{ad}$ is also bounded or if λ_f is positive, then the optimal control problem* (OCP) *admits at least one optimal control $\bar{\mathbf{f}}\in\mathbf{F}_{ad}$.*

Proof. Let $\{\mathbf{f}_n\}_{n=1}^\infty\subset\mathbf{F}_{ad}$ be an infimal sequence, i.e. $J(\mathbf{y}_n,\mathbf{f}_n)=J(G(\mathbf{f}_n),\mathbf{f}_n)\to j$, as $n\to\infty$, where

$$j=\inf_{\mathbf{f}\in\mathbf{F}_{ad}} J(G(\mathbf{f}),\mathbf{f}).$$

If $\mathbf{F}_{ad}$ is bounded, then its closedness and convexity imply that $\mathbf{F}_{ad}$ is weakly sequentially compact. Therefore, we can assume w.l.o.g. that $\mathbf{f}_n\rightharpoonup\bar{\mathbf{f}}$ in $\mathbf{L}^2(0,T;\mathcal{H})$. Thanks to Lemma 3.1, we have $\mathbf{y}_n\to\mathbf{y}_{\bar{f}}$ in $\mathbf{L}^2(0,T;\mathbf{V})$. Moreover, an inspection of the proof of this Lemma shows that $\mathbf{y}_n\rightharpoonup\bar{\mathbf{y}}:=\mathbf{y}_{\bar{f}}$ in $\mathbf{W}(0,T)$, hence $\mathbf{y}_n(T)\rightharpoonup\bar{\mathbf{y}}(T)$ in $\mathbf{L}^2(\Omega)$. The lower semicontinuity of the functional J finally yields that

$$j=\liminf_{n\to\infty}J(\mathbf{y}_n,\mathbf{f}_n)\geq J(\bar{\mathbf{y}},\bar{\mathbf{f}}),$$

and hence $\bar{\mathbf{f}}$ is an optimal control.

Let now λ_f be positive. The infimum j satisfies $j\leq J(\mathbf{y_0},\mathbf{0})$, hence we can restrict the search for an optimal solution to the set of all controls $\mathbf{f}$ with $J(\mathbf{y_f},\mathbf{f})\leq J(\mathbf{y_0},\mathbf{0})$. By

$$\frac{\lambda_f}{2}\|\mathbf{f}\|^2\leq J(\mathbf{y_f},\mathbf{f})\leq J(\mathbf{y_0},\mathbf{0}),$$

an optimal solution must belong to the set $\mathbf{F}_{ad}\cap\{\mathbf{f}\in\mathbf{L}^2(0,T;\mathcal{H}):\|\mathbf{f}\|^2\leq 2(\lambda_f)^{-1}J(\mathbf{y_0},\mathbf{0})\}$. Again, this is a weakly sequentially compact set so that the proof can be finished in the same way as above. □

3.1. **Necessary optimality conditions.**

3.1.1. *Differentiability of the control-to-state mapping.* For necessary optimality conditions, we first have to discuss the differentiability of the control-to-state mapping G. To this aim, following [19], we require some more assumptions.

Assumption 3.3 (Differentiability)**.** For almost all $x \in \Omega$, the mapping $\mathbb{R}^3 \ni \mathbf{s} \mapsto \mathcal{F}(x,\mathbf{s}) \in \mathbb{R}^3$ is continuously differentiable with respect to $\mathbf{s}$.

Remark 3.4. Continuous differentiability of the mapping $\mathbf{s} \mapsto \mathcal{N}(x,\mathbf{s})$ is sufficient for Assumption 3.3. In this case, we write

$$\nabla_{\mathbf{s}}\mathcal{N}(x,\mathbf{s}) := \left[\frac{\partial \mathcal{N}}{\partial \mathbf{s}}(x,\mathbf{s})\right]^\top .$$

Note that by (2.6), the Jacobian matrix $\partial\mathcal{F}(x,\mathbf{s})/\partial\mathbf{s}$ satisfies

$$\left|\frac{\partial \mathcal{F}_i}{\partial s_j}(x,\mathbf{s})\right| \le L \text{ for a.a. } x \in \Omega, \quad \forall \mathbf{s} \in \mathbb{R}^3,\ i,\ j \in \{1,2,3\}. \tag{3.4}$$

Lemma 3.5. *The mapping G is weakly Gâteaux differentiable from $\mathbf{L}^2(\text{div}{=}0,Q)$ to $\mathbf{W}(0,T)$ and (strongly) Gâteaux differentiable from $\mathbf{L}^2(\text{div}{=}0,Q)$ to $\mathbf{L}^2(0,T;\mathcal{H})$. The weak derivative $\mathbf{z} = G'(\mathbf{f})\mathbf{h}$ at $\mathbf{f} \in \mathbf{F}_{ad}$ in the direction $\mathbf{h} \in \mathbf{L}^2(\text{div}{=}0,Q)$ is the unique weak solution in $\mathbf{W}(0,T)$ to the problem*

$$\begin{aligned} \sigma\frac{\partial \mathbf{z}}{\partial t} + \operatorname{curl}\left(\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot,\operatorname{curl}\mathbf{y}_\mathbf{f})\operatorname{curl}\mathbf{z}\right) &= \mathbf{h} \\ \mathbf{z}(0) &= 0. \end{aligned} \tag{3.5}$$

Proof. We first mention that the weak solution $z \in \mathbf{W}(0,T)$ of (3.5) is unique. This is a consequence of the inequality

$$\mathbf{s}^\top \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(x,\operatorname{curl}\mathbf{y}_\mathbf{f}(x,t))\,\mathbf{s} \ge \underline{\nu}\,|\mathbf{s}|^2 \quad \forall \mathbf{s} \in \mathbb{R}^3, \text{ for a.a. } (x,t) \in Q \tag{3.6}$$

that follows from [19], Proposition 3.7.

Now we select a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of nonzero real numbers tending to zero and consider the solutions $\mathbf{y} := \mathbf{y}_\mathbf{f}$ and $\mathbf{y}_{\tau_n} := \mathbf{y}_{\mathbf{f}+\tau_n\mathbf{h}}$ associated with the controls $\mathbf{f}$ and $\mathbf{f}+\tau_n\mathbf{h}$, respectively. Let us write for convenience $\mathbf{y}_n := \mathbf{y}_{\tau_n}$. Subtracting the state equations for $\mathbf{y}$ and $\mathbf{y}_\mathbf{f}$ (written in strong form), we find

$$\sigma\,\frac{\partial(\mathbf{y}_n - \mathbf{y})}{\partial t} + \operatorname{curl}\big(\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y}_n) - \mathcal{F}(\cdot,\operatorname{curl}\mathbf{y})\big) = \tau_n\,\mathbf{h}. \tag{3.7}$$

Testing the variational formulation with $\mathbf{y}_n - \mathbf{y}$ yields

$$\begin{aligned} &\int_0^T \left\langle \sigma\frac{\partial(\mathbf{y}_n - \mathbf{y})}{\partial t}, \mathbf{y}_n - \mathbf{y}\right\rangle dt \\ &\qquad + \int_Q (\mathcal{F}(\cdot,\operatorname{curl}\mathbf{y}_n) - \mathcal{F}(\cdot,\operatorname{curl}\mathbf{y}))\cdot \operatorname{curl}(\mathbf{y}_n - \mathbf{y})\,dxdt \\ &\qquad\qquad = \int_Q \tau_n\,\mathbf{h}\cdot(\mathbf{y}_n - \mathbf{y})\,dxdt, \end{aligned}$$

and hence, by (2.5),

$$\|\mathbf{y}_n(T) - \mathbf{y}(T)\|^2 + \|\operatorname{curl}(\mathbf{y}_n - \mathbf{y})\|_Q^2 \lesssim \tau_n\,\|\mathbf{h}\|_Q\|\mathbf{y}_n - \mathbf{y}\|_Q. \tag{3.8}$$

As in the proof of Lemma 3.1, this yields

$$\|\operatorname{curl}(\mathbf{y}_n - \mathbf{y})\|_Q \lesssim \tau_n\|\mathbf{h}\|_Q.$$

We already know that $\|\mathbf{v}\|_Q \lesssim \|\operatorname{curl}\mathbf{v}\|_Q$, hence

$$\|\mathbf{y}_n - \mathbf{y}\|_{\mathbf{L}^2(0,T;\mathbf{V})} \lesssim \tau_n \|\mathbf{h}\|_Q \tag{3.9}$$

and therefore

$$\|\mathbf{y}_n - \mathbf{y}\|_Q + \|\operatorname{curl}(\mathbf{y}_n - \mathbf{y})\|_Q \to 0, \quad n \to \infty. \tag{3.10}$$

By the differential equation (3.7), we have

$$\int_0^T \left\langle \sigma \frac{\partial(\mathbf{y}_n - \mathbf{y})}{\partial t}, \mathbf{v} \right\rangle dt = -\int_Q (\mathcal{F}(\cdot, \operatorname{curl}\mathbf{y}_n) - \mathcal{F}(\cdot, \operatorname{curl}\mathbf{y}))\cdot \operatorname{curl}\mathbf{v}\, dxdt$$
$$+ \int_Q \tau_n\, \mathbf{h}\cdot\mathbf{v}\, dxdt, \forall \mathbf{v} \in \mathbf{V}.$$

From (3.9), we obtain

$$\|\mathbf{y}_n - \mathbf{y}\|_{\mathbf{L}^2(0,T;\mathbf{V}')} \lesssim \tau_n \|\mathbf{h}\|_Q,$$

and therefore

$$\|\mathbf{y}_n - \mathbf{y}\|_{\mathbf{W}(0,T)} \lesssim \tau_n \|\mathbf{h}\|_Q.$$

Hence, after dividing by τ_n,

$$\left\| \frac{\mathbf{y}_n - \mathbf{y}}{\tau_n} \right\|_{\mathbf{W}(0,T)} \lesssim \|\mathbf{h}\|_Q.$$

In view of this boundedness, we can assume, possibly after selecting a subsequence, that $(\mathbf{y}_n - \mathbf{y})/\tau_n \rightharpoonup \mathbf{z} \in \mathbf{W}(0,T)$, $n \to \infty$. Let us proceed further using the strong formulation of the associated pde. After dividing the pde by τ_n, we have

$$\sigma \frac{\partial}{\partial t} \frac{\mathbf{y}_n - \mathbf{y}}{\tau_n} + \operatorname{curl}\left(\frac{\mathcal{F}(\cdot, \operatorname{curl}\mathbf{y}_n) - \mathcal{F}(\cdot, \operatorname{curl}\mathbf{y})}{\tau_n} \right) = \mathbf{h},$$

and, by the mean value theorem in integral form,

$$\sigma \frac{\partial}{\partial t} \frac{\mathbf{y}_n - \mathbf{y}}{\tau_n} + \operatorname{curl}\left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}(\mathbf{y} + \vartheta(\mathbf{y}_n - \mathbf{y}))\, d\vartheta \cdot \frac{\operatorname{curl}(\mathbf{y}_n - \mathbf{y})}{\tau_n} \right) = \mathbf{h},$$

hence

$$\sigma \frac{\partial}{\partial t} \frac{\mathbf{y}_n - \mathbf{y}}{\tau_n} + \operatorname{curl}\left(\left(\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}\mathbf{y}) + \mathbf{r}_n \right) \cdot \frac{\operatorname{curl}(\mathbf{y}_n - \mathbf{y})}{\tau_n} \right) = \mathbf{h}, \tag{3.11}$$

where

$$\mathbf{r}_n = \int_0^1 \left[\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}(\mathbf{y} + \vartheta(\mathbf{y}_n - \mathbf{y})) - \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}\mathbf{y}) \right] d\vartheta.$$

Re-arranging the variational formulation of (3.11), we find

$$\int_0^T \sigma \left\langle \frac{\partial}{\partial t} \frac{\mathbf{y}_n - \mathbf{y}}{\tau_n}, \mathbf{v} \right\rangle dt + \int_Q \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}\mathbf{y}) \frac{\operatorname{curl}(\mathbf{y}_n - \mathbf{y})}{\tau_n} \cdot \operatorname{curl}\mathbf{v}\, dxdt$$
$$+ \int_Q \frac{\operatorname{curl}(\mathbf{y}_n - \mathbf{y})}{\tau_n} \cdot \mathbf{r}_n^\top \operatorname{curl}\mathbf{v}\, dxdt = \int_Q \mathbf{h}\cdot\mathbf{v}\, dxdt \quad \forall \mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{V}).$$

We will confirm below that $\mathbf{r}_n^\top \operatorname{curl}\mathbf{v} \to 0$ in $\mathbf{L}^2(Q)$, $n \to \infty$, up to a subsequence. Since $\tau_n^{-1}(\mathbf{y}_n - \mathbf{y}) \rightharpoonup \mathbf{z}$ in $\mathbf{W}(0,T)$, we can pass to the limit and obtain

$$\int_0^T \left\langle \sigma \frac{\partial \mathbf{z}}{\partial t}, \mathbf{v} \right\rangle dt + \int_Q \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl}\mathbf{y}) \operatorname{curl} z \cdot \operatorname{curl}\mathbf{v}\, dxdt$$
$$= \int_Q \mathbf{h}\cdot\mathbf{v}\, dxdt \quad \forall \mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{V}).$$

A simple inspection of the proof reveals that we have obtained a little bit more: Any subsequence of $\{\tau_n^{-1}(\mathbf{y}_n - \mathbf{y})\}$ contains a weakly convergent subsequence, and all these subsequences converge weakly to the same limit z. Notice that z is a solution of the linearized equation and hence is unique. Therefore, the whole sequence $\{\tau_n^{-1}(\mathbf{y}_n - \mathbf{y})\}$ converges weakly and we have proven the desired result of weak Gâteaux-differentiability.

Moreover, since the embedding of $\mathbf{W}(0,T)$ in $\mathbf{L}^2(0,T;\mathcal{H})$ is compact, we even know that all subsequences of $\{\tau_n^{-1}(\mathbf{y}_n - \mathbf{y})\}$ contain a subsequence that converges strongly in $\mathbf{L}^2(0,T;\mathcal{H})$, again with the same limit. This yields the result on strong Gâteaux differentiability in $\mathbf{L}^2(0,T;\mathcal{H})$.

It remains to confirm the strong convergence of a subsequence of $\mathbf{r}_n^\top \operatorname{curl} \mathbf{v}$ to $\mathbf{0}$ in $\mathbf{L}^2(Q)$. Thanks to the estimate (3.4), all entries of $\mathbf{r}_n^\top$ are functions of $L^\infty(Q)$. Moreover, we know that $\operatorname{curl} \mathbf{y}_n \to \operatorname{curl} \mathbf{y}$ in $\mathbf{L}^2(Q)$. Therefore, a subsequence $\{\operatorname{curl} \mathbf{y}_{n_k}\}_k$ tends to $\operatorname{curl} \mathbf{y}$ almost everywhere in Q. In view of this, and since $\mathcal{F}$ is continuously differentiable w.r. to $\mathbf{s}$, the entries of $\mathbf{r}_{n_k}$ converge to zero a.e. in Q. We have

$$\left|\mathbf{r}_{n_k}(x,t)^\top \operatorname{curl} \mathbf{v}(x,t)\right|^2 \le C\,|\operatorname{curl} \mathbf{v}(x,t)|^2 \qquad a.e. \text{ in } Q$$

with some $C > 0$. The right-hand side is integrable on Q. Now, the pointwise convergence of $\mathbf{r}_{n_k}$ along with Lebesgue's dominated convergence theorem ensure that $\|\mathbf{r}_{n_k}^\top \operatorname{curl} \mathbf{v}\|_{\mathbf{L}^2(Q)} \to 0$, $k \to \infty$. □

3.1.2. *Adjoint equation and necessary optimality conditions.* Let us first introduce the reduced objective functional

$$\hat{J}(\mathbf{f}) := J(\mathbf{y}_\mathbf{f}, \mathbf{f}). \tag{3.12}$$

It is well known that an optimal control $\bar{\mathbf{f}}$ minimizing $\hat{J}$ in $\mathbf{F}_{ad}$ has to obey the variational inequality

$$\hat{J}'(\bar{\mathbf{f}})(\mathbf{f} - \bar{\mathbf{f}}) \ge 0 \quad \forall \mathbf{f} \in \mathbf{F}_{ad}, \tag{3.13}$$

provided that $\hat{J}$ is (strongly) Gâteaux-differentiable at $\bar{\mathbf{f}}$. In our case, the mapping $\mathbf{f} \to \mathbf{y}_\mathbf{f}(T)$ is only weakly differentiable. Therefore, let us prove that $\hat{J}$ is Gâteaux-differentiable although the control-to-state mapping is only weakly Gâteaux-differentiable.

Lemma 3.6. *The mapping* $\mathbf{f} \mapsto \hat{J}(\mathbf{f})$ *is Gâteaux-differentiable.*

Proof. We have

$$\hat{J}(\mathbf{f}) = \frac{\lambda_T}{2}\|G(\mathbf{f})(T) - \mathbf{y}_T\|^2 + \frac{\lambda_Q}{2}\|G(\mathbf{f}) - \mathbf{y}_Q\|_Q^2 + \frac{\lambda_f}{2}\|\mathbf{f}\|_Q^2.$$

The second and the third part of $\hat{J}$ are obviously differentiable. Notice that the mapping $\mathbf{f} \to \mathbf{y}$, considered with range $\mathbf{L}^2(0,T;\mathcal{H})$, is (strongly) Gâteaux-differentiable by Lemma 3.5. Therefore, it suffices to prove the Lemma for the first term $\hat{J}_T(\mathbf{f})$,

$$\hat{J}_T(\mathbf{f}) := \frac{\lambda_T}{2}\|G(\mathbf{f})(T) - \mathbf{y}_T\|^2.$$

The mapping $G : \mathbf{f} \mapsto \mathbf{y}_\mathbf{f}$ is weakly Gâteaux-differentiable with range in $\mathbf{W}(0,T)$ by Lemma 3.5, and the mapping $\mathbf{y} \mapsto \mathbf{y}(T)$ is linear and continuous from $\mathbf{W}(0,T)$ to $\mathbf{L}^2(\Omega)$.

Consider the sequence $\mathbf{z}_n = (\mathbf{y}_n - \mathbf{y})/t_n$ that converges to some $\mathbf{z}$ weakly in $\mathbf{W}(0,T)$ as $t_n \to 0$, i.e.

$$\frac{1}{t_n}(G(\mathbf{f}+t_n\mathbf{h}) - G(\mathbf{f})) \rightharpoonup \mathbf{z}, \qquad \text{in } \mathbf{W}(0,T).$$

To compute $\hat{J}_T'(\mathbf{f})\mathbf{h}$, we consider

$$\begin{aligned}
&\frac{1}{t_n}(\hat{J}_T(\mathbf{f}+t_n\mathbf{h}) - \hat{J}_T(\mathbf{f})) = \frac{1}{t_n}(\|\mathbf{y}_n(T) - \mathbf{y}_T\|^2 - \|\mathbf{y}(T) - \mathbf{y}_T\|^2)\\
&= \frac{1}{t_n}\left(\mathbf{y}_n(T) - \mathbf{y}(T)\,,\, \mathbf{y}_n(T) + \mathbf{y}(T) - 2\,\mathbf{y}_T\right)\\
&= \Big(\frac{G(\mathbf{f}+t_n\mathbf{h})(T) - G(\mathbf{f})(T)}{t_n}\,,\, G(\mathbf{f}+t_n\mathbf{h})(T) + G(\mathbf{f})(T) - 2\mathbf{y}_T\Big).
\end{aligned}$$

The embedding $\mathbf{W}(0,T) \subset C([0,T],\mathcal{H})$ is linear and continuous. Moreover, the mapping $\mathbf{y} \mapsto \mathbf{y}(T)$ is linear and continuous from $\mathbf{W}(0,T)$ to $\mathbf{L}^2(\Omega)$. Let us denote the mapping $\mathbf{y} \mapsto \mathbf{y}(T)$ from $\mathbf{W}(0,T)$ to $\mathbf{L}^2(\Omega)$ by E_T. As a continuous linear mapping, E_T is also weakly continuous. We proceed by

$$\begin{aligned}
&\Big(\frac{G(\mathbf{f}+t_n\mathbf{h})(T) - G(\mathbf{f})(T)}{t_n}\,,\, G(\mathbf{f}+t_n\mathbf{h})(T) + G(\mathbf{f})(T) - 2\mathbf{y}_T\Big)\\
&= \Big(E_T\frac{G(\mathbf{f}+t_n\mathbf{h}) - G(\mathbf{f})}{t_n}\,,\, E_T(G(\mathbf{f}+t_n\mathbf{h}) + G(\mathbf{f})) - 2\mathbf{y}_T\Big)\\
&= \Big(E_T\,\mathbf{z}_n\,,\, E_T(G(\mathbf{f}+t_n\mathbf{h}) + G(\mathbf{f})) - 2\mathbf{y}_T\Big).
\end{aligned}$$

By $\mathbf{z}_n \rightharpoonup \mathbf{z}$ in $\mathbf{W}(0,T)$, the weak continuity of E_T and the strong convergence of $G(\mathbf{f}+t_n\mathbf{h})(T)$ to $G(\mathbf{f})(T)$ in $\mathbf{L}^2(\Omega)$, cf. (3.8), we can pass to the limit and find

$$\hat{J}_T'(\mathbf{f})\mathbf{h} = \Big(E_T\,\mathbf{z}\,,\, E_T(G(\mathbf{f}) + G(\mathbf{f})) - 2\mathbf{y}_T\Big) = 2\,\Big(\mathbf{z}(T)\,,\, \mathbf{y}(T) - \mathbf{y}_T\Big). \tag{3.14}$$

□

In view of (3.14), the derivative of $\hat{J}'$ at $\bar{\mathbf{f}}$ in the direction $\mathbf{h} \in \mathbf{L}^2(\mathrm{div}{=}0, Q)$ is given by

$$\begin{aligned}
\hat{J}'(\bar{\mathbf{f}})\mathbf{h} &= \lambda_T\left(G(\bar{\mathbf{f}})(T) - \mathbf{y}_T\,,\, (G'(\bar{\mathbf{f}})\mathbf{h})(T)\right)\\
&\qquad + \lambda_Q\left(G(\bar{\mathbf{f}}) - \mathbf{y}_Q\,,\, G'(\bar{\mathbf{f}})\mathbf{h}\right)_Q + \lambda_f\left(\bar{\mathbf{f}}\,,\, \mathbf{h}\right)_Q\\
&= \lambda_T\left(\bar{\mathbf{y}}(T) - \mathbf{y}_T\,,\, \mathbf{z}(T)\right) + \lambda_Q\left(\bar{\mathbf{y}} - \mathbf{y}_Q\,,\, \mathbf{z}\right)_Q + \lambda_f\left(\bar{\mathbf{f}}\,,\, \mathbf{h}\right)_Q,
\end{aligned} \tag{3.15}$$

where $\mathbf{z} = G'(\bar{\mathbf{f}})\mathbf{h}$ is the solution of the linearized equation (3.5).

By an adjoint equation, we are able to transform this expression to one, where the increment $\mathbf{h}$ appears explicitely.

Definition 3.7 (Adjoint equation)**.** Let $\mathbf{f} \in \mathbf{L}^2(\mathrm{div}{=}0, Q)$ be given and let $\mathbf{y}_\mathbf{f}$ be the associated state. Then the equation

$$\begin{aligned}
-\sigma\frac{\partial\boldsymbol{\varphi}}{\partial t} + \mathrm{curl}\left(\frac{\partial\mathcal{F}}{\partial\mathbf{s}}(\cdot,\mathrm{curl}\,\mathbf{y}_\mathbf{f})^\top\,\mathrm{curl}\,\boldsymbol{\varphi}\right) &= \lambda_Q\left(\mathbf{y}_\mathbf{f} - \mathbf{y}_Q\right)\\
\sigma\,\boldsymbol{\varphi}(T) &= \lambda_T\left(\mathbf{y}_\mathbf{f}(T) - \mathbf{y}_T\right)
\end{aligned} \tag{3.16}$$

is said to be the *adjoint equation* associated with $\mathbf{y}_\mathbf{f}$. The unique solution of (3.16) is called *adjoint state* associated with $\mathbf{f}$ and denoted by $\boldsymbol{\varphi}_\mathbf{f}$. Existence and uniqueness of $\boldsymbol{\varphi}_\mathbf{f}$ will be discussed below.

Let us briefly confirm that the adjoint equation has a unique solution $\boldsymbol{\varphi}$. This follows immediately by the estimate (3.6) that was used to prove the unique solvability of the linearized equation. Obviously, the same inequality is satisfied for the matrix $\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(x, \operatorname{curl} \mathbf{y_f}(x,t))^\top$, hence the differential operator in the adjoint equation (3.16) is coercive and the existence of a unique $\boldsymbol{\varphi}_\mathbf{f}$ of $\mathbf{W}(0,T)$ is a fairly standard conclusion.

Theorem 3.8. *If $\bar{\mathbf{f}}$ is optimal for the optimal control problem (3.1), then there exists a unique adjoint state $\boldsymbol{\varphi}_{\bar{\mathbf{f}}} \in \mathbf{W}(0,T)$ such that the variational inequality*

$$\int_Q (\boldsymbol{\varphi}_\mathbf{f}(x,t) + \lambda_f \, \bar{\mathbf{f}}(x,t)) \cdot (\mathbf{f}(x,t) - \bar{\mathbf{f}}(x,t)) \, dxdt \geq 0 \quad \forall \mathbf{f} \in \mathbf{F}_{ad} \tag{3.17}$$

is fulfilled.

Proof. First, we consider the linearized state equation (3.5) for $\mathbf{z}$, where $\mathbf{h} := \mathbf{f} - \bar{\mathbf{f}}$ is taken as right-hand side. We insert the adjoint state $\boldsymbol{\varphi}_{\bar{\mathbf{f}}} \in \mathbf{W}(0,T)$ associated with $\mathbf{f} := \bar{\mathbf{f}}$ and $\mathbf{y_f} := \mathbf{y}_{\bar{\mathbf{f}}}$ as test function in the weak formulation of the linearized equation. In this way, we obtain

$$\begin{aligned} \int_0^T \left\langle \sigma \frac{\partial \mathbf{z}}{\partial t} \, , \, \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \right\rangle dt + \int_Q \left(\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl} \mathbf{y}_{\bar{\mathbf{f}}}) \operatorname{curl} \mathbf{z} \right) \cdot \operatorname{curl} \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \, dxdt & \\ = \int_Q (\mathbf{f} - \bar{\mathbf{f}}) \cdot \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \, dxdt & \\ \mathbf{z}(0) = 0. & \end{aligned} \tag{3.18}$$

Moreover, we insert the solution $\mathbf{z} = G'(\bar{\mathbf{f}})\mathbf{h}$ as test function in the weak formulation of the adjoint equation (3.16). Then

$$\begin{aligned} - \int_0^T \left\langle \sigma \frac{\partial \boldsymbol{\varphi}_{\bar{\mathbf{f}}}}{\partial t} \, , \, \mathbf{z} \right\rangle dt + \int_Q \left(\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl} \mathbf{y}_{\bar{\mathbf{f}}})^\top \operatorname{curl} \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \right) \cdot \operatorname{curl} \mathbf{z} \, dxdt & \\ = \int_Q \lambda_Q \, (\bar{\mathbf{y}} - \mathbf{y}_Q) \cdot \mathbf{z} \, dxdt & \\ \sigma \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(T) = \lambda_T \, (\bar{\mathbf{y}}(T) - \mathbf{y}_T) & \end{aligned}$$

is found. Performing an integration by parts with respect to the time in (3.18) yields

$$\begin{aligned} (\mathbf{z}(T) \, , \, \sigma \, \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(T)) - \int_0^T \left\langle \sigma \frac{\partial \boldsymbol{\varphi}_{\bar{\mathbf{f}}}}{\partial t} \, , \, \mathbf{z} \right\rangle dt & \\ + \int_Q \left(\frac{\partial \mathcal{F}}{\partial \mathbf{s}}(\cdot, \operatorname{curl} \mathbf{y}_{\bar{\mathbf{f}}})^\top \operatorname{curl} \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \right) \cdot \operatorname{curl} \mathbf{z} \, dxdt = \int_Q (\mathbf{f} - \bar{\mathbf{f}}) \cdot \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \, dxdt. & \end{aligned}$$

Inserting the terminal condition $\sigma \, \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(T) = \lambda_T \, (\mathbf{y}_{\bar{\mathbf{f}}}(T) - \mathbf{y}_T)$, we arrive in view of the adjoint equation (3.16) at

$$(\mathbf{z}(T) \, , \, \lambda_T \, (\mathbf{y}_{\bar{\mathbf{f}}}(T) - \mathbf{y}_T)) + (\lambda_Q \, (\mathbf{y}_{\bar{\mathbf{f}}} - \mathbf{y}_Q) \, , \, \mathbf{z})_Q = \int_Q (\mathbf{f} - \bar{\mathbf{f}}) \cdot \boldsymbol{\varphi}_{\bar{\mathbf{f}}} \, dxdt$$

In view of the representation (3.15), this is equivalent to

$$\hat{J}(\bar{f})\mathbf{h} = \int_Q (\boldsymbol{\varphi}_{\bar{\mathbf{f}}} + \lambda_f \, \bar{\mathbf{f}}) \cdot (\mathbf{f} - \bar{\mathbf{f}}) \, dxdt.$$

The claim follows from the general variational inequality (3.13) and from our setting $\mathbf{h} = \mathbf{f} - \bar{\mathbf{f}}$. □

This variational inequality (3.17) is a quite general result that can be discussed in more detail for particular cases of $\mathbf{F}_{ad}$. We consider the following three particular cases:
(i) At first, the unconstrained case is of interest,

$$\mathbf{F}_{ad} = \mathbf{L}^2(0,T;\mathcal{H}). \tag{3.19}$$

(ii) Let us define for convenience the closed ball of $\mathbf{L}^2(\Omega)$ with radius $r > 0$,

$$\mathbf{B}_r(0) = \{\mathbf{v} \in \mathcal{H} : \|\mathbf{v}\| \le r\}.$$

We may also consider, for given $r > 0$, the admissible set

$$\mathbf{F}_{ad} = \{\mathbf{f} \in \mathbf{L}^2(0,T;\mathcal{H}) : \mathbf{f}(t) \in \mathbf{B}_r(0) \text{ for a.a. } t \in (0,T)\}. \tag{3.20}$$

(iii) Another interesting set is defined as follows: Fix $k \in \mathbb{N}$, functions $\mathbf{e}_i \in \mathcal{H}$ with $\mathbf{e}_i \neq 0$, real numbers $\alpha_i < \beta_i$, $i = 1, \dots, k$, and define

$$\begin{aligned} \mathbf{F}_{ad} = \Big\{ \mathbf{f} \in \mathbf{L}^2(0,T;\mathcal{H}) : \mathbf{f}(x,t) = \sum_{i=1}^{k} \mathbf{e}_i(x)\, u_i(t),\ u_i \in L^\infty(0,T), \\ \alpha_i \le u_i(t) \le \beta_i \ \text{ for a.a. } t \in (0,T),\ i = 1, \dots, k \Big\}. \end{aligned} \tag{3.21}$$

Let us detail the variational inequality (3.17) for these particular sets of admissible controls.

Case (i): If $\mathbf{F}_{ad}$ is given by (3.19), then obviously the variational inequality is equivalent to

$$\bar{\mathbf{f}}(x,t) = -\frac{1}{\lambda_f}\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(x,t) \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T), \tag{3.22}$$

provided that $\lambda_f > 0$. The positivity of λ_f is needed anyway to guarantee the solvability of the optimal control problem. For $\lambda_f = 0$, the problem is solvable, if the desired state is attainable.

Case (ii): If $\mathbf{F}_{ad}$ is defined by (3.20), then the variational inequality is equivalent to the pointwise inequality

$$(\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f\, \bar{\mathbf{f}}(t)\,,\, \mathbf{v} - \bar{\mathbf{f}}(t)) \ge 0 \qquad \forall \mathbf{v} \in \mathbf{B}_r(0).0$$

Therefore, $\bar{\mathbf{f}}(t)$ should be a negative multiple of $\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f\, \bar{\mathbf{f}}(t)$ with (maximal) norm r. For $\lambda_f = 0$, this implies

$$\bar{\mathbf{f}}(t) = -\frac{r}{\|\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(\cdot,t)\|}\, \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) \text{ for a.a. } t \text{ with } \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) \neq 0.$$

If $\lambda_f > 0$, then we find in a standard way for a.a. $t \in (0,T)$ that

$$\bar{\mathbf{f}}(t) = \begin{cases} -\dfrac{\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)}{\lambda_f} & \text{if } -\dfrac{\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)}{\lambda_f} \in \mathbf{B}_r(0), \\ -r\, \dfrac{\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)}{\|\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(\cdot,t)\|} & \text{if } -\dfrac{\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)}{\lambda_f} \notin \mathbf{B}_r(0). \end{cases}$$

In other words, we have almost everywhere

$$\bar{\mathbf{f}}(t) = \mathbb{P}_{B_r}\left(-\lambda_f^{-1}\, \boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)\right),$$

where $\mathbb{P}_{B_r} : \mathcal{H} \to \mathcal{H}$ denotes the $\mathbf{L}^2(\Omega)$-projection operator onto $\mathbf{B}_r(0)$.

Case (iii): Here, we expand the terms in the variational inequality as follows: (3.17) holds if and only if,

$$\sum_{i=1}^{k} (\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f \bar{\mathbf{f}}(t)\,,\,\mathbf{e}_i)(v_i - \bar{u}_i(t)) \geq 0 \quad \text{ for a.a. } t \in (0,T)$$

and for all real numbers v_i with $\alpha_i \leq v_i \leq \beta_i$, $i = 1, \ldots, k$. Obviously, this splits in k independent variational inequalities

$$(\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f \bar{\mathbf{f}}(t)\,,\,\mathbf{e}_i)(v - \bar{u}_i(t)) \geq 0 \qquad \forall v \in [\alpha_i, \beta_i], \tag{3.23}$$

$i = 1, \ldots, k$. Therefore, we have for all $i = 1, \ldots, k$

$$\bar{u}_i(t) = \begin{cases} \alpha_i & \text{if } (\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f \bar{\mathbf{f}}(t)\,,\,\mathbf{e}_i) > 0 \\ \beta_i & \text{if } (\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t) + \lambda_f \bar{\mathbf{f}}(t)\,,\,\mathbf{e}_i) < 0. \end{cases} \tag{3.24}$$

Under the additional assumption $\lambda_f > 0$ and

$$(\mathbf{e}_i\,,\,\mathbf{e}_j) = 0 \quad \text{ if } i \neq j,$$

we get

$$\bar{u}_i(t) = \mathbb{P}_{[\alpha_i,\beta_i]}\left(-\frac{1}{\lambda_f \|\mathbf{e}_i\|^2}(\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)\,,\,\mathbf{e}_i)\right). \tag{3.25}$$

Here, $\mathbb{P}_{[\alpha_i,\beta_i]} : \mathbb{R} \to [\alpha_i, \beta_i]$ is the projection mapping defined by

$$\mathbb{P}_{[\alpha_i,\beta_i]}(z) = \max(\alpha_i, \min(\beta_i, z)).$$

Let us briefly sketch this result: We insert the representation

$$\bar{\mathbf{f}}(t) = \sum_{j=1}^{k} \mathbf{e}_j\, \bar{u}_j(t)$$

in (3.23) and obtain, for $i = 1, \ldots, k$ and almost all $t \in (0,T)$

$$\Big((\boldsymbol{\varphi}_{\bar{\mathbf{f}}}(t)\,,\,\mathbf{e}_i) + \lambda_f \,\|\mathbf{e}_i\|^2\, \bar{u}_i(t)\Big)\,(v - \bar{u}_i(t)) \geq 0 \qquad \forall v \in [\alpha_i, \beta_i],$$

since the functions $\mathbf{e}_i$ were assumed to be mutually orthogonal. This last variational inequality is equivalent to relation (3.25). We refer to [16, Thm. 2.28] for an analogous situation.

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