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DYNAMICAL STATE AND CONTROL RECONSTRUCTION FOR A PHASE FIELD MODEL

V. Maksimov[1] and F. Tröltzsch[2]

[1]Institute of Mathematics and Mechanics
Ural Branch of Russian Academy of Sciences
S.Kovalevskoi Str. 16, Ekaterinburg, 620219 Russia
[2]Technische Universität Berlin
Fakultät II — Mathematik und Naturwissenschaften
Berlin, Germany

Abstract. The problem of reconstructing the state and control functions of a distributed parameter system from measurements of its observable part is considered for a phase field model. An algorithm is suggested that is stable with respect to measurement errors and computational errors. It is based on the ideas of the theory of feedback control.
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1 Introduction

In this paper, we consider the problem to reconstruct an unobservable part of the state of a nonlinear parabolic control system together with the associated unknown control from certain measurements. The reconstruction process is dynamical with respect to the time, i.e., it exploits information on the state that have been obtained up to the current instant of time.

We present our method for the following phase field model that has been introduced in [2, 3, 5, 7, 8],

$$\frac{\partial}{\partial t}\psi + l\frac{\partial}{\partial t}\varphi = \Delta\psi + u \quad \text{in} \quad \Omega \times (0, \vartheta], \tag{1.1}$$

$$\frac{\partial}{\partial t}\varphi = \Delta\varphi + g(\varphi) + \psi \quad \text{in} \quad \Omega \times (0, \vartheta], \tag{1.2}$$

with boundary conditions

$$\frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0 \quad \text{on} \quad \partial\Omega \times (0, \vartheta], \tag{1.3}$$

and initial conditions

$$\psi(0) = \psi_0, \quad \varphi(0) = \varphi_0 \quad \text{in} \quad \Omega. \tag{1.4}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain, the state functions are φ (phase function) and ψ (temperature), and the control function is u. All these functions depend on $\eta \in \Omega$ and $t \in [0, \vartheta]$. By Δ, the Laplace operator with respect to the spatial coordinate η is denoted. More details are given in the next section.

Compared with [2, 3, 5, 7, 8], this model is slightly simplified. The original version contains some additional constants that have been normalized here to one for convenience.

To explain our problem, let us assume that the phase function φ can be measured during the time interval $[0, \vartheta]$, while the temperature ψ and the chosen control function u are unknown. However, we have the additional information that u is taken from a set U_{ad} of admissible controls.

Our aim is to reconstruct ψ and u from measurements of φ. The associated algorithm should work *dynamically,* i.e., the reconstruction at the time t should exploit the information gained in $[0, t]$ and this information must be processed sufficiently fast – real time in some sense.

Moreover, the algorithm has to be *stable* with respect to the inevitable measurement errors of φ. This means that the computed approximations tend to ψ and u, respectively, if the measurement error tends to zero.

In this way, the problem under discussion is related to inverse problems for controlled dynamical distributed parameter systems and, in a more general context, to the theory of ill-posed problems [19, 4].

Let us denote by $(\cdot, \cdot)_H$ and $|\cdot|_H$ the scalar product and the norm of $H = L^2(\Omega)$, respectively. In all what follows, we consider φ, ψ, u etc. as abstract functions on $[0, \vartheta]$ with values in H.

The main scope of the paper can be described as follows: An unobserved control $u = u_*$ acts upon the system (1.1)–(1.4) that is denoted by S.

We assume that the set of admissible controls is of the form:

$$U_{ad} = \{u \in L_2(T; H) : u(t) \in U \quad \text{for a.a.} \quad t \in T\},$$

where $T = [0, \vartheta]$ and $U \subset H$ is a given convex, bounded, and closed set. At discrete instants of time τ_i, $i = 0, \ldots, m$, the phase function φ is measured. The results of the measurements are functions $\xi_i \in H$. We assume that they satisfy the inequalities

$$|\varphi(\tau_i) - \xi_i|_H \leq h. \tag{1.5}$$

Here, $h \in (0, 1)$ stands for a level of informational noise. It is necessary to construct an algorithm that allows to calculate approximations to the unknown coordinate ψ as well as to the unknown input $u = u_*$.

Since exact reconstruction is impossible (in particular, due to inaccurate measurements of $\varphi(\tau_i)$), the algorithm should determine approximations v^h and u^h such that the following norms tend to zero as $h \downarrow 0$:

$$|v^h - \psi|_{C(T;H)} = \sup_{0 \leq t \leq \vartheta} |v^h(t) - \psi(t)|_H;$$

$$|v^h - \psi|_{L_2(T;H^1(\Omega))} = \Big(\int\limits_0^{\vartheta} |v^h(t) - \psi(t)|^2_{H^1(\Omega)}\, dt \Big)^{1/2};$$

$$|u^h - u_*|_{L_2(T;H)} = \Big(\int\limits_0^{\vartheta} |u^h(t) - u_*(t)|^2_H\, dt \Big)^{1/2}.$$

Our method to solve this problem can be briefly sketched as follows: We introduce an auxiliary control system Σ, the so-called *model*:

$$\begin{aligned} \frac{\partial w_1}{\partial t} &= \Delta w_1 + g(w_1) + p \text{ in } \Omega \times (0, \vartheta], \\ \frac{\partial w_2}{\partial t} + l \frac{\partial w_3}{\partial t} &= \Delta w_2 + u, \\ \frac{\partial w_3}{\partial t} &= \Delta w_3 + g(w_3) + w_2 \end{aligned} \tag{1.6}$$

with boundary conditions

$$\frac{\partial}{\partial n} w_1 = \frac{\partial}{\partial n} w_2 = \frac{\partial}{\partial n} w_3 = 0 \quad \text{on} \quad \partial\Omega \times (0, \vartheta]$$

and initial conditions

$$w_2(0) = \psi_0, \quad w_1(0) = w_3(0) = \varphi_0 \quad \text{in} \quad \Omega.$$

Here, the functions p and u play the role of auxiliary controls, and the vector $w(t) = \{w_1(t), w_2(t), w_3(t)\} \in H \times H \times H$ is said to be the *model state* at the time t. In (1.6), the first equation is an auxiliary one, while the remaining equations are a copy of (1.1), (1.2).

Roughly speaking, following [10], the auxiliary controls are defined by certain feedback formulas depending on the current value $w(t)$ and the current measurement $\xi(t)$, $p(t) = \mathcal{V}(t, w_1(t), \xi(t))$ and $u(t) = \mathcal{U}(t, w_2(t), w_3(t), \xi(t), p(t))$. The functions $\mathcal{V}$ and $\mathcal{U}$ are called *control strategies* for the model Σ and will be defined later.

These formulas for p and u are inserted into the model described above. The functions $v = w_2$ and u computed by this scheme, are considered as approximations to the unknown temperature ψ and the unknown control u_*.

To make this idea work, we have to perform a discretization in time. We introduce an equidistant partition of $[0, \vartheta]$ by $\delta = \vartheta/m$ and $\tau_i = i\,\delta$, $i = 0, \ldots, m$. The reconstruction algorithm is decomposed into $m-1$ identical steps. For $i \in [0 : m-1]$, the i-th step of the algorithm proceeds as follows:

Initiating from $w(\tau_i)$ compute

$$p_i = \mathcal{V}(\tau_i, w_1(\tau_i), \xi_i), \tag{1.7}$$

$$u_i = \mathcal{U}(\tau_i, w_2(\tau_i), w_3(\tau_i), \xi_i, p_i). \tag{1.8}$$

Then define for all $t \in (\tau_i, \tau_{i+1}]$

$$p(t) = p_i, \qquad u(t) = u_i. \tag{1.9}$$

Insert these auxiliary control functions $p(\cdot)$ *and* $u(\cdot)$ *into the model system (1.6) and determine its solution* w *on* $(\tau_i, \tau_{i+1}]$.

The value $w(\tau_{i+1})$ is taken as initial value for the next step. As the reader will have noticed, the functions obtained in this way depend on the discretization parameter δ. Moreover, δ depends on the accuracy h of the measurements, hence $\delta = \delta(h)$. Therefore, all computed functions finally depend on h. This is indicated by a superscript h as in p^h, u^h, w^h, v^h.

In this way, piecewise constant auxiliary control functions p^h and u^h as well as the model trajectory $w^h = \{w_1^h, w_2^h, w_3^h\}$ are determined. The functions $v^h = w_2^h$ and u^h are considered as approximations to the functions ψ and u_* to be reconstructed.

The approach presented here follows conceptually the theory of stable dynamical inversion developed in [9, 11, 13–18]. This theory is based on combining methods of the theory of ill-posed problems [19, 4] with results of the theory of positional control [10].

In the present paper, the approach outlined above is applied to a new class of distributed systems, namely, the phase field model. Note also that we discuss the problem of reconstruction of an unknown control by measuring only a part of the state of the system, namely φ. When doing so, we also will reconstruct the unknown system's state, ψ. In previous investigations on reconstruction problems for nonlinear distributed parameter systems, our approach required measurements of the entire state of a system.

The structure of the paper is as follows. Section 2 deals with the state equations. Section 3 is of auxiliary character. It suggests a procedure to reconstruct an unmeasured coordinate of the system. The main section, Section 4, presents the main algorithm for solving the problem together with a rigorous mathematical foundation. Moreover, estimates of the convergence rate of the algorithm are derived.

2 The phase field equations

Let us briefly survey results concerning the phase field equations (1.1)–(1.4) that we will need for proving the convergence of our reconstruction algorithm. We discuss the model,

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi + l\frac{\partial}{\partial t}\varphi = \Delta\psi + u \quad \text{in} \quad \Omega \times (0, \vartheta], \\
&\frac{\partial}{\partial t}\varphi = \Delta\varphi + g(\varphi) + \psi \quad \text{in} \quad \Omega \times (0, \vartheta], \\
&\frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0 \quad \text{on} \quad \partial\Omega \times (0, \vartheta], \\
&\psi(0) = \psi_0, \quad \varphi(0) = \varphi_0 \quad \text{in} \quad \Omega.
\end{aligned}$$

The function $-g$ is the derivative of a so-called double well potential G. Often, G is taken as $G(z) = \frac{1}{8}(z^2-1)^2$. We assume that g admits the form $g(z) = az + bz^2 - cz^3$ with bounded coefficient functions a, b, c, where c is strictly positive.

As mentioned above, the state functions of this system are the order parameter φ (also called the phase function) and the temperature ψ. In contrast to the classical Stefan problem that models a sharp solid-liquid interface, phase field models are applicable for a mushy region. The phases are identified by φ. Under appropriate normalization, $\{\eta \in \Omega : \varphi(\eta) = 1\}$ is the liquid region while $\{\eta \in \Omega : \varphi(\eta) = -1\}$ is the solid one. The interface is formed by the points $\eta \in \Omega$, where the order parameter takes values in $(-1, 1)$.

Throughout this paper, we impose the following assumptions:

(A1) The domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded C^2-domain.

(A2) The coefficients a and b are elements of the space $L_\infty(\Omega \times [0, \vartheta])$.

(A3) The initial data satisfy $\psi_0, \varphi_0 \in W^2_\infty(\Omega)$, together with the compatibility conditions

$$\frac{\partial}{\partial n}\psi_0 = \frac{\partial}{\partial n}\varphi_0 = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, we use the following notation: The space-time domain is denoted by $Q = \Omega \times (0, \vartheta)$. For $p \in [1, \infty)$ we define

$$W^{2,1}_p(Q) = \left\{ u :\ u, \frac{\partial u}{\partial \eta_i}, \frac{\partial^2 u}{\partial \eta_i \partial \eta_j}, \frac{\partial u}{\partial t} \in L^p(Q) \right\}.$$

The space $W^{2,1}_p(Q)$ endowed with the norm

$$\|u\|_{W^{2,1}_p(Q)} = \left(\int\limits_Q \left(|u|^p + \sum_{i=1}^n \left| \frac{\partial u}{\partial \eta_i} \right|^p + \sum_{i,j=1}^n \left| \frac{\partial^2 u}{\partial \eta_i \partial \eta_j} \right|^p + \left| \frac{\partial u}{\partial t} \right|^p \right) d\eta dt \right)^{1/p}$$

is known to be a Banach space.

Let an initial state $x_0 = \{\psi_0, \varphi_0\}$ and a control $u \in L_2(T; H)$ be fixed. A solution of the system S, $x(\cdot; 0, x_0, u) = \{\psi(\cdot; 0, x_0, u), \varphi(\cdot; 0, x_0, u)\}$ is said to be a unique function

$$x = x(\cdot; 0, x_0, u) \in (W^{2,1}_2(Q))^2$$

satisfying relations (1.1)–(1.4). In virtue of a known embedding theorem [6, 12], we can assume without loss of generality that the space $W^{2,1}_2(Q)$ is embedded into $C(T; X)$, where $X = H \times H$. Therefore, for each $t \in T$, the element $x(t) = \{\psi(t), \varphi(t)\} \in X$ is defined correctly. By the symbol X_T we denote the set of all solutions of S:

$$X_T = \{x(\cdot; 0, x_0, u) : u \in U_{ad}\},$$

and the symbols Ψ_T and Φ_T stand for projections of the set X_T onto the spaces $W_2^{2,1}(Q)$ of coordinates ψ and φ, respectively.

Lemma 1 [8, 7] *If the assumptions (A1)–(A3) are satisfied then for any $u \in L_2(T;H)$ there exists a unique solution $x = \{\psi, \varphi\} \in (W_2^{2,1}(Q))^2$ of the system S. Moreover, the estimate*

$$\sup_{t\in T} \int_\Omega \Big\{\psi^2(t,\eta) + |\nabla\psi(t,\eta)|^2 + |\nabla\varphi(t,\eta)|^2 + \varphi^2(t,\eta)\Big\}\, d\eta \, +$$

$$\int_0^\vartheta\!\!\int_\Omega \Big\{\psi_t^2(t,\eta) + (\Delta\psi(t,\eta))^2 + \varphi_t^2(t,\eta) + (\Delta\varphi(t,\eta))^2\Big\}\, d\eta\, dt \le$$

$$d_* = C\,\mu(\varphi_0,\psi_0)$$

is fulfilled uniformly in $x = \{\psi,\varphi\} \in X_T$, where C and μ are defined below.

The constant C depends on $|a|_{L_\infty(Q)}$, $|b|_{L_\infty(Q)}$, ϑ, l, and

$$\mu(\varphi_0,\psi_0) = |\varphi_0|_{W^2_\infty(\Omega)} + |\psi_0|_{W^2_\infty(\Omega)} + d(U),$$

$$d(U) = \sup\{|u|_H : u \in U\}.$$

The symbol $\nabla\varphi$ stands for the gradient of function φ with respect to η, and symbol $|\nabla\varphi|$ denotes the Euclidean norm of vector $\nabla\varphi$.

To reconstruct ψ, we have introduced in (1.6) the following nonlinear parabolic equation for $w = w_1 \in W_2^{2,1}(Q)$ (see the first equation of (1.6))

$$\frac{\partial w}{\partial t} = \Delta w + p + g(w) \quad \text{in} \quad \Omega \times (0,\vartheta] \tag{2.10}$$

with boundary condition

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta]$$

and initial condition

$$w(0) = \varphi_0 \in W^2_\infty(\Omega).$$

This is an auxiliary system (a model according to terminology of [11, 14]); the role of the unknown control is played by the function p.

Take any $d > 0$ with

$$d \ge \sup_{t\in T}\{|\psi(t;0,x_0,u)|_H : u \in U_{ad}\}.$$

Notice that d depends on the set U. Define

$$U_d = \{u \in H : |u|_H \le d\},$$

$$U^d = \{u \in L_2(T;H) : u(t) \in U_d \text{ for a.a. } t \in T\}$$

and put

$$\mu_1(\varphi_0) = |\varphi_0|_{W^2_\infty(\Omega)} + d.$$

Existence and uniqueness of a solution of equation (2.10) have been discussed in [8], where the following Lemma was proved.

Lemma 2 *For any $p \in U^d$, there exists a unique solution w of equation (2.10), i.e., a function $w = w(\cdot; 0, \varphi_0, p) \in W_2^{2,1}(Q)$. In addition, the estimate*

$$\sup_{t\in T} \int\limits_{\Omega} \Big\{ (w(t,\eta))^2 + |\nabla w(t,\eta)|^2 \Big\}\, d\eta + \tag{2.11}$$

$$\int\limits_0^{\vartheta}\int\limits_{\Omega} \Big\{ (w_t(t,\eta))^2 + (\Delta w(t,\eta))^2 \Big\}\, d\eta\, dt \le d_1 = C_1\mu_1(\varphi_0)$$

is fulfilled uniformly with respect to $p \in U^d$, where the constant C_1 depends on the same quantities as the constant C in Lemma 1.

3 Reconstruction of ψ in the L^2-norm

In this section, we start by reconstructing only the coordinate ψ in the mean square metric. The algorithm presented here is of auxiliary character. It will be used in the next section to solve the whole problem under consideration.

Let us first explain the algorithm in more detail. We use the partition $\Theta_h = \{\tau_0, \dots, \tau_m\}$ of T introduced in the last section, where $\tau_{i+1} = \tau_i + \delta$, $\tau_i = \tau_{i,h}$, $\delta = \delta(h)$, $m = m(h)$. Moreover, we define the following quantities and functions:

$$B = \|a + \frac{1}{3}b^2\|_{L^\infty(Q)},$$
$$Q(\alpha,\tau,u,s) = \exp(-2B\tau)(s,u)_H + \alpha\,|u|^2_H,$$

$Q : R^2_+ \times H \times H \to R$. Here, $\alpha > 0$ is a regularization parameter. It will depend on h, hence we fix a function $\alpha(h) : (0,1) \to R_+$ that is said to be a *regularizator*.

The algorithm of reconstruction is decomposed into $m-1$ identical steps. In the i-th step, performed on the time interval $T_i = [\tau_i, \tau_{i+1})$, the following operations are carried out. First, we determine the value

$$\begin{aligned} s_i &= w^h(\tau_i) - \xi_i, \\ p_i^h &= p^h(\xi_i, w^h(\tau_i)) = \arg\min\nolimits_{u\in U_d} Q(\alpha,\tau_i,u,s_i). \end{aligned} \tag{3.1}$$

Then, for all $t \in T_i$, the control of the form

$$p^h(t,\eta) = p_i^h(\eta), \quad \eta\in\Omega, \quad i \in [0 : m(h)-1], \tag{3.2}$$

is fed onto the input of the model (2.10). In this context, the control strategy $\mathcal{U}$ mentioned above is given by

$$\mathcal{U}(\tau, w, \xi) = \arg\min\nolimits_{u \in U_d} Q(\alpha, \tau, u, w - \xi).$$

As a result, under the action of this control p^h, the model passes from the state $w^h(\tau_i)$ to the state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), p^h)$. At the next step $(i+1)$, analogous actions are repeated. The procedure stops at the moment $t = \vartheta$.

It follows from the theorem presented below that for an appropriate relation between the parameters h, $\alpha(h)$, and $\delta(h)$, the control p^h computed according to the algorithm above is a "good" mean square approximation to the coordinate ψ.

This algorithm of reconstructing ψ is based on the following considerations that are visible from the proof of Lemma 3. A smoothing mapping $\varepsilon_h(t) = \varepsilon_h(t, w^h, \varphi)$ is introduced. It is a special Lyapunov mapping. Then the law $\mathcal{U}$ of forming a control of the model (2.10) is constructed. This law, defined by (1.7), (3.1), and (3.2) ensures weak growth of $\varepsilon_h(t)$ in time, see (3.13). In virtue of the structure of $\varepsilon_h(t)$, this growth allows to guarantee that p^h is close to ψ in the L^2-norm. Moreover, we obtain an upper estimate for $|p^h - \psi|_{L_2(T;H)}$, see (3.2). This is essential for the proof of the main result in the next section.

Let $\Xi(\varphi, h)$ be the set of all measurements that are compatible with $\varphi \in \Phi_T$, i.e., $\Xi(\varphi, h)$ is the set of all piecewise constant functions $\xi^h : T \to H$ such that $\xi^h(t) = \xi_i$ for $t \in T_i$, $i \in [0 : m(h)-1]$, satisfying (1.5). The symbol W_1^h stands for the set of all solutions of equation (2.10), i.e.,

$$W_1^h = \{w^h(\cdot; 0, \varphi_0, p^h) : p^h \in U^d\}.$$

Theorem 1 *Let h, $\delta(h)$, and $\alpha(h)$ satisfy the condition*

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \quad (h + \delta(h))\alpha^{-1}(h) \to 0 \quad \text{as } h \downarrow 0.$$

Then it holds

$$\lim_{h \downarrow 0} |p^h - \psi|_{L_2(T;H)} = 0.$$

Theorem 1 is a direct consequence of Lemma 3 proved below.

Lemma 3 *For all $h \in (0,1)$, $\xi^h \in \Xi(\varphi, h)$, $x = \{\psi, \varphi\} \in X_T$, the inequalities*

$$|p^h - \psi|^2_{L_2(T;H)} \le K\nu(h), \tag{3.3}$$

$$|\varphi(t) - w^h(t)|_H^2 + \int_0^t \int_\Omega |\nabla(\varphi(\tau,\eta) - w^h(\tau,\eta))|^2 \, d\eta \, d\tau \le \tag{3.4}$$

$$\le K_0(h + \delta(h) + \alpha(h)), \quad t \in T,$$

are valid, where the constants K and K_0 may depend on X_T but are independent of h, ξ^h, and x and

$$\nu(h) = (h + \delta(h) + \alpha(h))^{1/2} + (h + \delta(h))\alpha^{-1}(h).$$

Proof. (i) Estimation of a Lyapunov mapping. Let us fix $h \in (0,1)$, $\xi^h \in \Xi(\varphi, h)$, $x \in X_T$, $w^h \in W_1^h$. Introduce the Lyapunov mapping $(w^h, \varphi) \mapsto \varepsilon_h$ from $W_1^h \times \Phi_T$ to $C[0,\vartheta]$ defined by

$$\varepsilon_h(t) = \frac{1}{2}\exp(-2Bt)|\mu^h(t)|_H^2 + \int_0^t \exp(-2B\tau)|\nabla\mu^h(\tau)|_H^2\, d\tau +$$

$$\alpha(h)\int_0^t \{|p^h(\tau)|_H^2 - |\psi(\tau)|_H^2\}\, d\tau,$$

where

$$\mu^h(t) = w^h(t) - \varphi(t), \quad t \in T,$$

and p^h is defined by (3.1) and (3.2). The function μ^h is the solution of the nonlinear equation

$$\frac{\partial\mu^h(t,\eta)}{\partial t} - \Delta\mu^h(t,\eta) = \tag{3.5}$$

$$D^h(t,\eta)\mu^h(t,\eta) + p^h(t,\eta) - \psi(t,\eta) \quad \text{in} \quad \Omega \times (0,\vartheta],$$

$$\mu^h(0) = 0, \quad \frac{\partial\mu^h}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta],$$

where

$$D^h(t,\cdot) = a + (w^h + \varphi)\, b - ((w^h)^2 + w^h\varphi + \varphi^2).$$

Multiplying both sides of (3.5) by $\exp(-2Bt)\mu^h(t,\eta)$, we obtain after integration over Ω and adding $\alpha\{|p^h(t)|_H^2 - |\psi(t)|_H^2\}$ at both sides

$$\exp(-2Bt)(\mu_t^h(t), \mu^h(t))_H + \exp(-2Bt)\int_\Omega |\nabla\mu^h(t,\eta)|^2\, d\eta + \tag{3.6}$$

$$\alpha(h)\{|p^h(t)|_H^2 - |\psi(t)|_H^2\} = \exp(-2Bt)\Big\{(D^h(t,\cdot)\mu^h(t), \mu^h(t))_H +$$

$$(p^h(t) - \psi(t), \mu^h(t))_H\Big\} + \alpha(h)\{|p^h(t)|_H^2 - |\psi(t)|_H^2\} \quad \text{for a.a.} \quad t \in T.$$

Note that the relations

$$|\exp(-2Bt) - \exp(-B\tau_i)| \le k_0(t - \tau_i), \tag{3.7}$$

$$|\mu^h(t) - s_i|_H = |w^h(t) - \varphi(t) - w^h(\tau_i) + \xi_i|_H \le$$

$$|w^h(t) - w^h(\tau_i)|_H + |\varphi(t) - \varphi(\tau_i)|_H + |\varphi(\tau_i) - \xi_i|_H$$

are true uniformly with respect to all i and $t \in T_i$. In virtue of the Lemmas 1, 2 and inequalities (1.5), (3.7), the following estimates

$$\Big| \exp(-2Bt)(p^h(t) - \psi(t), \mu^h(t))_H \Big| \le \tag{3.8}$$

$$\exp(-2B\tau_i)\Big|(p^h(t) - \psi(t), s_i)_H\Big| + k_0\rho_i(t,h) \quad \text{for a.a.} \quad t \in T_i,$$

hold, where

$$\rho_i(t,h) = h + \int_{\tau_i}^{t} \{1 + |w^h_\tau(\tau)|_H + |\varphi_\tau(\tau)|_H\}\, d\tau.$$

In addition, for any v_1, $v_2 \in R$ the inequality

$$\operatorname*{vraimax}_{(t,\eta)\in T\times\Omega} \{a(t,\eta) + b(t,\eta)(v_1 + v_2) - (v_1^2 + v_1 v_2 + v_2^2)\} \le B$$

is true. Therefore,

$$|(D^h(t,\cdot)\mu^h(t), \mu^h(t))_H| \le B|\mu^h(t)|_H^2, \quad t \in T. \tag{3.9}$$

Note that for all t_1, $t_2 \in T$, $t_1 \le t_2$ we have

$$\int_{t_1}^{t_2} \exp(-2B\tau)(\mu^h(\tau), \mu^h_\tau(\tau))_H\, d\tau = \frac{1}{2}\Big[\exp(-2Bt)|\mu^h(t)|_H^2\Big]\Big|_{t_1}^{t_2} + \tag{3.10}$$

$$+ B\int_{t_1}^{t_2} \exp(-2B\tau)|\mu^h(\tau)|_H^2\, d\tau.$$

Next, we integrate equality (3.6) and use the estimates (3.8)–(3.10). We obtain for all $t \in T_i$

$$\frac{1}{2}\exp(-2Bt)|\mu^h(t)|_H^2 + \int_{\tau_i}^{t}\int_{\Omega} \exp(-2B\tau)|\nabla\mu^h(\tau,\eta)|_H^2\, d\eta\, d\tau + \tag{3.11}$$

$$B\int_{\tau_i}^{t} \exp(-2B\tau)|\mu^h(\tau)|_H^2\, d\tau + \alpha(h)\int_{\tau_i}^{t} \{|p^h(\tau)|_H^2 - |\psi(\tau)|_H^2\}\, d\tau \le$$

$$B\int_{\tau_i}^{t} \exp(-2B\tau)|\mu^h(\tau)|_H^2\, d\tau + \alpha(h)\int_{\tau_i}^{t} \{|p^h(\tau)|_H^2 - |\psi(\tau)|_H^2\}\, d\tau +$$

$$\exp(-2B\tau_i)\int\limits_{\tau_i}^{t}(p_i^h-\psi(\tau),s_i)_H\,d\tau+k_1\rho_i(t,h)+\frac{1}{2}\exp(-2B\tau_i)|\mu^h(\tau_i)|_H^2,$$

where $k_1=k_0\delta$. Taking into account (3.1), we derive

$$\exp(-2B\tau_i)(p_i^h,s_i)+\alpha(h)|p_i^h|_H^2\le$$

$$\exp(-2B\tau_i)(\psi(t),s_i)+\alpha(h)|\psi(t)|_H^2,\quad t\in T_i,$$

since p_i^h is the $\arg\min$ of Q. This yields

$$\exp(-2B\tau_i)(p_i^h-\psi(t),s_i)+\alpha(h)\{|p_i^h|^2-|\psi(t)|_H^2\}\le 0,\quad t\in\delta_i. \tag{3.12}$$

We integrate (3.12) from τ_i to t. Then it follows from (3.11), (3.12) that

$$\frac{1}{2}\exp(-2Bt)|\mu^h(t)|_H^2+\int\limits_{\tau_i}^{t}\Big(\exp(-2B\tau)\int\limits_{\Omega}|\nabla\mu^h(\tau,\eta)|^2\,d\eta\Big)d\tau+$$

$$\alpha(h)\int\limits_{\tau_i}^{t}\{|p^h(\tau)|_H^2-|\psi(\tau)|_H^2\}\,d\tau\le k_1\rho_i(t;h)+\frac{1}{2}\exp(-2B\tau_i)|\mu^h(\tau_i)|_H^2.$$

After adding $\int\limits_0^{\tau_i}\Big\{\exp(-2B\tau)|\nabla\mu^h(\tau,\eta)|^2+\alpha(h)\{|p^h(\tau)|_H^2-|\psi(\tau)|_H^2\}\Big\}\,d\tau$ at both sides we get by definition of ε_h

$$\varepsilon_h(t)\le\varepsilon_h(\tau_i)+k_1\rho_i(t;h),\quad t\in T_i.$$

For instance, taking $t=\tau_{i+1}$, we can estimate $\varepsilon_h(\tau_{i+1})$ by an expression with $\varepsilon_h(\tau_i)$. Repeating the same for $\varepsilon_h(\tau_i)$ etc., we finally arrive at

$$\varepsilon_h(t)\le\varepsilon_h(0)+k_2(h+\delta\int\limits_0^{t}\{1+|w_{1\tau}^h(\tau)|_H+|\varphi_\tau(\tau)|_H\}\,d\tau)\le \tag{3.13}$$

$$\varepsilon_h(0)+k_3(h+\delta),\quad t\in T,$$

with certain constants k_j. However, $\varepsilon_h(0)=0$. The Lemmas 1, 2, the boundedness of the set U, and inequality (3.13) yield (3.4).

(ii) Estimation of $|p^h-\psi|_{L_2(T;H)}$. Let us verify (3.3). Multiplying both sides of (3.5) by $\psi(t)$, we obtain after integration over Ω

$$(\mu_t^h(t),\psi(t))_H+\int\limits_{\Omega}\nabla\mu^h(t,\eta)\cdot\nabla\psi(t,\eta)\,d\eta=(D^h(t,\cdot)\mu^h(t),\psi(t))_H+ \tag{3.14}$$

$$(p^h(t) - \psi(t), \psi(t))_H \quad \text{for a.a.} \quad t \in T.$$

In virtue of (3.13), we have

$$|p^h|^2_{L_2(T;H)} \le |\psi|^2_{L_2(T;H)} + k_3(h+\delta)\alpha^{-1}. \tag{3.15}$$

Then, integrating (3.14), after rather simple transformations, we derive the estimate

$$\Big| \int_0^\vartheta (p^h(t) - \psi(t), \psi(t))_H \, dt \Big| \le k_4 \int_0^\vartheta |\mu^h(\tau)|_H \, d\tau + \tag{3.16}$$

$$\Big(\int_0^\vartheta \Big(\int_\Omega |\nabla \mu^h(t,\eta)|^2 \, d\eta \int_\Omega |\nabla \psi(t,\eta)|^2 \, d\eta \Big) \, dt \Big)^{1/2} +$$

$$|(\mu^h(t), \psi(t))_H \Big|_0^\vartheta | + \Big| \int_0^\vartheta (\mu^h(t), \psi_t(t))_H \, dt \Big| \le k_5(h+\delta+\alpha)^{1/2},$$

which is uniform with respect to $h \in (0,1)$, $\xi^h \in \Xi(\varphi, h)$, $x(\cdot) = \{\psi, \varphi\} \in X_T$. In virtue of (3.15), (3.16), we obtain

$$|p^h - \psi|^2_{L_2(T;H)} \le 2|\psi|^2_{L_2(T;H)} - 2 \int_0^\vartheta (p^h(t), \psi(t))_H \, dt + \tag{3.17}$$

$$k_3(h+\delta)\alpha^{-1} \le 2k_5(h+\delta+\alpha)^{1/2} + k_3(h+\delta)\alpha^{-1} \le K\nu(h).$$

The lemma is proven. □

In this way, we have designed an algorithm for reconstructing the unknown coordinate ψ together with the estimate (3.3) of its convergence rate.

4 Reconstruction of the pair $\{\psi, u_*\}$

Let us proceed to solving the problem of reconstruction of ψ and u_*. Namely, we design an algorithm of reconstruction of a pair $\{\psi, u_*\}$ based on the ideas of feedback stabilization of some special Lyapunov functional. In addition, we obtain estimates of the algorithm's convergence rate. Following the scheme described in Section 2, we should choose a model and a law of forming a model control (see (1.7), (1.8)). As a model, we take the system of equations

(1.6), i.e., the system

$$\begin{aligned} &\frac{\partial w_1^h(t,\eta)}{\partial t} = \Delta w_1^h(t,\eta) + g(w_1^h(t,\eta)) + p^h(t,\eta) \quad \text{in} \quad \Omega \times (0,\vartheta], \\ &\frac{\partial w_2^h(t,\eta)}{\partial t} + l\frac{\partial w_3^h(t,\eta)}{\partial t} = \Delta w_2^h(t,\eta) + u^h(t,\eta), \\ &\frac{\partial w_3^h(t,\eta)}{\partial t} = \Delta w_3^h(t,\eta) + g(w_3^h(t,\eta)) + w_2^h(t,\eta) \end{aligned} \tag{4.1}$$

with boundary conditions

$$\frac{\partial}{\partial n} w_1^h = \frac{\partial}{\partial n} w_2^h = \frac{\partial}{\partial n} w_3^h = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta] \tag{4.2}$$

and initial conditions

$$w_2^h(0) = \psi_0, \quad w_1^h(0) = w_3^h(0) = \varphi_0 \quad \text{in} \quad \Omega. \tag{4.3}$$

Here, p^h and u^h are controls to be reconstructed.

Since the solution w^h of this system depends on $w^h(0) = \{w_1^h(0), w_2^h(0), w_3^h(0)\}$, $w_j^h(0) \in W_\infty^2(\Omega)$, $j \in [1:3]$, and on the controls $\{u^h, p^h\} \in U_{ad} \times U^d$, we indicate this dependence by the notation

$$w = w(\cdot; 0, w^h(0), u^h, p^h) = \{w_1^h(\cdot; 0, w^h(0), u^h, p^h),$$

$$w_2^h(\cdot; 0, w^h(0), u^h, p^h), w_3^h(\cdot; 0, w^h(0), u^h, p^h)\} \in (W_2^{2,1}(Q))^3.$$

It follows from Lemmas 1, 2 that this solution exists and is unique. Recall the definition of U^d in the previous section. By the symbol W^h we denote the set of all solutions of system (4.1)–(4.3), i.e.,

$$W^h = \{w^h(\cdot; 0, w_0^h, u, p) : u \in U_{ad},\ p \in U^d\}.$$

The algorithm of reconstruction of the pair $\{\psi, u_*\}$ is similar to the one described in the previous section. We assume that $l \geq B$. First, we fix two functions $\alpha(h) : (0,1) \to R^+$ and $\beta(h) : (0,1) \to R^+$ (regularizators) as well as the family of partitions Θ_h, $h \in (0,1)$, used in Section 3, of the interval T. We assume that these two functions and the family Θ_h satisfy the conditions:

$$\alpha(h) \to 0, \quad \beta(h) \to 0, \quad \delta(h) \to 0, \quad (h + \delta(h) + \alpha(h))^{1/4}\beta^{-1}(h) \to 0, \tag{4.4}$$

$$(h + \delta(h))\alpha^{-1}(h)\beta^{-2}(h) \to 0 \quad \text{as} \quad h \to 0.$$

Then we organize the process of synchronous feedback control of model (4.1)–(4.3) and the real system S in such a way that the function u^h approximates the unknown input u_* in $L_2(T; H)$ and the function w_2^h approximates the component ψ in $C(T; H)$ for sufficiently small h. The algorithm is decomposed into $m - 1$, $m = m(h)$, identical steps. At the ith step carried out on the time interval $T_i = [\tau_i, \tau_{i+1})$, the following operations are performed. First, the values p_i^h and u_i^h are calculated by the formulas

$$p_i^h = p^h(\xi_i, w_1^h(\tau_i)) = \arg\min\{Q(\alpha, \tau_i, u, s_i) : u \in U_d\}, \tag{4.5}$$

$$u_i^h = u^h(\xi_i, w_2^h(\tau_i), w_3^h(\tau_i), p_i^h) = u_{1,i}^h + u_{2,i}^h. \tag{4.6}$$

Here Q and s_i were defined in Section 3,

$$u_{1,i}^h = \arg\min\{L(\beta, u, z_i^h) : u \in U\}, \tag{4.7}$$

$$u_{2,i}^h = -c_*(w_3^h(\tau_i) - \xi_i), \tag{4.8}$$

$$L(\beta, u, z) = (z, u)_H + \beta|u|_H^2,$$

$$z_i^h = w_2^h(\tau_i) - p_i^h + l(w_3^h(\tau_i) - \xi_i), \quad c_* > 0.25l^2.$$

Then we define for $t \in T_i$

$$p^h(t) = p_i^h, \quad u^h(t) = u_i^h. \tag{4.9}$$

Hereinafter, for $t \in T_i$, the controls (4.9) are taken as input of the model (4.1)–(4.3). As a result, under the action of such controls, the model passes from the state $w^h(\tau_i)$ to the state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), u_i^h, p_i^h)$. At the next step $i+1$, analogous operations are repeated. The procedure stops at $t = \vartheta$.

Thus (see (4.5)–(4.9)), the control strategies $\mathcal{U}$ and $\mathcal{V}$ used in (1.7), (1.8) are of the following form:

$$\mathcal{V}(\tau_i, w_1^h(\tau_i), \xi_i) = \arg\min\{Q(\alpha, \tau_i, u, s_i) : u \in U_d\},$$

$$\mathcal{U}(\tau_i, w_2^h(\tau_i), w_3^h(\tau_i), \xi_i, p_i^h) = \arg\min\{L(\beta, u, z_i^h) : u \in U\} - c_*(w_3^h(\tau_i) - \xi_i).$$

The algorithm of reconstruction of the pair $\{\psi, u_*\}$ is based on the ideas presented in the previous section. The control strategy for the model (4.1)–(4.3) that is given by (4.5), (4.6) provides weak growth of the Lyapunov mapping $L(t, x, w^h, u^h)$, see the last estimate in the proof of Lemma 5.

Theorem 2 *Let the relations* (4.4) *be satisfied. Then the following convergence properties hold:*

$$u_1^h \to u_* \quad in \quad L_2(T; H),$$

$$w_2^h \to \psi \quad in \quad C(T; H) \cap L_2(T; H^1(\Omega)) \quad as \quad h \to 0,$$

where

$$u_1^h(t) = u_{1,i}^h \quad for \quad t \in T_i, \quad i \in [0 : m(h) - 1]. \tag{4.10}$$

The proof of the theorem is subdivided into several steps. At the beginning, we establish some auxiliary statements.

Lemma 4 *The estimate*

$$\sup_{t\in T}\int\limits_{\Omega}\sum_{j=1}^{3}\{(w_j^h(t,\eta))^2+|\nabla w_j^h(t,\eta)|^2\}\,d\eta+ \tag{4.11}$$

$$+\int\limits_0^{\vartheta}\int\limits_{\Omega}\sum_{j=1}^{3}\{(w_{jt}^h(t,\eta))^2+(\Delta w_j^h(t,\eta))^2\}\,d\eta\,dt\le d_*=C_2\mu(\varphi_0,\psi_0)$$

holds uniformly with respect to all $w^h=\{w_1^h,w_2^h,w_3^h\}\in W^h$*, where* μ *is defined in section 2.*

Here the constant C_2 depends on the same values as the constant C in Lemma 1. The proof of this Lemma is presented in the Appendix.

Introduce the Lyapunov mapping

$$L(t,x,w^h,u^h)=\Lambda^0(t,x,w^h)+\beta(h)\int\limits_0^t\{|u_1^h(\tau)|_H^2-|u_*(\tau)|_H^2\}\,d\tau\quad\text{for}\quad t\in T,$$

where

$$\Lambda^0(t,x,w^h)=|g^h(t)|_H^2+0.25l^2(1-\lambda)^{-1}|\nu^h(t)|_H^2+$$

$$+\lambda\int\limits_0^t\int\limits_{\Omega}|\nabla\pi^h(\tau,\eta)|^2\,d\eta\,d\tau,$$

$$\lambda=1-0.25c_*^{-1}l^2\in(0,1),\quad g^h(t)=\pi^h(t)+l\nu^h(t),$$

$$\pi^h(t)=w_2^h(t)-\psi(t),\quad \nu^h(t)=w_3^h(t)-\varphi(t).$$

Lemma 5 *For all* $h\in(0,1)$*,* $\xi^h\in\Xi(\varphi,h)$*,* $x=\{\psi,\varphi\}\in X_T$*, the inequalities*

$$|u_1^h|_{L_2(T;H)}^2\le|u_*|_{L_2(T;H)}^2+K_*\nu^{1/2}(h)\beta^{-1}(h),$$

$$\Lambda^0(t,x,w^h)\le K^*\{\nu^{1/2}(h)+\beta(h)\},\quad\forall t\in T,$$

are valid.

Here the constants K_* and K^* depend on X_T but not on h, ξ^h, x, u.

Proof. The proof of the Lemma is along the lines of the proof of Lemma 3. Let us fix $h\in(0,1)$, $\xi^h\in\Xi(\varphi,h)$, $x\in X_T$, $w^h\in W^h$. Then the functions π^h and ν^h are the solutions of the system

$$\frac{\partial\pi^h(t,\eta)}{\partial t}+l\frac{\partial\nu^h(t,\eta)}{\partial t}=\Delta\pi^h(t,\eta)+u^h(t,\eta)-u_*(t,\eta)$$

$$\text{in}\quad\Omega\times(0,\vartheta], \tag{4.12}$$

$$\frac{\partial\nu^h(t,\eta)}{\partial t}=\Delta_L\nu^h(t,\eta)+R^h(t,\eta)\nu^h(t,\eta)+\pi^h(t,\eta)$$

with initial and boundary conditions

$$\pi^h(0) = \nu^h(0) = 0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial \pi^h}{\partial n} = \frac{\partial \nu^h}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0, \vartheta],$$

where

$$R^h(t, \cdot) = a + b(w_3^h + \varphi) - ((w_3^h)^2 + w_3^h \varphi + \varphi^2).$$

To prove the Lemma, we mainly have to estimate the Lyapunov mapping. Multiplying scalarly the first equation of (4.12) by $g^h(t)$, and the second one by $\nu^h(t)$, we obtain

$$(g^h(t), g_t^h(t))_H + \int_\Omega \{|\nabla \pi^h(t,\eta)|^2 + l \nabla \pi^h(t,\eta) \cdot \nabla \nu^h(t,\eta)\}\, d\eta = \tag{4.13}$$

$$= (g^h(t), u^h(t) - u_*(t))_H,$$

$$(\nu^h(t), \nu_t^h(t))_H + \int_\Omega |\nabla \nu^h(t,\eta)|^2\, d\eta \le (\pi^h(t), \nu^h(t))_H + B|\nu^h(t)|_H^2, \quad t \in T.$$

Note that the inequality

$$\int_\Omega l(\nabla \pi^h(t,\eta), \nabla \nu^h(t,\eta))\, d\eta \ge -\int_\Omega \{(1-\lambda)|\nabla \pi^h(t,\eta)|^2 + \tag{4.14}$$

$$+ 0.25 l^2 (1-\lambda)^{-1} |\nabla \nu^h(t,\eta)|^2\}\, d\eta$$

holds. From (4.13) and (4.14) we derive

$$(g^h(t), g_t^h(t))_H + 0.25 l^2 (1-\lambda)^{-1} (\nu^h(t), \nu_t^h(t))_H + \tag{4.15}$$

$$+ \lambda \int_\Omega |\nabla \pi^h(t,\eta)|^2\, d\eta \le (g^h(t), u^h(t) - u_*(t))_H +$$

$$+ 0.25 l^2 (1-\lambda)^{-1} (\pi^h(t), \nu^h(t))_H + 0.25 B l^2 (1-\lambda)^{-1} |\nu^h(t)|_H^2.$$

In virtue of the rule (4.8) of forming $u_{2,i}^h$, the following inequality is valid for a.a. $t \in T_i$:

$$(g^h(t), u_{2,i}^h)_H = -c_*(g^h(t), w_3^h(\tau_i) - \xi_i) \le -c_*(g^h(t), \nu^h(t))_H + L(t;\tau_i) =$$

$$= -c_*(\pi^h(t), \nu^h(t))_H - c_* l |\nu^h(t)|_H^2 + L(t;\tau_i),$$

where

$$L(t;\tau_i) = c_* |g^h(t)|_H \Big(h + \int_{\tau_i}^t \{|w_{3\tau}^h(\tau)|_H + |\varphi_\tau(\tau)|_H\}\, d\tau \Big).$$

By definition of the number c_*, we find

$$c_* = 0.25l^2(1-\lambda)^{-1}.$$

Therefore, since $l \geq B$, we obtain

$$(g^h(t), u_{2,i}^h)_H + 0.25l^2(1-\lambda)^{-1}\{(\pi^h(t), \nu^h(t))_H + B|\nu^h(t)|_H^2\} \leq L(t;\tau_i).$$

In this case, we have for a.a. $t \in T_i$

$$(g^h(t), g_t^h(t))_H + 0.25l^2(1-\lambda)^{-1}(\nu^h(t), \nu_t^h(t))_H + \quad (4.16)$$
$$+ \lambda \int_\Omega |\nabla \pi^h(t,\eta)|^2\, d\eta \leq (g^h(t), u_{1,i}^h - u_*(t))_H + L(t;\tau_i).$$

It is easily seen that

$$|g^h(t) - z_i^h|_H \leq \int_{\tau_i}^{t} \{|w_{2\tau}^h(\tau)|_H + l|w_{3\tau}^h(\tau)|_H + l|\varphi_\tau(\tau)|_H\}\, d\tau + \quad (4.17)$$
$$+ |p_i^h - \psi(t)|_H + l|\varphi(\tau_i) - \xi_i|_H.$$

In virtue of (1.5), the uniform boundedness of $\{u_{1,i}^h\}$, and the Lemmas 3, 4, we deduce from (4.17) that

$$\sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} M(t;\tau_i)\, dt \leq k_1(h+\delta) + \int_0^{\vartheta} |p^h(\tau) - \psi(\tau)|_H\, d\tau \leq k_2\nu^{1/2}(h), \quad (4.18)$$

$$\sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} L(t;\tau_i)\, dt \leq k_3(h+\delta), \quad (4.19)$$

where

$$M(t;\tau_i) = |g^h(t) - z_i^h|_H\{|u_{1,i}^h|_H + |u_*(t)|_H\}.$$

Then, by (4.16) we obtain for all $t \in T_i$

$$(g^h(t), g_t^h(t))_H + 0.25l^2(1-\lambda)^{-1}(\nu^h(t), \nu_t^h(t))_H + \lambda \int_\Omega |\nabla \pi^h(t,\eta)|^2\, d\eta + \quad (4.20)$$
$$+ \beta(h)\{|u_1^h(t)|_H^2 - |u_*(t)|_H^2\} \leq (z_i^h, u_{1,i}^h - u_*(t))_H +$$
$$+ M(t;\tau_i) + L(t;\tau_i) + \beta(h)\{|u_1^h(t)|_H^2 - |u_*(t)|_H^2\}.$$

Taking into account the formulas (4.7), (4.9) for the control $u_1^h(t)$, we conclude from (4.20) that

$$(g^h(t), g^h_t(t))_H + 0.25l^2(1-\lambda)^{-1}(\nu^h(t), \nu^h_t(t))_H + \lambda \int_\Omega |\nabla \pi^h(t,\eta)|^2 \, d\eta + \tag{4.21}$$

$$+ \beta(h)\{|u_1^h(t)|_H^2 - |u_*(t)|_H^2\} \le M(t;\tau_i) + L(t;\tau_i), \quad t \in T_i.$$

Using (4.21), (4.18), (4.19) we have found our main estimate,

$$L(t, x, w^h, u^h) \le \sum_{i=0}^{m-1} \{M(\tau_{i+1};\tau_i) + L(\tau_{i+1};\tau_i)\} \le k_4 \nu^{1/2}(h).$$

The assertion of the lemma follows from the last inequality. □

Introduce the functional

$$\Lambda(x, w^h) = \max_{t\in T} \Lambda^0(t, x, w^h).$$

Lemma 6 *Assume that u_* is a real admissible control with associated state $x = \{\psi, \varphi\}$. Let $u^{h_k} = u_1^{h_k} + u_2^{h_k}$ be the result of our algorithm, based on measurements $\xi^{h_k} \in \Xi(\varphi, h_k)$. Assume further that $h_k \to 0$,*

$$\Lambda(x, w^{h_k}) \to 0, \tag{4.22}$$

$$u_1^{h_k} \to u^0 \quad \textit{weakly in} \quad L_2(T;H) \quad \textit{as} \quad k \to \infty. \tag{4.23}$$

Then

$$u^0 = u_*.$$

Proof. Assuming the contrary, we conclude that there exist a control $u_1^{h_k}$, a measurement $\xi^{h_k} \in \Xi(\varphi, h_k)$, and a sequence of numbers $\{h_k\}$, $h_k \to 0$ as $k \to \infty$, such that (4.22), (4.23) hold, but

$$u^0 \ne u_*.$$

Note that ψ is a solution of the parabolic equation

$$\begin{aligned} &\frac{\partial \psi_2(t,\eta)}{\partial t} = \Delta\psi_2(t,\eta) + f_1(t,\eta) \quad \text{in} \quad \Omega \times (0,\vartheta], \\ &\frac{\partial \psi_2}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta], \\ &\psi_2(0) = \psi_0 \quad \text{in} \quad \Omega \end{aligned}$$

with the right hand side

$$f_1(t,\eta) = -l\frac{\partial \varphi(t,\eta)}{\partial t} + u^0(t,\eta).$$

Consequently, we can indicate a number $t_* \in (0, \vartheta]$ such that

$$|w_2(t_*) - \psi(t_*)|_H > 0, \tag{4.24}$$

where $w_2 = w_2(\cdot; \psi_0, u^0)$ is a solution of the parabolic equation

$$\frac{\partial w_2(t,\eta)}{\partial t} = \Delta w_2(t,\eta) - l\frac{\partial \varphi(t,\eta)}{\partial t} + u^0(t,\eta) \quad \text{in} \quad \Omega \times (0,\vartheta],$$
$$\frac{\partial w_2}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta],$$
$$w_2(0) = \psi_0 \quad \text{in} \quad \Omega.$$

Estimate the value $|w_2(t_*) - \psi(t_*)|_H^2$. We have

$$|w_2(t_*) - \psi(t_*)|_H^2 \le 2|w_2(t_*) - w_2^{h_k}(t_*)|_H^2 + 2|w_2^{h_k}(t_*) - \psi(t_*)|_H^2.$$

Here $w_2^{h_k} = w_2^{h_k}(\cdot; \psi_0, u^{h_k})$ is a solution of the equation

$$\frac{\partial w_2^{h_k}(t,\eta)}{\partial t} = \Delta w_2^{h_k}(t,\eta) - l\frac{\partial w_3^{h_k}(t,\eta)}{\partial t} + u^{h_k}(t,\eta) \quad \text{in} \quad \Omega \times (0,\vartheta],$$
$$\frac{\partial w_2^{h_k}}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta],$$
$$w_2^{h_k}(0) = \psi_0 \quad \text{in} \quad \Omega.$$

In this case the function $w_k(t) = w_2(t) - w_2^{h_k}(t)$ satisfies the equation

$$\frac{\partial w_k(t,\eta)}{\partial t} = \Delta w_k(t,\eta) + l\Big(\frac{\partial w_3^{h_k}(t,\eta)}{\partial t} - \frac{\partial \varphi(t,\eta)}{\partial t}\Big) + u^0(t,\eta) - u^{h_k}(t,\eta)$$
$$\text{in} \quad \Omega \times (0,\vartheta],$$
$$\frac{\partial w_k}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\vartheta],$$
$$w_k(0) = \psi_0 \quad \text{in} \quad \Omega.$$

Multiplying scalarly both sides of this equation by $w_k(t)$ we obtain after integration:

$$\frac{1}{2}|w_k(t_*)|_H^2 = \frac{1}{2}|w_2(t_*) - w_2^{h_k}(t_*)|_H^2 \le$$

$$l\int_0^{t_*}(\varphi_t(t) - w_{3t}^{h_k}(t), w_2(t) - w_2^{h_k}(t))_H\,dt + \int_0^{t_*}(w_2(t) - w_2^{h_k}(t), u^0(t) - u^{h_k}(t))_H\,dt.$$

Analogously, we have

$$\frac{1}{2}|\psi(t_*) - w_2^{h_k}(t_*)|_H^2 \le$$

$$l \int_0^{t_*} (\varphi_t(t) - w_{3t}^{h_k}(t), \psi(t) - w_2^{h_k}(t))_H \, dt + \int_0^{t_*} (w_2^{h_k}(t) - \psi(t), u^{h_k}(t) - u_*(t))_H \, dt.$$

In this case we have

$$|w_2(t_*) - \psi(t_*)|_H^2 \le \lambda_{1,k} + \lambda_{2,k} + \lambda_{3,k}, \tag{4.25}$$

where

$$\lambda_{1,k} = 2 \int_0^{t_*} (w_2(t) - w_2^{h_k}(t), u^0(t) - u^{h_k}(t))_H \, dt,$$

$$\lambda_{2,k} = 2 \int_0^{t_*} (w_2^{h_k}(t) - \psi(t), u^{h_k}(t) - u_*(t))_H \, dt,$$

$$\lambda_{3,k} = 2l \Big| \int_0^{t_*} (\varphi_t(t) - w_{3t}^{h_k}(t), w_2(t) - w_2^{h_k}(t))_H \, dt \Big| +$$

$$+ 2l \Big| \int_0^{t_*} (\varphi_t(t) - w_{3t}^{h_k}(t), w_2^{h_k}(t) - \psi(t))_H \, dt \Big|.$$

Taking into account the form of the Lyapunov function from the second inequality of Lemma 5 we obtain the inequality

$$|w_3^{h_k}(t) - \varphi(t)|_H^2 \le K_1 \{\nu^{1/2}(h) + \beta(h)\}, \quad \forall t \in T.$$

Due to (4.4) we deduce:

$$w_3^{h_k} \to \varphi \quad \text{in} \quad C(T;H) \quad \text{as} \quad k \to \infty.$$

Therefore, in virtue of the formula (4.8),

$$\sup_{t \in T} |u_2^{h_k}(t)|_H \to 0 \quad \text{in} \quad C(T;H) \quad \text{as} \quad k \to \infty, \tag{4.26}$$

$$u_2^{h_k}(t) = u_{2,i}^{h_k}, \quad t \in [\tau_{h_k,i}, \tau_{h_k,i+1}), \quad i \in [0 : m(h_k) - 1].$$

Here, $u_{2,i}^{h_k}$ is defined by formula (4.8). Using condition (4.23) and the convergence (4.26), we obtain

$$u^{h_k} \to u^0 \quad \text{weakly in} \quad L_2(T;H) \quad \text{as} \quad k \to \infty.$$

Therefore,

$$\lim_{k \to \infty} \lambda_{1,k} = 0. \tag{4.27}$$

It follows from (4.22) that

$$\sup_{t\in T}|w_2^{h_k}(t)-\psi(t)|_H \to 0 \quad \text{as} \quad k\to\infty.$$

In this case

$$\lim_{k\to\infty}\lambda_{3k}=0, \quad \lim_{k\to\infty}\lambda_{2,k}=0. \tag{4.28}$$

Combining (4.25), (4.27), and (4.28), we obtain

$$|w_2(t_*)-\psi(t_*)|_H=0. \tag{4.29}$$

However, relation (4.29) contradicts inequality (4.24). □

Proof of Theorem 2. Due to Lemma 5, $w_2^h \to \psi$ in $C(T;H)\cap L_2(T;H^1(\Omega))$ as $h\to 0$. We have only to prove the convergence $u_1^h\to u_*$ in $L_2(T;H)$. This is performed following a standard scheme (see, for example, the proof of lemma 2.2, [14]). Assuming the contrary, we conclude that there exist sequences $(u_1^{h_k})_{k=0}^\infty$, $(\xi^{h_k})_{k=0}^\infty \in \Xi(\varphi,h_k)$ ($h_k\to 0$ as $k\to\infty$) and associated model trajectories w^{h_k} such that

$$\inf_k |u_1^{h_k}-u_*|_{L_2(T;H)}>0. \tag{4.30}$$

Due to weak compactness of the set $U_{ad}\subset L_2(T;H)$, we can assume without loss of generality that

$$u_1^{h_k}\to\overline{u} \quad \text{weakly in} \quad L_2(T;H) \quad \text{as} \quad k\to\infty, \quad \overline{u}\in U_{ad}. \tag{4.31}$$

In virtue of Lemma 5, the convergence (4.22) holds. In addition,

$$\overline{\lim_{k\to\infty}}|u_1^{h_k}|_{L_2(T;H)}\le |u_*|_{L_2(T;H)}. \tag{4.32}$$

Taking into account (4.31) and Lemma 6, we conclude that

$$\overline{u}=u_*.$$

Thus,

$$u_1^{h_k}\to u_* \quad \text{weakly in} \quad L_2(T;H) \quad \text{as} \quad k\to\infty. \tag{4.33}$$

Hence, by lower semicontinuity, we derive

$$\underline{\lim_{k\to\infty}}\,|u_1^{h_k}|_{L_2(T;H)}\ge |u_*|_{L_2(T;H)}. \tag{4.34}$$

It follows from (4.32), (4.34) that

$$\lim_{k\to\infty}|u_1^{h_k}|_{L_2(T;H)}=|u_*|_{L_2(T;H)}. \tag{4.35}$$

By (4.33), (4.35), and theorem 5.12, [6], we obtain

$$u_1^{h_k} \to u_* \quad \text{in} \quad L_2(T;H) \quad \text{as} \quad k \to \infty. \tag{4.36}$$

However, (4.36) contradicts (4.30). □

Under some additional conditions on the function u_*, convergence rates can be derived. Denote

$$W^{1,2}(T;H) = \{x \in L_2(T;H) : x_t \in L_2(T;H)\}.$$

Theorem 3 *(Convergence rate) Assume*

$$u_* \in W^{1,2}(T,H) \cap L_2(T;H^1(\Omega)). \tag{4.37}$$

Then the following estimates are valid:

$$|u_* - u_1^h|_{L_2(T;H)} \le K_0\{\nu^{1/2}(h)\beta^{-1}(h) + \nu^{1/4}(h) + \beta^{1/2}(h)\}, \tag{4.38}$$

$$|w_2^h - \psi|_{L_2(T;H^1(\Omega))} + |w_2^h - \psi|_{C(T;H)} \le K_1\{\beta(h) + \nu^{1/2}(h)\}^{1/2}. \tag{4.39}$$

Proof. We multiply the first equality of (4.12) by $u_*(t)$. After integration we have

$$\Big|\int_0^\vartheta (u^h(t) - u_*(t), u_*(t))_H \, dt\Big| \le \tag{4.40}$$

$$\le \Big(\int_0^\vartheta \Big(\int_\Omega |\nabla \pi^h(t,\eta)|^2 \, d\eta \int_\Omega |\nabla u_*(t,\eta)|^2 \, d\eta\Big) \, dt\Big)^{1/2} + g^h(t), u_*(t))_H\Big|_0^\vartheta +$$

$$+ \Big|\int_0^\vartheta (g^h(t), u_{*t}(t))_H \, dt\Big| \le k^{(1)}\{\nu^{1/4}(h) + \beta^{1/2}(h)\}.$$

In addition, the estimate

$$|u_1^h|^2_{L_2(T;H)} \le |u_*|^2_{L_2(T;H)} + K_*\nu^{1/2}(h)\beta^{-1}(h), \tag{4.41}$$

which is uniform with respect to $h \in (0,1)$, $\xi^h \in \Xi(\varphi, h)$, $x = \{\psi, \varphi\} \in X_T$, follows from Lemma 5. Thus, from (4.40), (4.41) we deduce that

$$|u_* - u_1^h|^2_{L_2(T;H)} \le 2|u_*|^2_{L_2(T;H)} - 2\int_0^\vartheta (u_1^h(t), u_*(t))_H \, dt +$$

$$+ K_*\nu^{1/2}(h)\beta^{-1}(h) \le K_0\{\nu^{1/2}(h)\beta^{-1}(h) + \nu^{1/4}(h) + \beta^{1/2}(h)\}.$$

Relation (4.38) is proved. Estimate (4.39) follows from Lemma 5. □

Let us summarize the steps to be performed for solving the problem. Select functions α and β, δ as well as a family of partitions with properties (4.4). Fix the value h of informational noise to obtain $\alpha(h)$ and $\beta(h)$, $\delta(h)$, together with the tuning parameter c_*, a partition $\Theta = \Theta_h = \{\tau_i\}_{i=0}^m$ of the interval T. Choose the initial state w_0 of the model (4.1). Then perform the following steps for $i = 0$ until $m-1$.

1. *Calculate*
$$s_i = w_1^h(\tau_i) - \xi_i.$$
2. *Solve the minimization problem (4.5) and find the unique solution*
$$p_i^h = \arg\min\{\exp(-2B\tau_i)(s_i, u)_H + \alpha|u|_H^2 : u \in U_d\}.$$
3. *Determine*
$$z_i^h = w_2^h(\tau_i) - p_i^h + l(w_3^h(\tau_i) - \xi_i).$$
4. *Solve the minimization problem (4.7) to find the element u_i^h (see (4.6)–(4.8))*
$$u_i^h = u_{i,1}^h + u_{i,2}^h,$$
$$u_{i,1}^h = \arg\min\{(z_i^h, u)_H + \beta|u|_H^2 : u \in U\},$$
$$u_{i,2}^h = -c_*(w_3^h(\tau_i) - \xi_i).$$
5. *Define*
$$p^h(t) = p_i^h, \quad u^h(t) = u_i^h, \quad t \in T_i.$$
6. *Solve the system (4.1)–(4.3) to obtain w^h.*

At the end of this algorithm we have computed the functions u_1^h and w_2^h that approximate u_* and ψ.

5 Appendix

Proof of Lemma 4. By Lemma 2, the estimate (2.11) for the first component w_1^h is valid. Multiplying the second equation of system (4.1) by $z^h(t) = w_2^h(t) + lw_3^h(t)$, and the third one by $w_3^h(t)$, we obtain

$$(z_t^h(t), z^h(t))_H + \int_\Omega \{|\nabla w_2^h(t,\eta)|^2 + l\nabla w_2^h(t,\eta)\cdot\nabla w_3^h(t,\eta)\}\, d\eta = (z^h(t), u^h(t))_H,$$

$$(w_{3t}^h(t), w_3^h(t))_H + \int_\Omega |\nabla w_3^h(t,\eta)|^2\, d\eta = (g(w_3^h(t)) + w_2^h(t), w_3^h(t))_H, \quad t \in T.$$

This implies the estimate

$$(z_t^h(t), z^h(t))_H + 0.25l^2(w_{3t}^h(t), w_3^h(t))_H \le (z^h(t), u^h(t))_H + \rho_1,$$

where

$$\rho_1 = 0.25l^2(w_2^h(t), w_3^h(t))_H + 0.25Bl^2|w_3^h(t)|_H^2.$$

Then we have for $t \in T_i$

$$(z^h(t), u_{2,i}^h)_H = -c_*(z^h(t), w_3^h(\tau_i) - \xi_i) \le \rho_2 + L_1(t; \tau_i),$$

where

$$\rho_2 = -c_*(z^h(t), w_3^h(t) - \varphi(t))_H,$$

$$L_1(t;\tau_i) = c_*|z^h(t)|_H\Big(h + \int_{\tau_i}^{t} \{|w_{3\tau}^h(\tau)|_H + |\varphi_\tau(\tau)|_H\}\,d\tau\Big).$$

Therefore we obtain for $t \in T_i$

$$(z_t^h(t), z^h(t))_H + 0.25l^2(w_{3t}^h(t), w_3^h(t))_H \le (z^h(t), u_i^{h(1)}) + L_2(t;\tau_i),$$

where

$$L_2(t;\tau_i) = c_0(|z^h(t)|^2 + 1 + h^2 + (t-\tau_i)\int_{\tau_i}^{t} \{|w_{3\tau}^h(\tau)|_H^2 + |\varphi_\tau(\tau)|_H^2\}\,d\tau + |w_3^h(t)|_H^2).$$

By (4.7) and boundedness of the set U, we derive from the last two inequalities for $t \in T$

$$|z^h(t)|_H^2 + 0.25l^2|w_3^h(t)|_H^2 \le c_1(1 + \int_0^t |z^h(\tau)|_H^2\,d\tau + \qquad (5.1)$$

$$\delta \int_0^t \{|w_{3\tau}^h(\tau)|_H^2 + |\varphi_\tau(\tau)|_H^2\}\,d\tau).$$

Multiplying the third equation of system (4.1) by $w_{3t}^h(t)$, we have after integration

$$\int_0^t |w_{3\tau}^h(\tau)|_H^2\,d\tau + \frac{1}{2}\int_\Omega |\nabla w_3^h(t,\eta)|^2\,d\eta \le \int_0^t (g(w_3^h(\tau)) + w_2^h(\tau), w_{3\tau}^h(\tau))_H\,d\tau.$$

By the inequality

$$|g(w_3^h(t,\eta))| \le B|w_3^h(t,\eta)|, \quad (t,\eta) \in T \times \Omega,$$

we obtain with $ab \le a^2 + b^2/4$

$$\int_0^t |w_{3\tau}^h(\tau)|_H^2\,d\tau \le B\int_0^t |w_3^h(\tau)|_H|w_{3\tau}^h(\tau)|_H\,d\tau + \int_0^t |w_2^h(\tau)|_H|w_{3\tau}^h(\tau)|_H\,d\tau \le$$

$$\leq \int_0^t \{0,5|w_{3\tau}^h(\tau)|_H^2 + (1+B^2)|w_3^h(\tau)|_H^2 + |w_2^h(\tau)|_H^2\}\, d\tau.$$

Consequently, it holds

$$\int_0^t |w_{3\tau}^h(\tau)|_H^2\, d\tau \leq 2(1+B^2) \int_0^t |w_3^h(\tau)|_H^2\, d\tau + 2\int_0^t |w_2^h(\tau)|_H^2\, d\tau. \tag{5.2}$$

Note that

$$w_2^h(t) = z^h(t) - lw_3^h(t), \quad t \in T.$$

This relation, (5.1), and (5.2) imply

$$|z^h(t)|_H^2 + 0.25l^2|w_3^h(t)|_H^2 \leq c_2 + c_3\int_0^t |z^h(\tau)|_H^2\, d\tau + c_4\int_0^t |w_3^h(\tau)|_H^2\, d\tau +$$

$$+ c_5\int_0^t |z^h(\tau) - lw_3^h(\tau)|_H^2\, d\tau.$$

From the last inequality, in virtue of the Gronwall inequality, we get

$$\sup_{t\in T}\{|w_2^h(t)|_H + |w_3^h(t)|_H\} \leq c^{(1)}\mu(\varphi_0, \psi_0). \tag{5.3}$$

In this way, we have obtained the estimate for the first item (4.11). The second item can be estimated along the lines of [8], Theorem 3.1. □

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