ON CONVERGENCE OF REGULARIZATION METHODS FOR NONLINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS WITH CONTROL AND STATE CONSTRAINTS

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Abstract. Moreau-Yosida and Lavrentiev type regularization methods are considered for nonlinear optimal control problems governed by semilinear parabolic equations with bilateral pointwise control and state constraints. The convergence of optimal controls of the regularized problems is studied for regularization parameters tending to infinity or zero, respectively. In particular, the strong convergence of global and local solutions is addressed. Moreover, it is shown that, under certain assumptions, locally optimal solutions of the Lavrentiev regularized problems are locally unique. This analysis is based on a second-order sufficient optimality condition and a separation assumption on almost active sets.

Key words. optimal control, semilinear parabolic equation, pointwise state constraints, Moreau-Yosida regularization, Lavrentiev regularization, convergence, local uniqueness

1. Introduction.

In this paper, we consider a class of optimal control problems for parabolic partial differential equations, where pointwise constraints are imposed on the control and on the state. Problems of this type were discussed extensively in the recent past because of specific difficulties of their numerical analysis. Optimal control problems with pointwise state constraints are difficult in particular, because the associated Lagrange multipliers are measures.

Different regularization methods were proposed to deal with this specific difficulty. For instance, Ito and Kunisch [12] and Bergounioux, Ito and Kunisch [3] suggested a Moreau-Yosida type regularization method, where the pointwise state constraints are penalized by a standard quadratic penalty functional. Later, Meyer, Rösch and Tröltzsch [15] suggested a Lavrentiev type method, where the compact control-to-state mapping G is substituted by $\lambda I + G$ with a small regularization parameter λ, we refer also to the nonlinear setting in Meyer and Tröltzsch [16].

Both techniques have been discussed in detail in the literature, mainly for elliptic problems with linear state equation and quadratic objective functional. Special emphasis was laid on the convergence analysis for the regularization parameter tending to zero. Only a few contributions were devoted to this issue in the nonlinear case. We mention Meyer and Hinze [11], who discuss some related questions for the Lavrentiev type regularization in the semilinear elliptic case and Meyer and Yousept [17], who study the convergence of the Moreau-Yosida type regularization for a semilinear elliptic problem that arises from the control of the growth of SiC bulk single crystals.

In our paper, we investigate both regularization techniques for the control of semilinear parabolic equations with bilateral pointwise control and state constraints. First, we concentrate on the Moreau-Yosida type approach and discuss the strong convergence of globally optimal solutions under a certain convexity assumption on the objective functional. Next, we dicuss under which conditions locally optimal controls of the unregularized problem can be approximated by sequences of locally optimal controls of the regularized problems. Here, we concentrate on the Lavrentiev type

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regularization, although the same analysis would also work for the Moreau-Yosida technique, cf. [17].

In general, locally optimal solutions need not be locally unique. It is obvious that local uniqueness is an important requirement for the convergence of numerical optimization algorithms. We study this problem for the parabolic case in the second part of the paper. Here, certain second-order optimality conditions are needed. Therefore, we discuss this issue in the context of the Lavrentiev type regularization, because here the regularized problems contain only twice continuously Fréchet differentiable quantities.

2. Parabolic optimal control problems with pointwise control and state constraints. In this paper, we are concerned with the analysis of a class of control and state constrained optimal control problems governed by parabolic PDEs. We consider the nonlinear control problem

$$(P) \qquad \text{Minimize } J(y,u) = \int_Q L(x,t,y,u)\,dxdt$$

subject to

$$\begin{aligned} y_t + Ay + d(x,t,y) &= u && \text{in } Q \\ y(\cdot,0) &= 0 && \text{in } \Omega \\ \partial_A y + \alpha y &= 0 && \text{on } \Sigma, \end{aligned}$$

$$\begin{aligned} u_a \le \ & u \le u_b && \text{in } Q \\ y_a \le \ & y \le y_b && \text{in } Q. \end{aligned}$$

In this setting, $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded domain which has $C^{1,1}$ boundary Γ if $N > 1$. For a fixed time $T > 0$ we denote by $Q := \Omega \times (0,T)$ the space-time-domain with boundary $\Sigma = \Gamma \times (0,T)$. Moreover, functions $L : Q \times \mathbb{R}^2 \to \mathbb{R}$, $d : Q \times \mathbb{R} \to \mathbb{R}$ are given. We consider bounds u_a, $u_b \in L^\infty(Q)$, $u_a \le u_b$ a.e. in Q, and $y_a, y_b \in C(\bar{Q})$, $y_a \le y_b$ in Q, such that $y_a(x,0) < 0 < y_b(x,0)$ holds for all $x \in \bar{\Omega}$.

A is a uniformly elliptic differential operator of the form

$$Ay(x) = -\sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x))$$

such that the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy

$$m_0\|\xi\|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega$$

for some $m_0 > 0$.

We will use the following notation: By $\|\cdot\|_\Omega$ we denote the usual norm in $L^2(\Omega)$, and $(\cdot,\cdot)_\Omega$ is the associated inner product. The norms and inner products in other L^2 spaces such as $L^2(Q)$ and $L^2(\Sigma)$ are denoted accordingly. Moreover, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norms and inner products in $L^2(Q)$. Also, $\langle\cdot,\cdot\rangle$ will denote the duality pairing in $C(\bar{Q})$ and $C(\bar{Q})^*$. The $L^\infty(Q)$-norm will be specified by $\|\cdot\|_\infty$. Norms, inner products, and duality pairings on subsets of Q will be denoted by indexing the appropriate set, e.g. $\|\cdot\|_M$ refers to the norm in $L^2(M)$. By ∂_A, the conormal derivative at Γ is denoted.

Throughout the paper, we rely on the following assumptions on the given setting.

ASSUMPTION 2.1. *The functions* $L = L(x,t,y,u) : Q \times \mathbb{R}^2 \to \mathbb{R}$ *and* $d = d(x,t,y) : Q \times \mathbb{R} \to \mathbb{R}$ *are measurable with respect to* $(x,t) \in Q$ *for all fixed* $(y,u) \in \mathbb{R}^2$ *or* $y \in \mathbb{R}$*, respectively, and twice continuously differentiable with respect to* (y,u) *or* y*, respectively, for almost all* $(x,t) \in Q$*. Moreover, for* $y = 0$ *they are bounded of order 2 with respect to* x*, i.e. for* d

$$\|d(\cdot,0)\|_\infty + \|\frac{\partial d}{\partial y}(\cdot,0)\|_\infty + \|\frac{\partial^2 d}{\partial y^2}(\cdot,0)\|_\infty \le C \tag{2.1}$$

is satisfied. Further, for a.a. $(x,t) \in Q$*, it holds that*

$$d_y(x,t,y) \ge 0.$$

The function L *is assumed to satisfy* (2.1) *with* $\|L(\cdot,0,0)\|_\infty$*,* $\|L'_{(y,u)}(\cdot,0,0)\|_\infty$*, and* $\|L''_{(y,u)}(\cdot,0,0)\|_\infty$*.*

Also, L *and* d *and their first- and second-order derivatives are uniformly Lipschitz on bounded sets, i.e. for all* $M > 0$ *there exists* $L_M > 0$ *such that* d *satisfies*

$$\begin{aligned}\|d(\cdot,y_1) - d(\cdot,y_2)\|_\infty + \|\frac{\partial d}{\partial y}(\cdot,y_1) - \frac{\partial d}{\partial y}(\cdot,y_2)\|_\infty \\ + \|\frac{\partial^2 d}{\partial y^2}(\cdot,y_1) - \frac{\partial^2 d}{\partial y^2}(\cdot,y_2)\|_\infty \le L_M|y_1 - y_2|.\end{aligned} \tag{2.2}$$

for all $y_i \in \mathbb{R}$ *with* $|y_i| \le M$*,* $i = 1,2$*. The function* L *has to satisfy* (2.2) *accordingly with respect to* (y_i,u_i) *instead of* y_i *for all* $|y_i| \le M$*,* $|u_i| \le M$*,* $i = 1,2$*.*

Moreover, the function L *is assumed to fulfill the Legendre-Clebsch condition*

$$\frac{\partial^2 L}{\partial u^2}(x,t,y,u) \ge \beta_0 > 0 \tag{2.3}$$

for almost all $(x,t) \in Q$*, all* $y \in \mathbb{R}$ *and all* $u \in [\inf\, ess\, u_a, \sup\, ess\, u_b]$*.*

Let us begin our analysis by discussing the PDEs governing (P).

THEOREM 2.2. *Under Assumption 2.1, the parabolic initial-boundary-value problem*

$$\begin{aligned} y_t + Ay + d(\cdot,y) &= f \\ y(\cdot,0) &= y_0 \\ \partial_A y + \alpha y &= g \end{aligned} \tag{2.4}$$

admits for every triple $(f,y_0,g) \in L^2(Q) \times L^2(\Omega) \times L^2(\Sigma)$ *a unique solution* $y \in W(0,T)$*. For more regular data* $(f,y_0,g) \in L^r(Q) \times C(\bar{\Omega}) \times L^s(\Sigma)$*,* $r > N/2 + 1$*,* $s > N+1$*, we obtain Hölder continuous solutions* $y \in W(0,T) \cap C^\nu(\bar{Q})$*, with some* $\nu \in (0,1)$*, where the space* $W(0,T)$ *is given by*

$$W(0,T) = \{y \in L^2(0,T,H^1(\Omega)) \mid y_t \in L^2(0,T,H^1(\Omega)^*)\}.$$

For the proof, we refer to [4] and to the results on Hölder continuity in [6].

For our further analysis we consider in particular problems with $y_0 \equiv 0$, $g \equiv 0$, and $f = u$, where u is a control satisfying $u_a \le u \le u_b$ almost everywhere in Q. For that reason, we introduce the following definition.

Definition 2.3. *We introduce the set of* admissible controls *for* (P) *by*

$$U_{ad} = \{u \in L^2(Q) \mid u_a(x,t) \le u(x,t) \le u_b(x,t) \ \text{a.e. in } Q\}.$$

Note that the admissible controls $u \in U_{ad}$ are automatically bounded since u_a and u_b are functions in L^∞, i.e. $U_{ad} \subset L^\infty(Q)$. Hence, by Theorem 2.2, the parabolic initial-boundary-value problem governing (P), admits for each $u \in U_{ad}$ a unique solution $y \in W(0,T) \cap C(\bar{Q})$. This allows us to introduce the control-to-state operator

$$\begin{aligned} G : L^2(Q) \cap U_{ad} &\rightarrow W(0,T) \cap C(\bar{Q}), \\ G : u &\mapsto y. \end{aligned}$$

Later, we will also consider G with range in $L^2(Q)$ wherever appropriate. For future reference, the next definitions will be helpful.

Definition 2.4. *We denote by*

$$U_{feas} = \{u \in U_{ad} \mid y_a(x,t) \le Gu(x,t) \le y_b(x,t) \ \text{in } Q\}$$

the set of feasible controls for (P).

In this paper, we will rely on separation conditions for the active sets, i.e. we will assume later that at most one of the bounds u_a, u_b, y_a, and y_b can be active at a time. For that purpose, we will define σ-active, or almost active, sets as in [20].

Definition 2.5. *Let* $\tilde{u}$ *be a fixed reference control with associated state* $\tilde{y} = G(\tilde{u})$ *and let* σ *be a positive real number. The* σ*-active sets of the control* $\tilde{u}$ *for problem* (P) *are given by*

$$\begin{aligned} M^\sigma_{u,a}(\tilde{u}) &:= \{(x,t) \in Q : \ \tilde{u}(x,t) \le u_a(x,t) + \sigma\} \\ M^\sigma_{u,b}(\tilde{u}) &:= \{(x,t) \in Q : \ \tilde{u}(x,t) \ge u_b(x,t) - \sigma\} \\ M^\sigma_{y,a}(\tilde{u}) &:= \{(x,t) \in Q : \ G\tilde{u}(x,t) \le y_a(x,t) + \sigma\} \\ M^\sigma_{y,b}(\tilde{u}) &:= \{(x,t) \in Q : \ G\tilde{u}(x,t) \ge y_b(x,t) - \sigma\}. \end{aligned}$$

With this general setting, we analyse the optimal control problem with respect to existence and uniqueness of solutions as well as first and second order optimality conditions. Let us reformulate the problem with the help of the solution operator G to obtain the reduced formulation

$$\min_{u \in U_{ad}} f(u) = J(Gu, u) = \int_Q L(x,t,Gu,u)\,dxdt$$

subject to $y_a \le G(u) \le y_b$.

By standard methods, the following existence theorem can be proven.

Theorem 2.6. *If the set of feasible controls,* U_{feas}*, is not empty, the optimal control problem* (P) *admits at least one (globally) optimal control* $\bar{u}$ *with associated optimal state* $\bar{y} = G(\bar{u})$.

Due to the nonconvexity of the problem, we cannot expect uniqueness of $\bar{u}$ in general, but we may encounter the existence of multiple locally optimal controls. Therefore we introduce the notation of a local solution.

Definition 2.7. *A feasible control* $\bar{u} \in U_{feas}$ *is called a local solution of* (P) *in the sense of* $L^\infty(Q)$ *if there exists a positive real number* ε *such that* $f(\bar{u}) \le f(u)$ *holds for all feasible controls* u *of* (P) *with* $\|u - \bar{u}\|_\infty \le \varepsilon$.

In order to formulate first order optimality conditions, we have to rely on additional assumptions.

ASSUMPTION 2.8. *We say that $\bar{u}$ satisfies the linearized Slater condition for (P) if there exists a point $u_0 \in U_{ad}$ such that*

$$\begin{array}{lclcll} u_a(x,t)+\rho & \leq & u_0(x,t) & \leq & u_b(x,t)-\rho & a.e.\ in\, Q \\ y_a(x,t)+\rho & \leq & G(\bar{u})(x,t)+G'(\bar{u})(u_0-\bar{u})(x,t) & \leq & y_b(x,t)-\rho & \forall (x,t)\in \bar{Q} \end{array}$$

with a fixed positive real number ρ.

REMARK 2.9. *Strictly speaking, $u_0 \in U_{ad}$ is sufficient for the existence of Lagrange multipliers. However, we will need the fact that u_0 is an interior point of U_{ad} when considering for example the Lavrentiev regularized problem, which is why we make this stronger assumption from the beginning.*

For $u \in L^\infty(Q)$ with associated $y = G(u)$ and $h \in L^\infty(Q)$ it is known that $G'(u)h = y_h$, where y_h is the solution to

$$\begin{aligned} (y_h)_t + Ay_h + d_y(\cdot,y)y_h &= h \\ y_h(\cdot,0) &= 0 \\ \partial_A y_h + \alpha y_h &= 0. \end{aligned}$$

For future reference, we point out that for $h_1, h_2 \in L^\infty(Q)$ it holds $G''(u)[h_1,h_2] = y_{h_1,h_2}$, where y_{h_1,h_2} solves

$$\begin{aligned} (y_{h_1,h_2})_t + Ay_{h_1,h_2} + d_y(\cdot,y)y_{h_1,h_2} &= -d_{yy}(y)G'(u)h_1G'(u)h_2 \\ y_{h_1,h_2}(\cdot,0) &= 0 \\ \partial_A y_{h_1,h_2} + \alpha y_{h_1,h_2} &= 0, \end{aligned}$$

i.e. $G''(u)[h_1,h_2] = G'(u)(-d_{yy}G'(u)h_1G'(u)h_2)$. Based on the linearized Slater condition, first order necessary optimality conditions for problem (P) can be proven that include the existence of regular Borel measures as Lagrange multipliers associated with the state constraints $y_a \leq y$ and $y \leq y_b$. We refer to [4]. We will not apply these optimality conditions in this paper, since we consider regularized versions of (P).

We assume a separation condition of the active sets in order to obtain unique Lagrange multipliers and adjoint states for a given locally optimal control $\bar{u}$.

ASSUMPTION 2.10. *There exists a positive real number $\sigma > 0$ such that the σ-active sets associated with the (locally) optimal control $\bar{u}$ of the unregularized problem (P) according to Definition 2.5 are pairwise disjoint.*

Due to the nonconvexity of the optimal control problem first order necessary optimality conditions are not sufficient for optimality. In the sequel, we later additionally assume a quadratic growth condition.

3. Moreau-Yosida regularization. In this section we aim at applying the penalization technique by Ito and Kunisch, [12], based on a Moreau-Yosida approximation of the Lagrange multipliers, to the control-and-state-constrained parabolic model problem (P), i.e. we are interested in analyzing the regularized problem formulation

$$(P_\gamma) \qquad \min_{u\in U_{ad}} f_\gamma(u) := f(u) + \frac{\gamma}{2}\left(\|\max(0,y_a - Gu)\|^2 + \|\max(0,Gu-y_b)\|^2\right),$$

where $\gamma > 0$ is a regularization parameter that is taken large. We hence consider a purely control-constrained problem formulation where the state constraints have been removed by penalization. Again the nonconvexity of the problem leads to possible multiple local optima. However, in this section, we will concentrate on the convergence of global solutions of (P_γ). A local analysis is possible with the techniques of Section 4, too, assuming the quadratic growth condition (4.3) at a selected locally optimal reference control $\bar{u}$. We point out the works by Meyer and Yousept, [17], were a local analysis for a specialized Moreau-Yosida-regularized control problem has been carried out for L^2 optimal controls.

It is easy to show the existence of at least one globally optimal control $\bar{u}_\gamma$ with associated optimal state $\bar{y}_\gamma = G(\bar{u}_\gamma)$ for (P_γ), because the set of admissible controls, U_{ad}, is not empty. The associated first-order necessary optimality conditions can be determined by a standard computation.

Theorem 3.1. *Let γ be a positive real number and denote by $\bar{u}_\gamma$ a solution to (P_γ). We define $\bar{\mu}_{a,\gamma} = \max(0, \gamma(y_a - \bar{y}_\gamma))$ and $\bar{\mu}_{b,\gamma} = \max(0, \gamma(\bar{y}_\gamma - y_b))$ in $C(\bar{Q})$ and introduce the adjoint state $p_\gamma \in W(0,T) \cap C(\bar{Q})$ as the weak solution of the adjoint equation*

$$\begin{aligned} -p_t + A^*p + d_y(\cdot, \bar{y}_\gamma)p &= L_y(\cdot, \bar{y}_\gamma, \bar{u}_\gamma) + \bar{\mu}_{b,\gamma} - \bar{\mu}_{a,\gamma} && \text{in } Q \\ p(\cdot, T) &= 0 && \text{in } \Omega \\ \partial_{A^*}p + \alpha p &= 0 && \text{on } \Sigma, \end{aligned}$$

where L_y denotes the partial derivative of L with respect to y. Then the variational inequality

$$(L_u(\cdot, \bar{y}_\gamma, \bar{u}_\gamma) + p_\gamma, u - \bar{u}_\gamma) \geq 0 \quad \forall u \in U_{ad}$$

is satisfied.

Now, we are interested in the convergence analysis as γ tends to infinity. We follow the principle steps shown in [12] and adapt them to our nonlinear setting.

Let $\{\gamma_n\}$ be a monotone sequence of positive real numbers tending to infinity as n goes to infinity and let $\{\bar{u}_n\}$ denote a sequence of globally optimal solutions to $(P\gamma_n)$. Due to the control constraints, it is uniformly bounded in $L^\infty(Q)$. Hence, there exists a subsequence which we denote w.l.o.g. by $\{\bar{u}_n\}$, converging weakly in $L^r(Q)$, $r > N/2 + 1$, to some $u^* \in U_{ad}$.

The next lemma proves that u^* is a feasible control for the original problem.

Lemma 3.2. *Assume that the feasible set of (P) is not empty. Let $\bar{u}_n$ be a sequence of optimal controls to $(P\gamma_n)$ converging weakly in $L^r(Q)$, $r > N/2 + 1$, to u^*. Then the state $y^* = G(u^*)$ associated with u^* satisfies*

$$y_a \leq y^* \leq y_b \quad \text{in } Q,$$

i.e. the weak limit u^* is feasible for (P).

Proof. According to our assumption, there exists a globally optimal control $\bar{u}$ for (P). Since $\bar{u}$ is feasible for (P) as well as for (P_γ), we have

$$f_\gamma(\bar{u}_n) \leq f_\gamma(\bar{u}) = f(\bar{u}) \quad \forall \gamma > 0,$$

which implies that $\frac{\gamma}{2}\int_Q \max(0, y_a - G(\bar{u}_n))$ as well as $\frac{\gamma}{2}\int_Q \max(0, G(\bar{u}_n) - y_b)$ are uniformly bounded. Notice that $f(\bar{u}_n)$ remains bounded, since U_{ad} is bounded in $L^\infty(Q)$. This implies that $\int_Q \max(0, y_a - G(\bar{u}_n))^2$ and $\int_Q \max(0, G(\bar{u}_n) - y_a)^2$ tend to zero as $n \to \infty$. In view of Hölder continuity, the sequence $\bar{y}_\gamma$ tends uniformly in $\bar{Q}$ to y^*.

By the continuity of the max-function we obtain

$$\max(0, y_a - y^*) = \lim_{n\to\infty} \max(0, y_a - \bar{y}_n) = 0.$$

Likewise, $y^* \le y_b$ holds in Q, which implies the feasibility of $y^* = G(u)^*$ for the unregularized problem (P). ◻

Next, we show that the convergence of $\bar{u}_\gamma$ is strong and that the limit is optimal for (P).

THEOREM 3.3. *Assume that the feasible set of* (P) *is not empty and that* $\bar{u}_n$ *is a sequence of optimal controls to* (P_{γ_n}) *converging weakly in* $L^r(Q)$*,* $r > N/2 + 1$*, to* u^**. Then* u^* *is optimal for* (P) *and the sequence* $\{\bar{u}_n\}$ *converges strongly in* $L^2(Q)$*.*

Proof. The sequence of optimal values $f_{\gamma_n}(\bar{u}_n)$ is monotone non-decreasing, because

$$f_{\gamma_n}(\bar{u}_n) \le f_{\gamma_n}(\bar{u}_{n+1}) \le f_{\gamma_{n+1}}(\bar{u}_{n+1})$$

(notice that an increase of γ for fixed control does not decrease f_γ). Moreover, it is bounded from above, since

$$f_\gamma(\bar{u}_n) \le f_\gamma(\bar{u}) = f(\bar{u}) \quad \forall \gamma > 0 \tag{3.1}$$

holds for all $\gamma > 0$, where $\bar{u}$ optimal for (P). Therefore, the sequence $\{f_{\gamma_n}(\bar{u}_n)\}$ is convergent. By the convexity of L with respect to u, the functional f is lower semicontinuous in $L^r(Q)$ (notice that $G(u) : L^r(Q) \to C(\bar{Q})$ is compact). Therefore

$$f(u^*) \le \liminf_{n\to\infty} f(\bar{u}_n) \le \liminf_{n\to\infty} f_{\gamma_n}(\bar{u}_n) = \lim_{n\to\infty} f_{\gamma_n}(\bar{u}_n) \le f(\bar{u})$$

follows from (3.1). This implies $f(u^*) = f(\bar{u})$ and the optimality of u^*, since u^* is feasible in view of the last lemma. Moreover, all inequalities in the formula above become equations so that, in particular,

$$\lim_{n\to\infty} f(\bar{u}_n) = f(u^*). \tag{3.2}$$

It remains to show the strong convergence of $\{\bar{u}_n\}$. In view of (3.1) we obtain

$$\begin{aligned}
0 &\le f(u^*) - f(\bar{u}_n) \\
&\le \int_Q L(\cdot, y^*, u^*) - L(\cdot, \bar{y}_n, u^*) + L(\cdot, \bar{y}_n, u^*) - L(\cdot, \bar{y}_n, \bar{u}_n)\, dxdt \\
&= I_n - \int_Q \Big(\frac{\partial L}{\partial u}(\cdot, \bar{y}_n, u^*)(\bar{u}_n - u^*) + \frac{1}{2} \int_0^1 \frac{\partial^2 L}{\partial u^2}(\cdot, \bar{y}_n, u^* + s(\bar{u}_n - u^*))(\bar{u}_n - u^*)^2\, ds \Big)\, dxdt,
\end{aligned}$$

where

$$I_n = \int_Q L(\cdot, y^*, u^*) - L(\cdot, \bar{y}_n, u^*)\, dxdt.$$

From the Legendre-Clebsch condition (2.3) it follows that

$$\frac{\beta_0}{2} \int_Q (\bar{u}_n - u^*)^2\, dxdt \le I_n - \int_Q \frac{\partial L}{\partial u}(\cdot, \bar{y}_n, u^*)(\bar{u}_n - u^*)\, dxdt,$$

where I_n tends to zero thanks to our assumptions on L and the integral in the right-hand side converges to zero, since $y_n \to y^*$ in $C(\bar{Q})$ and $\bar{u}_n \rightharpoonup u^*$. Therefore, $\bar{u}_n \to u^*$ holds as $n \to \infty$. □

The convergence results obtained so far are related to globally optimal solutions. From a numerical point of view, this consideration is not completely satisfactory. In numerical optimization algorithms, it should be expected to find local solutions to (P_γ) rather than to find a global one. Under natural assumptions, we expect that locally optimal controls of (P) can be approximated by associated local solutions of (P_γ).

This analysis can be worked out for the Moreau-Yosida regularization assuming the quadratic growth condition (4.3) at a local solution $\bar{u}$. For two reasons, we do not discuss this issue here. First, the technique is similar to the one we are going to explain for the Lavrentiev regularization in the next section. Second, the quadratic growth condition is usually deduced analytically and numerically from second-order sufficient optimality conditions. The functional f_γ is not twice differentiable, and therefore the classical second-order analysis fits better to the Lavrentiev regularization approach.

4. Lavrentiev type regularization. In this section we apply Lavrentiev type regularization to the semilinear control problem (P), hence, for a Lavrentiev parameter $\lambda > 0$ we consider the regularized problem

$$(P_\lambda) \qquad \text{Minimize } f(u) = \int_Q L(x,t,Gu,u)\,dxdt$$

subject to

$$u_a \le u \le u_b \quad \text{and} \quad y_a \le \lambda u + Gu \le y_b.$$

Again, existence of global solutions can be shown by standard arguments if a feasible control exists, but there may exist multiple local optima. For this type of regularization a global convergence analysis can be set up following the arguments used for the Moreau-Yosida regularization. This presentation would be completely analogous to the preceding section, hence we do not repeat it and concentrate on a local investigation.

Before we proceed, we will introduce the following definitions.

Definition 4.1. *For fixed $\lambda > 0$, we denote by*

$$U^\lambda_{feas} = \{u \in U_{ad} \mid y_a \le \lambda u + Gu \le y_b \ a.\ e.\ in\ Q\}$$

the set of feasible controls for (P_λ).

Definition 4.2. *Let $\lambda > 0$. A function $\bar{u}_\lambda \in U^\lambda_{feas}$ is called a local solution of (P_λ) in the sense of $L^p(Q)$, $N/2+1 < p \le \infty$, if*

$$f(\bar{u}_\lambda) \le f(u)$$

is satisfied for all $u \in U^\lambda_{feas}$ with $\|u - \bar{u}_\lambda\|_p \le \varepsilon$, for some $\varepsilon > 0$.

We rely on a linearized Slater condition and a separation condition for the almost active sets.

Lemma 4.3. *Let u_λ be a feasible control for (P_λ) with $\|u_\lambda - \bar{u}\|_\infty \le \varepsilon$. If $\varepsilon > 0$ is sufficiently small, then the linearized Slater condition*

$$u_a + \frac{\rho}{2} \le u_0 \le u_b - \frac{\rho}{2}$$
$$y_a + \frac{\rho}{2} \le \lambda u_0 + G(u_\lambda) + G'(u_\lambda)(u_0 - u_\lambda) \le y_b - \frac{\rho}{2}$$

is satisfied for ρ, u_0 from Assumption 2.8.

Proof. The first inequality is trivial. We consider only the case $\lambda u_0 + G(u_\lambda) + G'(u_\lambda)(u_0 - u_\lambda) \leq y_b - \frac{\rho}{2}$. We obtain

$$\begin{aligned}\lambda u_0 + G(u_\lambda) + G'(u_\lambda)(u_0 - u_\lambda) =& \lambda u_0 + G(\bar{u}) + G'(\bar{u})(u_0 - \bar{u}) + (G(u_\lambda) - G(\bar{u})) \\ &+ (G'(u_\lambda) - G'(\bar{u}))(u_0 - u_\lambda) + G'(\bar{u})(\bar{u} - u_\lambda).\end{aligned}$$

Due to u_0 being bounded, λ can be chosen small enough such that $\lambda u_0 \leq \lambda \|u_0\|_\infty \leq \frac{\rho}{8}$. Also, if ε is sufficiently small, we obtain $G(u_\lambda) - G(\bar{u}) \leq \frac{\rho}{8}$, as well as $(G'(u_\lambda) - G'(\bar{u}))(u_0 - u_\lambda) \leq \frac{\rho}{8}$ and $G'(\bar{u})(\bar{u} - u_\lambda) \leq \frac{\rho}{8}$, since G and G' are Lipschitz. Hence,

$$\lambda u_0 + G(u_\lambda) + G'(u_\lambda)(u_0 - u_\lambda) \leq G(\bar{u}) + G'(\bar{u})(u_0 - \bar{u}) + \frac{\rho}{2} \leq y_b - \frac{\rho}{2},$$

by the assumption of a Slater condition for the unregularized problem. □

DEFINITION 4.4. *Let $\tilde{u}$ be a reference control and let σ be a positive real number. The σ-active sets for the Lavrentiev-regularized problem are given by*

$$\begin{aligned}M_{u,a}^{\sigma,\lambda}(\tilde{u}) &:= \{(x,t) \in Q : \ \tilde{u}(x,t) \leq u_a(x,t) + \sigma\} \\ M_{u,b}^{\sigma,\lambda}(\tilde{u}) &:= \{(x,t) \in Q : \ \tilde{u}(x,t) \geq u_b(x,t) - \sigma\} \\ M_{y,a}^{\sigma,\lambda}(\tilde{u}) &:= \{(x,t) \in Q : \ \lambda\tilde{u}(x,t) + G\tilde{u}(x,t) \leq y_a(x,t) + \sigma\} \\ M_{y,b}^{\sigma,\lambda}(\tilde{u}) &:= \{(x,t) \in Q : \ \lambda\tilde{u}(x,t) + G\tilde{u}(x,t) \geq y_b(x,t) - \sigma\}.\end{aligned}$$

ASSUMPTION 4.5. *We assume that there exists $\sigma > 0$ such that the $\sigma-$active sets associated with $\bar{u}_\lambda$ according to Definition* (4.4) *are pairwise disjoint for all λ sufficiently small.*

We will see later that this condition can be proven under an additional assumption. Then we obtain the following theorem concerning first order optimality conditions by applying the results from [20]. The main statement is that the Lagrange multipliers associated with the regularized state constraints are bounded and measurable and unique.

THEOREM 4.6. *Let $\lambda > 0$ be fixed and sufficiently small and let $\bar{u}_\lambda$ be a fixed local solution to (P_λ). If the assumptions 2.8 and 4.5 are satisfied, then there exist unique Lagrange multipliers $\bar{\mu}_a^\lambda, \bar{\mu}_b^\lambda \in L^\infty(Q)$ and an adjoint state $p_\lambda \in W(0,T) \cap C(\bar{Q})$, such that*

$$\begin{aligned}-p_t + A^*p + d_y(\cdot, \bar{y}_\lambda) &= L_y(\cdot, \bar{y}_\lambda, \bar{u}_\lambda) + \bar{\mu}_b^\lambda - \bar{\mu}_a^\lambda \\ p(\cdot, T) &= 0 \\ \partial_{A^*}p + \alpha p &= 0\end{aligned} \tag{4.1}$$

$$\left(L_u(\cdot, \bar{y}_\lambda, \bar{u}_\lambda) + p_\lambda + \lambda(\bar{\mu}_b^\lambda - \bar{\mu}_a^\lambda), u - \bar{u}_\lambda\right) \geq 0 \quad \forall u \in U_{ad}$$

$$\begin{aligned}\bar{\mu}_a^\lambda \geq 0, \quad (\bar{\mu}_a^\lambda, \lambda\bar{u}_\lambda + \bar{y}_\lambda - y_a) = 0 \\ \bar{\mu}_b^\lambda \geq 0, \quad (\bar{\mu}_b^\lambda, \lambda\bar{u}_\lambda + \bar{y}_\lambda - y_b) = 0\end{aligned}$$

is satisfied.

Proof. Lemma 4.3 ensures with Assumption 2.8 that $\bar{u}_\lambda$ satisfies a linearized Slater condition for sufficiently small λ. The existence of regular L^∞-multipliers follows now directly from recent works by Rösch and the second author, [20], even under the weaker assumption that $(M_{u,a}^{\sigma,\lambda}(\bar u_\lambda) \cup M_{y,a}^{\sigma,\lambda}(\bar u_\lambda)) \cap (M_{u,b}^{\sigma,\lambda}(\bar u_\lambda) \cup M_{y,b}^{\sigma,\lambda}(\bar u_\lambda)) = \emptyset$. However, for the uniqueness of the multipliers we need the stronger separation condition. We prove the uniqueness result following [2] for linear-quadratic elliptic problems. We know that $\bar\mu_a^\lambda = 0$ on $Q \setminus M_{y,a}^{\sigma,\lambda}(\bar u_\lambda)$ as well as $\bar\mu_b^\lambda = 0$ on $Q \setminus M_{y,b}^{\sigma,\lambda}(\bar u_\lambda)$. Due to our separation assumption, on $M_{y,a}^{\sigma,\lambda}(\bar u_\lambda) \cup M_{y,b}^{\sigma,\lambda}(\bar u_\lambda)$ the control constraints cannot be active so that the variational inequality implies an associated equation on this set. This pointwise interpretation of the variational inequality leads to

$$\begin{aligned}\bar\mu_a^\lambda &= \begin{cases} \frac{1}{\lambda}(L_u(\cdot,\bar y_\lambda,\bar u_\lambda) + p_\lambda) & \text{on } M_{y,a}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases} \\ \bar\mu_b^\lambda &= \begin{cases} -\frac{1}{\lambda}(L_u(\cdot,\bar y_\lambda,\bar u_\lambda) + p_\lambda) & \text{on} M_{y,b}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases} .\end{aligned}$$

Inserting these expressions into the adjoint equation, we obtain

$$\begin{aligned} -p_t + A^*p + d_y(\cdot,\bar y_\lambda) + (c_a + c_b)\,p &= L_y(\cdot,\bar y_\lambda,\bar u_\lambda) + m_b - m_a \\ p(\cdot,T) &= 0 \\ \partial_{A^*}p + \alpha p &= 0, \end{aligned} \tag{4.2}$$

where $c_a(x,t)$, $c_b(x,t)$ are given as

$$c_a = \begin{cases} \frac{1}{\lambda} & \text{on } M_{y,a}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases}, \quad c_b = \begin{cases} -\frac{1}{\lambda} & \text{on } M_{y,b}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases},$$

and m_a, m_b are defined by

$$\begin{aligned} m_a &= \begin{cases} \frac{1}{\lambda}L_u(\cdot,\bar y_\lambda,\bar u_\lambda) & \text{on } M_{y,a}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases} \\ m_b &= \begin{cases} -\frac{1}{\lambda}L_u(\cdot,\bar y_\lambda,\bar u_\lambda) & \text{on } M_{y,b}^{\sigma,\lambda}(\bar u_\lambda) \\ 0 & \text{else} \end{cases} . \end{aligned}$$

Theorem 2.2 yields the existence of a unique solution p_λ to (4.2). Hence, with the variational inequality, we obtain unique Lagrange multipliers by a simple discussion. □

4.1. Convergence analysis. This section is devoted to the convergence analysis as λ tends to zero. We rely on the Slater condition and the quadratic growth condition for optimal solutions of the unregularized problem. Our aim is to show that under natural conditions local solutions of the unregularized problem can be approximated by local solutions of the regularized problem, hence we focus on the convergence of local solutions instead of global solutions. We refer to [10] for convergence of global solutions for semilinear elliptic problems without control constraints. We point out that we follow closely the arguments in [11], where a convergence result for local solutions is shown in a context that includes Lavrentiev type regularization.

Let therefore $\{\lambda_n\}$, $\lambda_n > 0$, be a sequence converging to zero. We follow an idea from [5] and consider the auxiliary problem

$$(P_\lambda^r) \qquad \min_{u \in U_{ad}^r} f(u)$$
$$y_a \le \lambda u + G(u) \le y_b,$$

where $r = \frac{\varepsilon}{2}$ and $U_{ad}^r = \{u \in U_{ad} \mid \|u - \bar u\|_\infty \le r\}$.

LEMMA 4.7. *Let $\bar u$ be a feasible control for (P) satisfying the linearized Slater condition of Assumption 2.8. Then there exists a sequence $\{u_n\}$ converging strongly in $L^\infty(Q)$ to $\bar u$ as $n \to \infty$ such that u_n is feasible for (P_{λ_n}) for all sufficiently large n.*

Proof. Let u_0 denote the Slater point from Assumption 2.8 and choose $u_n = \bar u + t_n(u_0 - \bar u)$, where $t_n = t_n(\lambda_n) \in [0,1]$ and $\lambda_n > 0$ is given sufficiently small. It is clear that $u_n \to \bar u$ in $L^\infty(Q)$ as $t_n \to 0$. It remains to show the feasibility for (P_{λ_n}). We obtain for the upper state constraint

$$\begin{aligned}\lambda_n u_n + G(u_n) &= \lambda_n\, u_n + G(\bar u + t_n(u_0 - \bar u))\\ &\le \lambda_n \|\bar u + t_n(u_0 - \bar u)\|_\infty + G(\bar u) + t_n G'(\bar u)(u_0 - \bar u) + o(t_n)\\ &\le c\lambda_n + (1-t_n)G(\bar u) + t_n(G(\bar u) + G'(\bar u)(u_0 - \bar u)) + o(t_n)\\ &\le c\lambda_n + (1-t_n)y_b + t_n y_b - t_n\rho + o(t_n)\\ &= y_b + c\lambda_n - t_n(\rho + \frac{o(t_n)}{t_n}).\end{aligned}$$

Take t_0 small enough to ensure $\rho + \frac{o(t_0)}{t_0} \ge \frac{\rho}{2}$. Setting $c\lambda_n - t_n\frac{\rho}{2} = 0$ we obtain $t_n = t_n(\lambda_n) = \frac{2c}{\rho}\lambda_n$, which for λ_n sufficiently small yields $t(\lambda_n) \le t_0$. Hence we obtain

$$\lambda_n u_n + G(u_n) \le y_b + c\lambda_n - t_n(\rho + \frac{o(t_n)}{t_n}) \le y_b \quad \forall 0 < \lambda_n \le \lambda_0,$$

since $c\lambda - t_n(\rho + \frac{o(t_n)}{t_n}) \le 0$. Analogously, we can deal with the lower state constraint. ◻

It follows from this lemma that the feasible set of $(P^r_{\lambda_n})$ is not empty for all sufficiently large n provided that $\bar u$ satisfies the linearized Slater condition. In this case, the auxiliary problem admits at least one global solution. Let in the following $\{\bar u^r_n\}$ denote a sequence of arbitrary globally optimal solutions to $(P^r_{\lambda_n})$. Due to the control constraints, it is uniformly bounded in $L^\infty(Q)$. Hence, there exists a subsequence which w.l.o.g. we assume to be $\{\bar u^r_n\}$, converging weakly in $L^p(Q)$ to u^*, $p > N/2 + 1$. Since the associated states converge uniformly to $y^* = G(u^*)$ it is easy to see that u^* is feasible for (P) and also belongs to U^r_{ad}.

LEMMA 4.8. *Let $\bar u^r_n$ be a globally optimal control of $(P^r_{\lambda_n})$ for $\lambda_n \downarrow 0$, $n \to \infty$. There exists a sequence of feasible controls v^r_n of (P) with $\|v^r_n - \bar u\|_\infty \le r$ converging strongly in $L^\infty(Q)$ to $\bar u^r_n$ as $n \to \infty$.*

Proof. (i) We first construct a Slater point $\hat u_0$ with $\|\hat u_0 - \bar u\|_\infty \le r$. To this aim, let $u_0 \in L^\infty(Q)$ be the Slater point from Assumption 2.8 and let $\rho > 0$ be the associated Slater parameter. We define $\hat u_0 = \bar u + \hat t(u_0 - \bar u)$ with $\hat t = \min\{1, \frac{r}{\|u_0 - \bar u\|_\infty}\}$. Then $\|\hat u_0 - \bar u\|_\infty \le r$ is fulfilled. We observe that

$$\hat u_0 = (1-\hat t)\bar u + \hat t u_0 \ge (1-\hat t)u_a + \hat t(u_a + \rho) \ge u_a + \hat t\rho =: u_a + \hat\rho.$$

Analogously, one shows an associated upper estimate $\hat u_0 \le u_b - \hat\rho$. Moreover, we have

$$\begin{aligned}G(\bar u) + G'(\bar u)(\hat u_0 - \bar u) &= G(\bar u) + G'(\bar u)(\bar u + \hat t(u_0 - \bar u) - \bar u)\\ &= (1-\hat t)G(\bar u) + \hat t(G(\bar u) + G'(\bar u)(u_0 - \bar u))\\ &\ge (1-\hat t)y_a + \hat t(y_a + \rho) = y_a + \hat t\rho =: y_a + \hat\rho.\end{aligned}$$

Altogether, we have shown that $\hat{u}_0$ satisfies

$$\begin{aligned} u_a + \hat{\rho} \le \hat{u}_0 \le u_b - \hat{\rho} \\ y_a + \hat{\rho} \le G(\bar{u}) + G'(\bar{u})(\hat{u}_0 - \bar{u}) \le y_b - \hat{\rho}. \end{aligned}$$

By the same arguments as in Lemma 4.3, we obtain for sufficiently small ε, hence for sufficiently small r that $y_a + \frac{\hat{\rho}}{2} \le \lambda \hat{u}_0 + G(\bar{u}_n^r) + G'(\bar{u}_n^r)(\hat{u}_0 - \bar{u}_n^r)$. An analogous estimate can be obtained for the upper bound.

(ii) Next, we define

$$v_n^r = \bar{u}_n^r + t_n(\hat{u}_0 - \bar{u}_n^r)$$

with $t_n = \frac{2c}{\hat{\rho}}\lambda_n$ and $c = \max\{\|\hat{u}_0\|_\infty, \|\bar{u}_n^r\|_\infty\}$ (notice that $\bar{u}_n^r$ is uniformly bounded). Then $t_n \downarrow 0$, v_n^r converges strongly to $\bar{u}_n^r$ and $\|v_n^r - \bar{u}\|_\infty \le r$ holds for n large enough. We obtain

$$\begin{aligned} -\lambda_n \|v_n^r\|_\infty + G(v_n^r) &\le \lambda_n v_n^r + G(v_n^r) \\ &= (1 - t_n)\lambda_n \bar{u}_n^r + t_n \lambda_n \hat{u}_0 + G(\bar{u}_n + t_n(\hat{u}_0 - \bar{u}_n^r)) \\ &= (1 - t_n)\lambda_n \bar{u}_n^r + (1 - t_n) G(\bar{u}_n^r) \\ &+ t_n(\lambda_n \hat{u}_0 + G(\bar{u}_n^r) + G'(\bar{u}_n^r)(\hat{u}_0 - \bar{u}_n^r)) + o(t_n) \\ &\le (1 - t_n) y_b + t_n(y_b - \frac{\hat{\rho}}{2}) + o(t_n) \le y_b - t_n \frac{\hat{\rho}}{2} + o(t_n). \end{aligned}$$

This implies $G(v_n^r) \le y_b - t\frac{\hat{\rho}}{2} + \lambda_n \|v_n^r\|_\infty \le y_b$ by the definition of t_n, hence v_n^r satisfies the upper state constraint of (P). Analogously, it satisfies the lower one. For $\lambda_n \downarrow 0$, t_n tends to zero so that $0 < t_n < 1$ holds for sufficiently large n. Therefore v_n^r, as the convex combination of two elements of $U_{ad} \cap B_r(\bar{u})$, belongs to the same set. □

Now we proceed to show that $\bar{u}$, the locally optimal reference control of (P), can be approximated by optimal controls of $(P_{\lambda_n}^r)$. To this aim, we impose an assumption of quadratic growth.

Assumption 4.9. *We assume there exist positive real numbers ε and α such that $\bar{u}$ satisfies the quadratic growth condition, i.e.*

$$f(\bar{u}) + \frac{\alpha}{2}\|u - \bar{u}\|^2 \le f(u) \tag{4.3}$$

holds for every feasible control u of (P) that satisfies $\|u - \bar{u}\|_p \le \varepsilon$ with some $N/2+1 < p \le \infty$.

In sufficiently regular cases, this growth condition can be expected from second order sufficient optimality conditions (SSC). If $N = 1$, then standard SSC can be derived from a definiteness property of the second derivative of the Lagrange function and the growth condition is satisfied with $p = \infty$, cf. [18]. If additionally L has the form

$$L(x,t,y,u) = \Phi(x,t,y) + \psi(x,t,y)\,u + \nu(x,t)u^2 \tag{4.4}$$

with ϕ, ψ satisfying the assumptions on d except the monotonicity and $\nu \in L^\infty(Q)$, $\nu(x,t) \ge \delta > 0$, then the objective functional is twice continuously differentiable in $L^p(Q)$ with $p > N/2 + 1$. Here, we can expect the growth condition for associated p, cf. [22], Section 4.9 for an analogous discussion. For $N > 1$, the Lagrange multipliers for (P) must have higher regularity to guarantee a quadratic growth condition.

THEOREM 4.10. *Let $\bar{u}$ be a locally optimal control of (P) satisfying the quadratic growth condition* (4.3) *and Assumption 4.3 (linearized Slater condition) and fix $r > 0$. Then, for all sufficiently large n, problem $(P^r_{\lambda_n})$ has an optimal control. If $\{\bar{u}^r_n\}$ is any sequence of (globally) optimal controls for $(P^r_{\lambda_n})$, then it converges strongly in $L^q(Q)$ to $\bar{u}$, for all $2 \le q < \infty$. Moreover, it converges in $L^2(Q)$ with rate $\sqrt{\lambda_n}$, i.e. there exists $c > 0$ such that*

$$\|\bar{u}^r_n - \bar{u}\|_{L^2(Q)} \le c\sqrt{\lambda_n}.$$

Proof. From the quadratic growth condition we find

$$f(u) \ge f(\bar{u}) + \alpha\|u - \bar{u}\|^2 \quad \forall u \in U_{ad} \cap B_r(\bar{u}),$$

for r sufficiently small, where $B_r(\bar{u})$ is the closed ball of radius r around $\bar{u}$ in $L^p(Q)$. The above inequality holds especially for $u = v^r_n$ constructed in Lemma 4.8, since this function is feasible for (P). This yields

$$\begin{aligned} f(v^r_n) &\ge f(\bar{u}) + \alpha\|v^r_n - \bar{u}\|^2 \\ &= f(\bar{u}) + \alpha\|\bar{u}^r_n - \bar{u}\|^2 + 2(v^r_n - \bar{u}^r_n, \bar{u}^r_n - \bar{u}) + \|v^r_n - \bar{u}^r_n\|^2 \\ &\ge f(\bar{u}) + \alpha\|\bar{u}^r_n - \bar{u}\|^2 - c\|v^r_n - \bar{u}^r_n\|. \end{aligned}$$

We obtain

$$\begin{aligned} f(\bar{u}^r_n) &= f(v^r_n) - (f(v^r_n) - f(\bar{u}^r_n)) \\ &\ge f(\bar{u}) + \alpha\|\bar{u}^r_n - \bar{u}\|^2 - c\|v^r_n - \bar{u}^r_n\| - (f(v^r_n) - f(\bar{u}^r_n)), \end{aligned}$$

which yields

$$\alpha\|\bar{u}^r_n - \bar{u}\|^2 \le f(\bar{u}^r_n) - f(\bar{u}) + c\|v^r_n - \bar{u}^r_n\| + f(v^r_n) - f(\bar{u}^r_n). \tag{4.5}$$

On the other hand, we have $f(\bar{u}^r_n) \le f(u_n)$, hence

$$f(\bar{u}^r_n) - f(\bar{u}) \le f(u_n) - f(\bar{u}).$$

Inserting this inequality in (4.5) yields

$$\alpha\|\bar{u}^r_n - \bar{u}\|^2 \le f(u_n) - f(\bar{u}) + f(v^r_n) - f(\bar{u}^r_n) + c\|v^r_n - \bar{u}^r_n\|.$$

By the definition of $v^r_n := \bar{u}^r_n + t(\hat{u}_0 - \bar{u}^r_n)$, of $u_n := \bar{u} + t(u_0 - \bar{u})$, and the choice of $t = \frac{2c}{\rho}\lambda_n$ as in the proofs of Lemma 4.7 and 4.8 we obtain with a generic constant c that $\|v^r_n - \bar{u}^r_n\|_\infty \le c\lambda_n$ and $\|u_n - \bar{u}\|_\infty \le c\lambda_n$. The functional f is Lipschitz w.r. to the L^∞-norm. This yields

$$\|\bar{u}^r_n - \bar{u}\|^2 \le \frac{c}{\alpha}\lambda_n,$$

which implies that $\bar{u}^r_n$ converges strongly in $L^2(Q)$ towards $\bar{u}$ with rate $\sqrt{\lambda_n}$. Since $\bar{u}^r_n$ belongs to U_{ad}, this sequence is uniformly bounded, hence it converges also in $L^q(Q)$ with $q < \infty$. Therefore, the associated states converge uniformly on $\bar{Q}$. □

The control $\bar{u}^r_n$ is not necessarily a local solution of (P_{λ_n}), since it might touch the boundary of $B_r(\bar{u})$. To have local optimality, we need an additional assumption.

THEOREM 4.11. *Let the assumptions of Theorem 4.10 be satisfied and assume in addition that Assumption 4.9 is satisfied with $p < \infty$. Then, for n sufficiently large, $\bar{u}_n^r$ is a local solution of (P_λ), hence there exists a sequence of local solutions to (P_λ) that converges strongly in $L^p(Q)$ to $\bar{u}$.*

Proof. The result is a simple conclusion from the last theorem, since $\bar{u}_n^r \to \bar{u}$ in $L^q(Q)$ for all $q < \infty$, in particular for p. Therefore, $\|\bar{u}_n^r - \bar{u}\|_p < r$ must hold for sufficiently large n. In this case, $\bar{u}_n^r$ is a solution to $(P_{\lambda_n}^r)$ that is in the interior of $B_r(\bar{u})$. Therefore, it is a local solution to (P_{λ_n}). $\square$

If Assumption 4.9 is only satisfied for $p = \infty$, then this convergence result is not applicable. Here, we have to assume that the convergence of $\bar{u}_n^r$ towards $\bar{u}$ is strong in $L^\infty(Q)$. Under this strong assumption, Theorem 4.11 remains true for $p = \infty$. Moreover, we can deduce the separation condition on active sets for (P_λ) from the one imposed on $\bar{u}$ in problem (P).

LEMMA 4.12. *Let $\{\bar{u}_n\}$ be a sequence of locally optimal controls of (P_λ) converging strongly in $L^\infty(Q)$ to a locally optimal control $\bar{u}$ for (P). Assume that there exists $\sigma > 0$ such that the σ-active sets associated with $\bar{u}$ for the unregularized control problem according to Definition 2.5 are pairwise disjoint. Then there exists $\tau > 0$ such that the τ-active sets for the Lavrentiev-regularized control problems according to Definition 4.4 are pairwise disjoint for all sufficiently small $\lambda > 0$.*

The proof is elementary.

The convergence analysis presented in this section seems to satisfy the requirements needed for a numerical analysis. We know that, under certain assumptions, each locally optimal control of (P) can be approximated by a sequence of locally optimal controls of (P_λ). However, this result is still not completely satisfactory.

If a quadratic growth condition is satisfied at $\bar{u}$, then $\bar{u}$ is locally optimal, but $\bar{u}$ might be the accumulation point of different local solutions (with larger objective value). We cannot exclude such a situation for (P) (except, perhaps, for $N = 1$, cf. Griesse [8] for an elliptic problem), but in the case of (P_λ), the situation is better. Under associated assumptions, the Lagrange multipliers are bounded and measurable so that second-order sufficient conditions can be expected to hold. Based on the separation condition, we are able to show the local uniqueness of local solutions to (P_λ). The associated analysis is presented in the rest of the paper.

5. Local uniqueness of local optima.

5.1. Generalized equations and strong regularity. In this section we prove our main result, the local uniqueness of local optima of the Lavrentiev-regularized problems. Let us emphasize here that throughout this section we consider a fixed Lavrentiev parameter $\lambda > 0$. We will make use of an implicit function theorem by Robinson from [19] for strongly regular generalized equations.

Considering optimality systems as generalized equations is a meanwhile standard technique. We refer, for instance, to Josephy [13], who considered the Newton method for generalized equations in finite-dimensional spaces and to generalizations by Dontchev [7] and Alt [1]. Moreover, we mention the work by Malanowski [14] on Lipschitz stability for the solution of optimal control problems. Working in the context of generalized equations we proceed as follows:

First, we write the first order optimality conditions for (P_λ) as nonlinear generalized equation. Second, we show strong regularity of this generalized equation. Third, we apply Robinson's implicit function theorem and deduce local uniqueness of local optima of the optimal control problem.

The main part of this section will be devoted to show strong regularity. This involves proving a Lipschitz stability result for a second-order Taylor approximation of the problem.

When considering this linearized problem, we proceed in principle as in [2], where a linear elliptic optimal control problem is considered. Following an idea of Malanowski [14], in [2] an auxiliary problem is introduced with the constraints restricted to the almost active set of optimal control. For the auxiliary problem, first L^2-stability is shown and next extended to an L^∞-result, which is then carried over to the original problem.

For our parabolic problem, we develop a slightly different technique. On the one hand, we do not consider Lagrange multipliers for the control constraints as in [2], since they remain unperturbed. On the other hand, the approach of [2] cannot directly be applied. The main reason is the lower regularity of solutions to parabolic equations. We need to apply a special bootstrapping technique to the auxiliary problem, which does not admit control constraints in the whole domain, to obtain optimal controls in $L^\infty(Q)$ without restriction on the dimension, cf. also [21]. Yet another bootstrapping technique is required to prove the Lipschitz stability result in L^∞.

During the following analysis, we rely strongly on a second order sufficient condition (SSC) for the solution of the Lavrentiev regularized problem (P_λ).

Assumption 5.1. *Let $\lambda > 0$ be sufficiently small and let $\bar{u}_\lambda$ denote a local solution of (P_λ) satisfying the first order necessary conditions stated in Theorem 4.6. We assume that there exists $\kappa > 0$ such that*

$$f''(\bar{u}_\lambda)h^2 + (G''(\bar{u}_\lambda)h^2, \bar{\mu}_b^\lambda - \bar{\mu}_a^\lambda) \geq \kappa \|h\|^2 \quad \forall h \in L^\infty(Q). \tag{5.1}$$

Let us first introduce the notation fitting into the context of generalized equations.

Definition 5.2. *Let the cones $N_{U_{ad}}(\bar{u}_\lambda)$, $K(\bar{\mu}_a^\lambda)$, and $K(\bar{\mu}_b^\lambda)$ be given as*

$$\begin{aligned}
N_{U_{ad}}(\bar{u}_\lambda) &= \{g \in L^\infty(Q) : (g, u - \bar{u}_\lambda) \leq 0 \ \forall u \in U_{ad}\} \\
K(\bar{\mu}_a^\lambda) &= \begin{cases} \{g \in L^2(Q) \mid (g, \mu_a - \bar{\mu}_a^\lambda) \leq 0 \ \forall \mu_a \in L^2(Q)\} & \text{if } \bar{\mu}_a^\lambda \geq 0 \\ \emptyset & \text{else.} \end{cases} \\
K(\bar{\mu}_b^\lambda) &= \begin{cases} \{g \in L^2(Q) \mid (g, \mu_b - \bar{\mu}_b^\lambda) \leq 0 \ \forall \mu_b \in L^2(Q)\} & \text{if } \bar{\mu}_b^\lambda \geq 0 \\ \emptyset & \text{else.} \end{cases}
\end{aligned}$$

Note that we do not write $\partial_{U_{ad}(\bar{u}_\lambda)}$ because this is commonly understood as a subset of $(L^\infty(Q))^*$. Instead, we identify $\partial_{U_{ad}(\bar{u}_\lambda)}$ with the set $N_{U_{ad}(\bar{u}_\lambda)}$. Likewise, we simply write $K(\bar{\mu}_a^\lambda)$ and $K(\bar{\mu}_b^\lambda)$. It is easily verified that the optimality system is equivalent to the generalized equation

$$0 \in F(\bar{u}_\lambda, \bar{\mu}_a^\lambda, \bar{\mu}_b^\lambda) + \begin{pmatrix} N_{U_{ad}}(\bar{u}_\lambda) \\ K(\bar{\mu}_a^\lambda) \\ K(\bar{\mu}_b^\lambda) \end{pmatrix},$$

with

$$F(u, \mu_a, \mu_b) = \begin{pmatrix} f'(u) + \lambda(\mu_b - \mu_a) + G'(u)^*(\mu_b - \mu_a) \\ u + G(u) - y_a \\ y_b - u - G(u) \end{pmatrix}.$$

Linearization at $(\bar u_\lambda, \bar\mu_a^\lambda, \bar\mu_b^\lambda)$ in the direction $(u^\delta - \bar u_\lambda, \mu_a^\delta - \bar\mu_a^\lambda, \mu_b^\delta - \bar\mu_b^\lambda)^T$ and perturbation by a parameter $\delta = (\delta_1, \delta_2, \delta_3) \in (L^\infty(Q))^3$ in order to verify strong regularity leads to the following system:

$$L(\delta) \quad \delta \in \left(\begin{array}{l} f'(\bar u_\lambda) + f''(\bar u_\lambda)(u^\delta - \bar u_\lambda) + G''(\bar u_\lambda)(u^\delta - \bar u_\lambda)^*(\bar\mu_b^\lambda - \bar\mu_a^\lambda) \\ +\lambda(\mu_b^\delta - \mu_a^\delta) + G'(\bar u_\lambda)^*(\mu_b^\delta - \mu_a^\delta) + N_{U_{ad}}(u^\delta) \\ \\ \lambda u^\delta + G'(\bar u_\lambda)(u^\delta - \bar u_\lambda) + G\bar u_\lambda - y_a + K(\mu_a^\delta) \\ \\ y_b - \lambda u^\delta - G'(\bar u_\lambda)(u^\delta - \bar u_\lambda) - G\bar u_\lambda + K(\mu_b^\delta) \end{array} \right). \qquad (5.2)$$

Note that $L(0)$ corresponds to the unperturbed linearized equation, in which case we will denote the corresponding optimal control by u^0. The reader may readily verify that (5.2) coincides with the first order necessary optimality conditions for the following linear-quadratic problem:

$$P(\delta) \qquad \min_{u \in U_{ad}} f_\delta(u) := \frac{1}{2}(f''(\bar u_\lambda)(u - \bar u_\lambda)^2 + (G''(\bar u_\lambda)(u - \bar u_\lambda)^2, \bar\mu_b^\lambda - \bar\mu_a^\lambda)) \\ +f'(\bar u_\lambda)(u - \bar u_\lambda) - (\delta_1, u - \bar u_\lambda)$$

subject to

$$y_a + \delta_2 - \lambda\bar u_\lambda - G(\bar u_\lambda) \le \lambda(u - \bar u_\lambda) + G'(\bar u_\lambda)(u - \bar u_\lambda) \le y_b - \delta_3 - \lambda\bar u_\lambda - G(\bar u_\lambda),$$

with optimal control u^δ and associated Lagrange multipliers $\mu_a^\delta, \mu_b^\delta$.

Hence, if u^δ with associated regular Lagrange multipliers $\mu_a^\delta, \mu_b^\delta$ solves $P(\delta)$, the linearized generalized equation is fulfilled. The converse is true since the second order sufficient condition (5.1) is a sufficient condition for the problem $P(\delta)$. Thanks to (5.1), the objective function of $P(\delta)$ is strictly convex for every $\delta \in L^\infty(Q)^3$ and tends to infinity as $\|u\| \to \infty$. Therefore, problem $P(\delta)$ has a unique optimal control u^δ. It remains to show that this solution and the associated Lagrange multipliers depend Lipschitz on the perturbation δ. For the following analysis, we simplify the notation for $P(\delta)$. Below, we write L_{yy}, L_{yu} for $\partial^2 L/\partial y^2$, $\partial^2 L/\partial y \partial u$.

Definition 5.3. *Let d_0 be given as $d_0 = d_y(\cdot, \bar y_\lambda)$ and consider a control $\tilde u = u - \bar u_\lambda$ with associated state $\tilde y = G'(\bar u_\lambda)\tilde u$, i.e. $\tilde y$ solves the linearized equation*

$$\begin{aligned} \tilde y_t + A\tilde y + d_0 \tilde y &= \tilde u \\ \tilde y(\cdot, 0) &= 0 \\ \partial_A \tilde y + \alpha\tilde y &= 0. \end{aligned} \qquad (5.3)$$

Further, we define $\varphi_1 = L_{yy}(\cdot, \bar y_\lambda, \bar u_\lambda) - p_\lambda d_{yy}(\cdot, \bar y_\lambda)$, where p_λ solves the adjoint equation from Theorem 4.6, as well as $\varphi_2 = L_{yu}(\cdot, \bar y_\lambda, \bar u_\lambda)$, $\varphi_3 = L_{uu}(\bar y_\lambda, \bar u_\lambda)$, $\varphi_4 = L_y(\cdot, \bar y_\lambda, \bar u_\lambda)$ and $\varphi_5 = L_u(\cdot, \bar y_\lambda, \bar u_\lambda) + p + \lambda(\mu_b - \mu_a)$. Last, we define

$$\tilde y_a = y_a - \bar y_\lambda - \lambda\bar u_\lambda, \quad \tilde y_b = y_b - \bar y_\lambda - \lambda\bar u_\lambda, \\ \tilde u_a = u_a - \bar u_\lambda, \quad \tilde u_b = u_b - \bar u_\lambda,$$

and

$$\tilde U_{ad} = \{u \in L^2(Q) \mid \tilde u_a \le u \le \tilde u_b\}.$$

With Definition 5.3, we obtain that $P(\delta)$ is equivalent to

$$\tilde{P}(\delta) \min_{\tilde{u}\in\tilde{U}_{ad}} J_\delta(\tilde{u},\tilde{y}) := \iint\limits_Q \left[\frac{1}{2}(\varphi_1\tilde{y}^2 + 2\varphi_2\tilde{y}\tilde{u} + \varphi_3\tilde{u}^2) + \varphi_4\tilde{y} + (\varphi_5 - \delta_1)\tilde{u}\right] dxdt$$

such that

$$\tilde{y}_a + \delta_2 \le \lambda\tilde{u} + \tilde{y} \le \tilde{y}_b - \delta_3,$$

To see this, only the objective function $J_\delta(u,y)$ needs consideration. Let us define the (formal) Lagrange function $\mathcal{L} = \mathcal{L}(u,y,p,\mu_a,\mu_b)$ associated with (P_λ),

$$\mathcal{L} = J(y,u) - (y_t + Ay + d(\cdot,y) - u, p) + (y_a - \lambda u - y, \mu_a) + (\lambda u + y - y_b, \mu_b).$$

It is known that the second derivatives standing in $f_\delta(u)$ can be computed from the Lagrange function by

$$f''(\bar{u}_\lambda)u^2 + (G''(\bar{u}_\lambda)u^2, \bar{\mu}_b^\lambda - \bar{\mu}_a^\lambda) = \mathcal{L}''(\bar{y}_\lambda, \bar{u}_\lambda, p_\lambda, \bar{\mu}_a^\lambda, \bar{\mu}_b^\lambda)(y,u)^2$$

where y and u are coupled by the linearized equation (5.3) and p_λ solves (4.1), cf. [22], Thm. 4.23 for elliptic equations. The second-order derivative $\mathcal{L}''$ is easy to compute and equals the first, quadratic part of J_δ. Moreover, the first order derivative $f'(\bar{u}_\lambda)$ is easily computed and the linear part of J_δ is hence easily obtained.

Definition 5.4. *Let $\hat{u}$ be a fixed reference control. For a fixed $\lambda > 0$, we define the τ-active sets for the linearized unperturbed problem $\tilde{P}(0)$ as*

$$\begin{aligned}
M_{u,a}^\tau(\hat{u}) &:= \{(x,t) \in Q : \ \hat{u}(x,t) \le \tilde{u}_a(x) + \tau\}\\
M_{u,b}^\tau(\hat{u}) &:= \{(x,t) \in Q : \ \hat{u}(x,t) \ge \tilde{u}_b(x) - \tau\}\\
M_{y,a}^\tau(\hat{u}) &:= \{(x,t) \in Q : \ \lambda\hat{u}(x,t) + G'(\bar{u}_\lambda)\hat{u}(x,t) \le \tilde{y}_a(x,t) + \tau\}\\
M_{y,b}^\tau(\hat{u}) &:= \{(x,t) \in Q : \ \lambda\hat{u}_(x,t) + G'(\bar{u}_\lambda)\hat{u}(x,t) \ge \tilde{y}_b(x,t) - \tau\}.
\end{aligned}$$

Formally, we will need a separation condition for the τ-active sets associated with the optimal control of the linearized problem $P(0)$. We will see in the next section that this is obtained directly from the separation condition for the nonlinear problem, with $\tau = \sigma$.

5.2. An auxiliary control problem. We point out that, since $u^\delta = \bar{u}_\lambda$ solves the linearized problem $P(0)$, i.e. $\tilde{u}^\delta \equiv 0$ solves the problem $\tilde{P}(0)$, the τ-active sets associated with the optimal control $\bar{u}_\lambda$ of (P_λ) and $\tilde{u}^\delta$ of $\tilde{P}(0)$ coincide. Thus, there exists $\tau > 0$ such that the τ-active sets are pairwise disjoint, and we obtain the existence of unique Lagrange multipliers $\tilde{\mu}_a^0, \tilde{\mu}_b^0 \in L^\infty(Q)$ associated with the optimal control $\tilde{u}^0$. We choose such a fixed τ and define

$$M_1 = M_{u,a}^\tau(\bar{u}_\lambda), \quad M_2 = M_{u,b}^\tau(\bar{u}_\lambda), \qquad M_3 = M_{y,a}^\tau(\bar{u}_\lambda), \quad M_4 = M_{y,b}^\tau(\bar{u}_\lambda),$$

Finally, we set $M = Q \setminus \{M_1 \cup M_2 \cup M_3 \cup M_4\}$.

For the Lipschitz-stability analysis we define an auxiliary problem, where we ignore all constraints outside the τ-active sets, as in [14] and [2]. Since this section is self-contained and the stability results are applicable to any control problem of the form of $(P(delta))$, we simplify the notation and set $\tilde{u}_a =: u_a$, $\tilde{u}_b =: u_b$, $\tilde{y}_a =: y_a$, $\tilde{y}_a =: y_a$,

$$U_{ad}^{aux} = \{u \in L^2(Q) \mid u_a \le u \text{ a.e. in } M_1, \ u \le u_b \text{ a.e. in } M_2\},$$

and consider a general problem of the form

$$P(\delta)^{aux} \min_{u \in U_{ad}^{aux}} J_\delta(u,y) := \iint\limits_Q \left[\frac{1}{2}(\varphi_1 y^2 + 2\varphi_2 yu + \varphi_3 u^2) + \varphi_4 y + (\varphi_5 - \delta_1)u\right] dxdt$$

such that y solves (5.3) for $\tilde{u} := u$ and

$$y_a + \delta_2 \le \lambda u_a + y \quad \text{a.e. in } M_3$$

$$\lambda u + y \le y_b - \delta_3 \quad \text{a.e. in } M_4,$$

for which we will carry out the stability analysis. In view of (5.1) we assume with some $\kappa > 0$, for all y defined above,

$$\iint\limits_Q \left(\varphi_1 y^2 + 2\varphi_2 yu + \varphi_3 u^2\right) dxdt \ge \kappa \|u\|^2. \tag{5.4}$$

Existence and regularity of Lagrange multipliers. Let us first note that the existence of a unique solution in U_{ad}^{aux} follows just like for $P(\delta)$. However, L^∞-regularity of the optimal control as well as the multipliers is not easily given because the control constraints are not present in all Q, i.e. outside $M_1 \cup M_2$. We will see, however, that the separation condition for the active sets allows for the required regularity. Let us initially state some helpful results for the associated differential equations.

THEOREM 5.5. *There is a real number $s > 0$ such that the operator $G'(\bar{u}_\lambda)$ as well as the adjoint operator $G'(\bar{u}_\lambda)^*$ are continuous from $L^r(Q)$ to $L^{r+s}(Q)$ for all $r \ge 2$.*

The assertion follows for example from Theorem 4.2 [18].

REMARK 5.6. *During the stability analysis of the auxiliary problem, we will simplify the notation in order to maintain readability. In the following, let $\delta, \delta' \in L^\infty(Q)^3$ be two perturbation parameters. Unless noted otherwise, we will denote the optimal controls of $P(\delta)^{aux}$ and $P(\delta')^{aux}$ by u^δ and $u^{\delta'}$, respectively. Likewise, y^δ and $y^{\delta'}$ refer to the associated optimal states, and we will obtain Lagrange multipliers p^δ and $p^{\delta'}$ as well as $\mu_a^\delta, \mu_b^\delta$ and $\mu_a^{\delta'}, \mu_b^{\delta'}$.*

We obtain the following regularity result for our optimal solution.

THEOREM 5.7. *To the solution of $P(\delta)^{aux}$, there exist unique Lagrange multipliers $\mu_a^\delta \in L^2(Q)$, $\mu_b^\delta \in L^2(Q)$ with $\mu_a^\delta = 0$ on $Q \setminus M_3$ and $\mu_b^\delta = 0$ on $Q \setminus M_4$, as well as an adjoint state $p^\delta \in W(0,T)$, such that*

$$(-\delta_1 + \varphi_3 u^\delta + \varphi_5 + \varphi_2 y^\delta + p^\delta + \lambda(\mu_b^\delta - \mu_a^\delta), u - u^\delta) \ge 0 \quad \forall u \in U_{ad}^{aux}, \tag{5.5}$$

and

$$-\delta_1 + \varphi_3 u^\delta + \varphi_5 + \varphi_2 y^\delta + p^\delta + \lambda(\mu_b^\delta - \mu_a^\delta) = 0 \quad \text{on } Q \setminus M_1 \cup M_2, \tag{5.6}$$

where the adjoint state p^δ solves

$$\begin{aligned} -p_t + A^* p + d_0 p &= \varphi_1 y^\delta + \varphi_2 u^\delta + \varphi_4 + \mu_b^\delta - \mu_a^\delta \\ p(\cdot, T) &= 0 \\ \partial_{A^*} p + \alpha p &= 0, \end{aligned} \tag{5.7}$$

and the complementarity conditions

$$\mu_a^\delta \geq 0, \quad (\mu_a^\delta, \lambda u^\delta + y^\delta - y_a - \delta_2) = 0, \quad \mu_b^\delta \geq 0, \quad (\mu_b^\delta, \lambda u^\delta + y^\delta - y_b + \delta_3) = 0$$

are satisfied.

Proof. Let us first express the constraints of $P(\delta)^{aux}$ in another form. The constraints read

$$\begin{array}{rcll} -u & \leq & -u_a & \text{on } M_1 \\ u & \leq & u_b & \text{on } M_2 \\ -\lambda u - G'(\bar{u}_\lambda)u & \leq & -y_a - \delta_2 & \text{on } M_3 \\ \lambda u + G'(\bar{u}_\lambda)u & \leq & y_b - \delta_3 & \text{on } M_4 \\ u(x,t) & \in & \mathbb{R} & \text{on } M. \end{array} \tag{5.8}$$

Define the linear operator $\mathbb{G} : L^2(Q) \to L^2(Q)$ by the left-hand side of (5.8), more precisely,

$$\mathbb{G}u = (\chi_2 - \chi_1 + \chi_M)u + (\chi_4 - \chi_3)(\lambda u + G'(\bar{u}_\lambda)u),$$

where χ_i is the characteristic function of the set M_i. Then (5.8) is equivalent to

$$(\mathbb{G}u)(x,t) \begin{cases} \leq c(x,t) & \text{on } Q \setminus M \\ \text{arbitrary} & \text{on } M, \end{cases}$$

where

$$c(x,t) = -\chi_1 u_a + \chi_2 u_b - \chi_3(-y_a - \delta_2) + \chi_4(y_b - \delta_3).$$

We now show that this system satisfies the well-known regularity condition by Zowe and Kurcyusz, [24]. To introduce it, we need the convex cone

$$K(\bar{v}) = \{\alpha(v - \bar{v}) \mid \alpha \geq 0, \quad v \geq 0 \text{ on } Q \setminus M, \quad v \in L^2(Q)\}.$$

Notice that v is arbitrary on M, since no constraints are given there. There is no further constraint imposed on u, hence the Zowe-Kurcyusz-regularity condition is

$$\mathbb{G}L^2(Q) + K(-\mathbb{G}u^\delta) = L^2(Q),$$

i.e. each $z \in L^2(Q)$ can be represented in the form $z = \mathbb{G}u + \alpha(v + \mathbb{G}u^\delta)$, with $v \geq 0$ on $Q \setminus M$, $u \in L^2(Q)$, $\alpha \geq 0$. This is equivalent to $\mathbb{G}u + v + \alpha\mathbb{G}u^\delta = z$ with the same restrictions on v. It turns out that $\alpha = 0$ can be taken and also $v = 0$, i.e. $\mathbb{G}u = z$. A comparison with (5.8) shows that we can take

$$u = \begin{cases} -z & \text{on } M_1 \\ z & \text{on } M_2 \cup M. \end{cases} \tag{5.9}$$

It remains to find u on $M_3 \cup M_4$. Define $\hat{u}$ as the function that satisfies (5.9) on $M_1 \cup M_2 \cup M$ and is zero on $M_3 \cup M_4$. Then $u = u_3 + u_4 + \hat{u}$, where $u_3 = 0$ on $Q \setminus M_3$ and $u_4 = 0$ on $Q \setminus M_4$. We get the equation

$$\mathbb{G}(u_3 + u_4 + \hat{u}) = z \quad \text{on } M_3 \cup M_4,$$

hence

$$\begin{array}{rcll} \lambda u_3 + G'(\bar{u}_\lambda)(u_3 + u_4 + \hat{u}) & = & -z & \text{on } M_3 \\ \lambda u_4 + G'(\bar{u}_\lambda)(u_3 + u_4 + \hat{u}) & = & z & \text{on } M_4. \end{array} \tag{5.10}$$

Given $\hat{u}$, $y = G'(\bar{u}_\lambda)(u_3 + u_4 + \hat{u})$ is the solution to

$$\begin{aligned} y_t + Ay + d_0 y &= u_3 + u_4 + \hat{u} \\ y(\cdot, 0) &= 0 \\ \partial_A y + \alpha y &= 0. \end{aligned} \tag{5.11}$$

Therefore, (5.10) can be written as $u_3 = \frac{1}{\lambda}(-z - y)$, $u_4 = \frac{1}{\lambda}(z - y)$. Inserting this in (5.11), y has to solve the equation

$$\begin{aligned} y_t + Ay + d_0 y + \frac{1}{\lambda}\chi_{M_3 \cup M_4} y &= \frac{1}{\lambda}(\chi_4 - \chi_3)z + \hat{u} \\ y(\cdot, 0) &= 0 \\ \partial_A y + \alpha y &= 0. \end{aligned} \tag{5.12}$$

This equation has a unique solution. On the other hand, given the solution of (5.12), $u_3 = \frac{1}{\lambda}(-z - y)$ and $u_4 = \frac{1}{\lambda}(z - y)$ satisfy, together with $\hat{u}$, the system (5.10). Therefore, the Kurcyusz-Zowe condition is satisfied. From the associated Lagrange multiplier rule, we obtain at least one Lagrange multiplier function $\mu^\delta \in L^2(Q)$ with $\mu^\delta \geq 0$ on $Q \setminus M$. Now, the Lagrange multipliers to the associated single constraints are obtained by restriction of μ^δ to the appropriate sets. Their uniqueness follows from the fact that the τ-active sets are pairwise disjoint. □

Lemma 5.8. *The Lagrange multipliers associated with the solution of $P(\delta)^{aux}$ fulfill the projection formula*

$$\mu_a^\delta = \max\{0, \frac{\varphi_3}{\lambda^2}(y_a + \delta_2 - y^\delta) + \frac{1}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + p^\delta)\} \quad \textit{on } M_3 \tag{5.13}$$

$$\mu_b^\delta = \max\{0, \frac{\varphi_3}{\lambda^2}(y^\delta + \delta_3 - y_b) - \frac{1}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + p^\delta)\} \quad \textit{on } M_4. \tag{5.14}$$

Proof. Note first that $\varphi_3 = L_{uu}(\cdot, \bar{y}_\lambda, \bar{u}_\lambda) \geq \beta_0 > 0$ on Q due to our general assumptions. The projection formulas can be shown analogously to [23]. The proof is based on the fact that the multipliers are represented by

$$\mu_a^\delta = \max\{0, \mu_a^\delta + c(y_a + \delta_2 - \lambda u^\delta - y^\delta)\} \quad \text{on } M_3$$

$$\mu_b^\delta = \max\{0, \mu_b^\delta + c(\lambda u^\delta + y^\delta + \delta_3 - y_b)\} \quad \text{on } M_4,$$

for an arbitrary $c = c(x,t) > 0$, which is an idea from [9]. Clearly, if the max is positive, the multiplier cancels out in the associated equation and we see that the inequality is active. If the max is negative, the multiplier is zero and the inequality is inactive. This representation, however, contains the control u^δ in the right-hand-side. The main idea is to represent u^δ in terms of the other quantities, especially containing μ_a^δ and μ_b^δ. With an adequate choice of c, the multipliers inside the max-function cancel out. The variational inequality is given as a gradient equation on M_3 and M_4. Hence,

$$-\delta_1 + \varphi_3 u^\delta + \varphi_5 + \varphi_2 y^\delta + \lambda(\mu_b^\delta - \mu_a^\delta) + p^\delta = 0 \quad \text{on } M_3 \cup M_4, \tag{5.15}$$

which yields

$$\lambda u^\delta = -\frac{\lambda}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + \lambda(\mu_b^\delta - \mu_a^\delta) + p^\delta) \quad \text{on } M_3 \cup M_4.$$

The last equation can be inserted into the maximum representation of the multipliers, since they are nonzero only on their respective active sets. Choosing $c = \frac{\varphi_3}{\lambda^2}$ then yields the assertion. ∎

THEOREM 5.9. *The optimal control u^δ of $P(\delta)^{aux}$ and the associated Lagrange multipliers $\mu_a^\delta, \mu_b^\delta$ are functions in $L^\infty(Q)$.*

Proof. We will use a bootstrapping argument to show L^∞-regularity of the control and the multipliers. Initially, we know that u^δ, μ_a^δ, and μ_b^δ are $L^2(Q)$-functions, u^δ is bounded on $M_1 \cup M_2$ due to the control constraints, and $\mu_a^\delta, \mu_b^\delta$ are zero, hence bounded, on $Q \setminus \{M_3 \cup M_4\}$. It remains to show boundedness of u^δ on $Q \setminus \{M_1 \cup M_2\}$ as well as boundedness of $\mu_a^\delta, \mu_b^\delta$ on $M_3 \cup M_4$.

From $u^\delta \in L^2(Q)$ we obtain with the help of Theorem 5.5 that $y^\delta \in L^{2+s}(Q)$, $s > 0$, which together with $\mu_a^\delta, \mu_b^\delta \in L^2(Q)$ ensures $p^\delta \in L^{2+s}(Q)$ by the same theorem, since all other expressions appearing in the right-hand-side of the adjoint equation are L^∞-functions by our assumptions. On M_3, M_4, respectively, we have the projection formulas (5.13) and (5.14), where all appearing functions are at least $L^{2+s}(Q)$ functions. Since the max-function preserves this regularity, we obtain L^{2+s}-regularity of the multipliers μ_a^δ and μ_b^δ in Q. Consequently, we obtain from the gradient equation (5.6), that $u^\delta_{Q\setminus\{M_1\cup M_2\}} \in L^{2+s}(Q \setminus \{M_1 \cup M_2\})$. Hence, $u^\delta, \mu_a^\delta, \mu_b^\delta \in L^{2+s}(Q)$. Repeating this argument, we obtain after finitely many steps that $u^\delta \in L^r(Q)$, $r > \frac{N}{2}+1$, which yields continuity of the state y^δ, hence also continuity of the adjoint state p^δ by Theorem 2.2. This implies in return boundedness of the Lagrange multipliers μ_a^δ and μ_b^δ by the projection formulas and finally boundedness of the optimal control u^δ due to the gradient equation. ∎

Stability analysis of $P(\delta)^{aux}$ in $L^2(Q)$. Let us start with the stability analysis of $P(\delta)^{aux}$ in $L^2(Q)$. We choose two perturbation vectors $\delta = (\delta_1, \delta_2, \delta_3) \in L^\infty(Q)^3$ and $\delta' = (\delta_1', \delta_2', \delta_3') \in L^\infty(Q)^3$ with associated optimal solutions u^δ and $u^{\delta'}$. The main result of this paragraph is Theorem 5.12 that states L^2-Lipschitz stability for the optimal control of $P(\delta)^{aux}$,

$$\|u^\delta - u^{\delta'}\|_{L^2(Q)} \le L_2^u \|\delta - \delta'\|_{L^2(Q)^3}.$$

We introduce the following short notation:

$$\delta u = u^\delta - u^{\delta'}, \quad \delta y = y^\delta - y^{\delta'}, \quad \delta p = p^\delta - p^{\delta'},$$
$$\delta\mu_a = \mu_a^\delta - \mu_a^{\delta'}, \quad \delta\mu_b = \mu_b^\delta - \mu_b^{\delta'}.$$

In the following, we will consider the optimality system for $P(\delta)^{aux}$ and $P(\delta')^{aux}$ and derive an estimate for $\|\delta u\|$ that does not depend on the Lagrange multipliers. This can be done following an idea by Griesse, [8]. For that purpose, we will prove several auxiliary results. Note again that $\|\cdot\|$ refers to the norm in $L^2(Q)$, unless denoted otherwise. We point out here that the solutions of the state and adjoint equations depend continuously on the right-hand-side, and we will use generic constants $c > 0$ for our estimates. Hence, Theorem 2.2 allows to estimate the L^2-norm of $y = G'(\bar{u}_\lambda)u$ by $\|y\| \le c\|u\|$, and similarly for the adjoint state.

LEMMA 5.10. *Let δu, δy, δp, as well as $\delta\mu_a$ and $\delta\mu_b$ be given as above. Then*

$$\kappa\|\delta u\|^2 \le (\delta_1 - \delta_1', \delta u) - (\lambda\delta u + \delta y, \delta\mu_b - \delta\mu_a)$$

is satisfied.

Proof. First, insert $u^{\delta'}$ into the variational inequality for u^δ, (5.5). Then, consider the variational inequality for $u^{\delta'}$, obtained from (5.5) by substituting δ' for δ, and insert u^δ. Adding both inequalities yields

$$(\varphi_3\delta u,\delta u)+(\varphi_2\delta y,\delta u)+(\delta p,\delta u)\le(\delta_1-\delta_1',\delta u)-\lambda(\delta\mu_b-\delta\mu_a,\delta u). \tag{5.16}$$

By standard calculations with the adjoint equation we obtain

$$(\delta p,\delta u)=(\varphi_1\delta y+\varphi_2\delta u+\delta\mu_b-\delta\mu_a,\delta y).$$

Inserting this in (5.16)

$$(\varphi_3\delta u,\delta u)+(\varphi_1\delta y,\delta y)+2(\varphi_2\delta y,\delta u)\le(\delta_1-\delta_1',\delta u)-(\lambda\delta u+\delta y,\delta\mu_b-\delta\mu_a).$$

With (5.4), the assertion is proven. □

Lemma 5.11. *The Lagrange multipliers satisfy*

$$(\lambda\delta u+\delta y,\delta\mu_a)\le(\delta_2-\delta_2',\delta\mu_a) \tag{5.17}$$

$$-(\lambda\delta u+\delta y,\delta\mu_b)\le(\delta_3-\delta_3',\delta\mu_b). \tag{5.18}$$

Moreover, there exists $c>0$ such that

$$\|\delta\mu_a\|\le c(\|\delta_1-\delta_1'\|+\|\delta u\|) \tag{5.19}$$

$$\|\delta\mu_b\|\le c(\|\delta_1-\delta_1'\|+\|\delta u\|). \tag{5.20}$$

Proof. We first prove (5.17). From the complementary slackness conditions, we obtain

$$(y_a+\delta_2-\lambda u^\delta-y^\delta,\mu_a^{\delta'}-\mu_a^\delta)\le 0,$$

as well as

$$(y_a+\delta_2'-\lambda u^{\delta'}-y^{\delta'},\mu_a^{\delta}-\mu_a^{\delta'})\le 0.$$

Adding these inequality yields

$$(\lambda\delta u+\delta y,\delta\mu_a)\le(\delta_2-\delta_2',\delta\mu_a).$$

The second inequality (5.18) is shown analogously. For the norm estimates (5.19) and (5.20) note first that by (5.15), we have

$$\begin{aligned}\lambda\|\delta\mu_a\|_{2,Q} &= \lambda\|\delta\mu_a\|_{2,M_3}=\|-(\delta_1-\delta_1')+\varphi_3\delta u+\varphi_2\delta y+\delta p\|_{2,M_3}\\ &\le \|\delta_1-\delta_1'\|+\|\varphi_3\|_\infty\|\delta u\|+\|\varphi_2\|_\infty\|\delta y\|+\|\delta p\|.\end{aligned} \tag{5.21}$$

To estimate $\|\delta p\|$, we note that the gradient equation (5.6) yields

$$\begin{aligned}\delta\mu_a&=\chi_3(-(\delta_1-\delta_1')+\varphi_2\delta y+\varphi_3\delta u+\delta p),\\ \delta\mu_b&=-\chi_4(-(\delta_1-\delta_1')+\varphi_2\delta y+\varphi_3\delta u+\delta p),\end{aligned}$$

where χ_i denotes the characteristic function of M_i. With (5.7), δp hence satisfies

$$\begin{aligned}-\delta p_t+A^*\delta p+(d_0+\chi_3+\chi_4)\delta p&=\varphi_1\delta y+\varphi_2\delta u+(\chi_3+\chi_4)((\delta_1-\delta_1')-\varphi_2\delta y-\varphi_3\delta u)\\ \delta p(\cdot,T)&=0\\ \partial_{A^*}\delta p+\alpha\delta p&=0.\end{aligned}$$

Applying Theorem 2.2, we obtain therefore

$$\|\delta p\| \leq c(\|\delta y\| + \|\delta u\| + \|\delta_1 - \delta_1'\|)$$

for some $c > 0$. Applying Theorem 2.2 to estimate $\|\delta y\| \leq c\|\delta u\|$ and collecting all estimates, we obtain from (5.21) $\|\delta\mu_a\| \leq c(\|\delta_1 - \delta_1'\| + \|\delta u\|)$. The estimate for $\|\delta\mu_b\|$ follows analogously. □

THEOREM 5.12. *Let δ and δ' be two perturbation vectors. Then*

$$\|u^\delta - u^{\delta'}\| \leq L_2^u \|\delta - \delta'\|_{L^2(Q)^3}$$

holds for the associated optimal controls u^δ and $u^{\delta'}$ of $P(\delta)^{aux}$.

Proof. Combining the results from Lemmas 5.10 and 5.11 we arrive at

$$\begin{aligned}
\kappa\|\delta u\|^2 &\leq (\delta_1 - \delta_1', \delta u) + (\delta_2 - \delta_2', \delta\mu_a) + (\delta_3 - \delta_3', \delta\mu_b) \\
&\leq \|\delta_1 - \delta_1'\|\|\delta u\| + \|\delta_2 - \delta_2'\|\|\delta\mu_a\| + \|\delta_3 - \delta_3'\|\|\delta\mu_b\| \\
&\leq \frac{\kappa}{2}\|\delta u\|^2 + c(\|\delta_1 - \delta_1'\|^2 + \|\delta_2 - \delta_2'\|^2 + \|\delta_3 - \delta_3'\|^2)
\end{aligned}$$

for $c > 0$ by Young's inequality. From this, the result follows. □

REMARK 5.13. *Let us point out that by the previous arguments, we obtain also $\|\delta y\| \leq L_2^y \|\delta - \delta'\|_{L^2(Q)^3}$, $\|\delta p\| \leq L_2^p \|\delta - \delta'\|_{L^2(Q)^3}$, as well as $\|\delta\mu_b\|, \|\delta\mu_a\| \leq L_2^\mu \|\delta - \delta'\|_{L^2(Q)^3}$.*

Stability analysis of $P(\delta)^{aux}$ in $L^\infty(Q)$. With the L^2-stability at hand, we are able to derive an associated L^∞-result.

THEOREM 5.14. *There exists a constant L_∞^u such that for any given $\delta, \delta' \in L^\infty(Q)^3$ the corresponding solutions of the auxiliary problem satisfy*

$$\|u^\delta - u^{\delta'}\|_\infty \leq L_\infty^u \|\delta - \delta'\|_{L^\infty(Q)^3}.$$

Proof. The proof requires a bootstrapping argument. We point out again that we will use a generic constant c whereever appropriate. We first prove a stability estimate for $\|\mu_a^\delta - \mu_a^{\delta'}\|_{2+s}$, where $s > 0$ as in Theorem 5.5. From the projection formula, we obtain on M_3

$$\begin{aligned}
\mu_a^\delta - \mu_a^{\delta'} &= \max\{0, \frac{\varphi_3}{\lambda^2}(y_a + \delta_2 - y^\delta) + \frac{1}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + p^\delta)\} \\
&\quad - \max\{0, \frac{\varphi_3}{\lambda^2}(y_a + \delta_2' - y^{\delta'}) + \frac{1}{\varphi_3}(-\delta_1' + \varphi_5 + \varphi_2 y^{\delta'} + p^{\delta'})\} \\
&\leq \max\{0, \frac{\varphi_3}{\lambda^2}(\delta_2 - \delta_2' - \delta y) + \frac{1}{\varphi_3}(-(\delta_1 - \delta_1') + \varphi_2 \delta y + \delta p)\}.
\end{aligned}$$

By considering the corresponding inequality for $\mu_a^{\delta'} - \mu_a^\delta$ we obtain

$$\|\delta\mu_a\|_{2+s} \leq c(\|\delta_1 - \delta_1'\|_\infty + \|\delta_2 - \delta_2'\|_\infty + \|\delta_y\|_{2+s} + \|\delta p\|_{2+s}) \tag{5.22}$$

for a constant $c > 0$. With arguments analogous to the proof of Lemma 5.11, we obtain $\|\delta p\|_{2+s} \leq c(\|\delta_1 - \delta_1'\|_\infty + \|\delta u\|_2 + \|\delta y\|_2)$. With Theorem 2.2, $\|\delta y\|_{2+s}$ can be estimated by $\|\delta y\|_{2+s} \leq c\|\delta u\|_2$. Obviously, we also have $\|\delta y\|_2 \leq c\|\delta u\|_2$. Hence, we obtain

$$\|\mu_a^\delta - \mu_a^{\delta'}\|_{2+s} \leq c(\|\delta - \delta'\|_{L^\infty(Q)^3} + \|\delta u\|_2). \tag{5.23}$$

We apply Theorem 5.12 and note that $\|\delta\|_{L^2(Q)^3} \le c\,\|\delta\|_{L^\infty(Q)^3}$, and obtain

$$\|\mu_a^\delta - \mu_a^{\delta'}\|_{2+s} \le c\|\delta - \delta'\|_{L^\infty(Q)^3}. \tag{5.24}$$

An analogous estimate holds for $\|\delta\mu_b\|_{2+s}$. Now, from the gradient equation (5.6) we deduce

$$\delta u = -\frac{1}{\varphi_3}(-(\delta_1 - \delta_1') + \varphi_2\delta y + \delta p + \lambda(\delta\mu_b - \delta\mu_a)) \quad \text{on } Q \setminus M_1 \cup M_2,$$

where $\delta\mu_a, \delta\mu_b \equiv 0$ outside M_3, M_4, respectively, hence

$$\|\delta u\|_{2+s,Q\setminus M_1\cup M_2} \le c(\|\delta_1 - \delta_1'\|_\infty + \|\varphi_2\|\|\delta y\|_{2+s} + \|\delta p\|_{2+s} + \lambda(\|\delta\mu_b\|_{2+s} + \|\delta\mu_a\|_{2+s}).$$

Inserting the estimate (5.24) and its analogon for the upper bound and reapplying the previous steps leads to

$$\|\delta u\|_{2+s,Q\setminus M_1\cup M_2} \le c(\|\delta\|_{L^\infty(Q)^3} + \|\delta u\|_2) \le c\|\delta\|_{L^\infty(Q)^3}. \tag{5.25}$$

It remains to estimate $\|\delta u\|_{2+s}$ on $M_1 \cup M_2$. It follows from the variational inequality (5.5) that u^δ satisfies the projection formula

$$u^\delta = P_{[u_a,u_b]}(-\frac{1}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + p^\delta)) \quad \text{on } M_1 \cup M_2,$$

and $u^{\delta'}$ satisfies an analogous formula.

The projection operator is Lipschitz with constant 1. Therefore, on $M_1 \cup M_2$ we obtain pointwisely

$$\begin{aligned}|\delta u| &= |P_{[u_a,u_b]}(-\frac{1}{\varphi_3}(-\delta_1 + \varphi_5 + \varphi_2 y^\delta + p^\delta)) - P_{[u_a,u_b]}(-\frac{1}{\varphi_3}(-\delta_1' + \varphi_5 + \varphi_2 y^{\delta'} + p^{\delta'}))| \\ &\le \frac{1}{\beta_0}\{|\delta_1' - \delta_1| + |\varphi_2||\delta y| + |\delta p|\}\end{aligned}$$

Hence

$$\|\delta u\|_{2+s,M_1\cup M_2} \le \frac{1}{\beta_0}(\|\delta_1 - \delta_1'\|_\infty + \|\varphi_2\|\|\delta y\|_{2+s} + \|\delta p\|_{2+s}). \tag{5.26}$$

Estimating the norms as before, we obtain $\|\delta u\|_{2+s} \le c\|\delta\|_{L^\infty(Q)^3}$. This allows to estimate $\|\delta\mu_a\|_{2+2s}$ in (5.22) which leads to $\|\delta u\|_{2+s} \le c\|\delta\|_{L^\infty(Q)^3}$. After finitely many steps, we obtain $\|\delta u\|_r \le c\,\|\delta\|_{L^\infty(Q)^3}$, where $r > \frac{N}{2} + 1$. In return, this allows to estimate $\|\delta\mu_a\|_\infty$ in (5.22), which finally yields the assertion with an appropriate L^u_∞. ∎

REMARK 5.15. *As for the L^2-stability analysis, we point out that we obtain also*

$$\|\delta y\|_\infty \le L^y_\infty\|\delta - \delta'\|_{L^\infty(Q)^3}, \quad \|\delta p\|_\infty \le L^p_\infty\|\delta - \delta'\|_{L^\infty(Q)^3}$$

as well as

$$\|\delta\mu_a\|_\infty, \|\delta\mu_b\|_\infty \le L^\mu_\infty\|\delta - \delta'\|_{L^\infty(Q)^3}.$$

5.3. Stability analysis for the original problem. Still following [2], it remains to carry out the stability analysis for the original problem. We rely on the observation that for sufficiently small δ the solutions of $P(\delta)$ and $P(\delta)^{aux}$ coincide. We therefore admit $\delta \in L^\infty(Q)^3$ that satisfy:

$$\|\delta\| \le \min(g_1(\tau), g_2(\tau)), \tag{5.27}$$

where $g_1(\tau) = \tau^{-1} L^u_\infty$ and $g_2(\tau) = \tau^{-1}(1 + \lambda L^u_\infty + L^y_\infty)$.

LEMMA 5.16. *Suppose that* $\|\delta\|_{L^\infty(Q)^3} \le g(\tau)$ *and that* (u^δ_{aux}) *is an optimal solution of* $P(\delta)^{aux}$ *with Lagrange multipliers* $\mu^\delta_{a,aux}, \mu^\delta_{b,aux}$. *Then the solution is feasible for the linearized original problem,* $P(\delta)$. *The triple* $(u^\delta_{aux}, \mu^\delta_{a,aux}, \mu^\delta_{b,aux})$ *satisfies the optimality system* $L(\delta)$, *and* u^δ_{aux} *is the unique optimal solution of* $P(\delta)$ *with associated unique multipliers* $\mu^\delta_{a,aux} = \mu^\delta_a$, $\mu^\delta_{b,aux} = \mu^\delta_b$.

Proof. The control u^δ_{aux} is feasible for $P(\delta)^{aux}$, hence it remains to show

$$\begin{aligned} u^\delta_{aux} &\ge \tilde{u}_a \quad \text{on } Q \setminus M_1, \quad u^\delta_{aux} \le \tilde{u}_b \quad \text{on } Q \setminus M_2, \\ \lambda u^\delta_{aux} + y^\delta_{aux} &\ge \tilde{y}_a + \delta_2 \quad \text{on } Q \setminus M_3, \quad \lambda u^\delta_{aux} + y^\delta_{aux} \le \tilde{y}_b - \delta_3 \quad \text{on } Q \setminus M_4. \end{aligned}$$

For the solution u^0_{aux} we know that $u^0_{aux} \ge \tilde{u}_a + \tau$ a.e. on $Q \setminus M_1$. Hence, we have

$$\begin{aligned} u^\delta_{aux} &= u^0_{aux} + u^\delta_{aux} - u^0_{aux} \ge u^0_{aux} - \|u^\delta_{aux} - u^0_{aux}\|_\infty \\ &\ge \tilde{u}_a + \tau - L^u_\infty \|\delta\|_{L^\infty(Q)^3} \ge \tilde{u}_a + \tau - L^u_\infty g_1(\tau) = \tilde{u}_a \end{aligned}$$

almost everywhere on $Q \setminus M_1$. The upper bound $\tilde{u}_b$ is treated similarly. For the mixed control-state constraints, we obtain $\lambda u^0_{aux} + y^0_{aux} \ge \tilde{y}_a + \tau$ a. e. on M_3. Consequently,

$$\begin{aligned} \lambda u^\delta_{aux} + y^\delta_{aux} - \delta_2 &= \lambda u^0_{aux} + y^0_{aux} + \lambda u^\delta_{aux} + y^\delta_{aux} - \delta_2 - \lambda u^0_{aux} - y^0_{aux} \\ &\ge \tilde{y}_a + \tau - \|\delta_2\|_\infty - \lambda \|u^\delta_{aux} - u^0_{aux}\|_\infty - \|y^\delta_{aux} - y^0_{aux}\|_\infty \\ &\ge \tilde{y}_a + \tau - \|\delta_2\|_\infty - \lambda L^u_\infty \|\delta\|_{L^\infty(Q)^3} - L^y_\infty \|\delta\|_{L^\infty(Q)^3} \\ &\ge \tilde{y}_a + \tau - (1 + \lambda L^u_\infty + L^y_\infty)\|\delta\|_{L^\infty(Q)^3} \\ &\ge \tilde{y}_a + \tau - (1 + \lambda L^u_\infty + L^y_\infty) g_2(\tau) \ge \tilde{y}_a \end{aligned}$$

almost everywhere on $Q \setminus M_3$. The upper bound and the control constraints are treated analogously. It is easy to see that $u^\delta_{aux}, \mu^\delta_{a,aux}, \mu^\delta_{b,aux}$ satisfy the optimality system $L(\delta)$ for $P(\delta)$, which is a sufficient condition for optimality, hence u^δ_{aux} with associated state y^δ_{aux} is the unique solution of $P(\delta)$. It remains to prove that $\mu^\delta_a, \mu^\delta_b$ are unique. Since $\mu^\delta_{a,aux}$ and $\mu^\delta_{b,aux}$ satisfy the optimality system, the assertion then follows. We consider a point $(x^*, t^*) \in A_3$, i.e. $\lambda u^\delta(x^*, t^*) + y^\delta(x^*, t^*) = \tilde{y}_a + \delta_2$. We know

$$\begin{aligned} \lambda u^0(x^*, t^*) + y^0(x^*, t^*) &= \lambda u^0(x^*, t^*) + y^0(x^*, t^*) - \lambda u^\delta(x^*, t^*) - y^\delta(x^*, t^*) \\ &\quad + \lambda u^\delta(x^*, t^*) + y^\delta(x^*, t^*) - \tilde{y}_a - \delta_2 + \tilde{y}_a + \delta_2 \\ &\le \lambda \|u^0 - u^\delta\|_\infty + \|y^0 - y^\delta\|_\infty + \|\delta_2\|_\infty + \tilde{y}_a \\ &\le (1 + L_\infty + cL_\infty)\|\delta\|_{L^\infty(Q)^3} + \tilde{y}_a \le \tau + \tilde{y}_a, \end{aligned}$$

where we used Lemma 5.16. Hence, $(x^*, t^*) \in M_3$. Applying analogous arguments to the other constraints, we obtain $A_1 \cup A_2 \cup A_3 \cup A_4 = \emptyset$, hence we can prove uniqueness of the Lagrange multipliers by arguments analogous to the proof of Theorem 4.6. □

THEOREM 5.17. *There exists a constant $L^u > 0$ such that for any δ, δ' satisfying* (5.27)*, the unique solution u^δ satisfies*

$$\|u^\delta - \bar{u}^{\delta'}\|_\infty \le L^u \|\delta - \delta'\|_{L^\infty(Q)^3}.$$

Proof. By the previous lemma, the solutions of the auxiliary and the original problem coincide, hence we can apply the Lipschitz stability result for $P(\delta)^{aux}$ to $P(\delta)$. □

Analogous results hold for adjoint state and Lagrange multipliers. Collecting all our results, we obtain that the linearized generalized equation is uniquely solvable with solution and regular Lagrange multipliers depending Lipschitzian on the perturbations. Applying Robinson's implicit function theorem this yields local uniqueness of local optima for the nonlinear Lavrentiev-regularized problem (P_λ). We have proven the following theorem:

THEOREM 5.18. *Let the general Assumption 2.1 be satisfied. If a locally optimal control $\bar{u}_\lambda$ to P_λ satisfies additionally the second order sufficient condition 5.1 and, for fixed λ, the separation condition 4.5 for the active sets, then it is locally unique.*

Notice that this result holds true for any dimension N of Ω. This indicates that the Lavrentiev regularized problems permit a deeper numerical analysis than the unregularized ones.

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