LIPSCHITZ STABILITY OF SOLUTIONS TO PARAMETRIC OPTIMAL CONTROL FOR ELLIPTIC EQUATIONS

Kazimierz Malanowski
Systems Research Institute
Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland
and
Fredi Tröltzsch
Fakultät für Mathematik
Technische Universität Chemnitz-Zwickau
D-09107 Chemnitz, Germany

Abstract. A family of parameter dependent elliptic optimal control problems with nonlinear boundary control is considered. The control function is subject to amplitude constraints. A characterization of conditions is given under which solutions to the problems exist, are locally unique and Lipschitz continuous in a neighborhood of the reference value of the parameter.

Keywords: Parametric optimal control, elliptic equation, nonlinear boundary control, Lipschitz stability of the solutions, generalized equations.

1 Introduction

This paper belongs to a series of papers, where the local Lipschitz stability of solutions to parametric optimal control problems for nonlinear systems is analyzed. Due to the presence of inequality type constraints, the problems are non-smooth. The main tool in stability analysis for such problems is Robinson's implicit function theorem for generalized equations [11]. Using this theorem it can be shown that a *sufficient* condition of Lipschitz stability for nonlinear system is that the solutions of the linear-quadratic accessory problems are Lipschitz continuous with respect to the additive perturbations. Usually, this last problem is much easier to investigate than the original one. This approach was applied to parametric mathematical programs in finite and infinite dimensions as well as to optimal control problems. However, Robinson's theorem does not provide information on the gap between the obtained sufficient and necessary conditions of Lipschitz stability. To get this information, Dontchev's extention of the theorem [3] can be used. It allows to get necessary conditions of Lipschitz stability, provided that the dependence of data on the parameter is sufficiently strong. Using this approach, a characterization of the Lipschitz stability property was recently obtained for mathematical programs [7], as well as for optimal control problems for systems described by nonlinear ordinary [6] and parabolic [10] equations, subject to control constraints.

The present paper concerns a nonlinear boundary control problem for the Laplace equation. We follow the approach of [10] and characterize local Lipschitz stability with respect to the parameter for the solutions of this problem. The main technical difference is that we eliminate the state, by introducing the solution map of the state equation, and we treat the problem as depending on the control alone. This approach is especially useful in the proof of necessity

*since it allows to weaken the required *strong dependence* on data. The approach can be applied, not only to the considered problem, but in general, to control constrained problems with different dynamics.*

As in [10], the crucial point in the stability analysis is to derive conditions of L^∞-Lipschitz stability of the solutions to the accessory linear-quadratic optimal control problems. Here we use the result obtained in the thesis of A.Unger [12].

The organization of the paper is the following. In Section 2 we introduce the considered optimal control problem, as well as the basic assumptions and we recall some regularity results for the solution of the state equation. In Section 3 we recall the abstract implicit function theorems and we use Robinson's theorem to derive *sufficient* conditions of Lipschitz stability. In Section 4 we use Dontchev's theorem to show that these conditions are also *necessary*, provided that the dependence of the data on the parameter is sufficiently strong.

2 Preliminaries

Let $\Omega \subset I\!\!R^n$ denote a bounded domain with boundary Γ. As usually, by Δy and $\partial_\nu y$ we denote the Laplace operator and the co-normal derivative of y at Γ, respectively. Moreover, let H be a Banach space of parameters and $G \subset H$ an open and bounded set of feasible parameters.

For any $h \in G$ consider the following elliptic optimal control problem:

$$(\mathrm{O}_h) \quad \text{Find } (y_h, u_h) \in Z^\infty := C(\bar{\Omega}) \times L^\infty(\Gamma) \text{ such that}$$

$$\begin{aligned} F(y_h, u_h, h) &= \min\{F(y,u,h) := \\ &:= \textstyle\int_\Omega \varphi(y(x),h)dx + \int_\Gamma \psi(y(x),u(x),h)dS_x\} \end{aligned} \tag{2.1}$$

subject to

$$\begin{aligned} -\Delta y(x) + y(x) &= 0 && \text{in } \Omega \\ \partial_\nu y(x) &= b(y(x),u(x),h) && \text{on } \Gamma, \end{aligned} \tag{2.2}$$

$$u \in \mathcal{U}^{ad} := \{v \in L^\infty(\Gamma) \mid q \le v(x) \le r \text{ a.e. in } \Gamma\}. \tag{2.3}$$

In this setting, $q < r$ are fixed real numbers, dS_x denotes the surface measure induced on Γ, and the subscript x indicates that the integration is performed w.r. to x. We assume:

(A1) The domain Ω has $C^{1,1}$-boundary Γ.

(A2) For any $h \in G$, the functions $\varphi(\cdot, h) : I\!R \mapsto I\!R$, $\psi(\cdot, \cdot, h) : I\!R \mapsto I\!R \times I\!R$ and $b(\cdot, \cdot, h) : I\!R \mapsto I\!R \times I\!R$ are of class C^2. Moreover, for any fixed $u \in I\!R$ and $h \in G$, $b(\cdot, u, h) : I\!R \mapsto I\!R$ is monotonically decreasing.

There is a bound $c_G > 0$ such that

$$|b(0,0,h)| + |D_{(y,u)}b(0,0,h)| + |D^2_{(y,u)}b(0,0,h)| \le c_G \quad \forall h \in G.$$

Moreover, for all $K > 0$ a constant $l(K)$ exists such that

$$||D^2_{(y,u)}(y_1, u_1, h) - D^2_{(y,u)}(y_2, u_2, h)|| \le l(K)(|y_1 - y_2| + |u_1 - u_2|)$$

for all y_i, u_i such that $|y_i| \le K, |u_i| \le K$, and all $h \in G$. The same conditions above are satisfied by φ and Ψ too.

(A3) For all fixed real y and u with $|y| \le K, |u| \le K$, there is a constant $l_H(K)$ such that

$$|b(y, u, h_1) - b(y, u, h_2)| \le l_H(K)||h_1 - h_2||_H \quad \forall h_i \in G,$$

$i = 1, 2$. This estimate holds for φ and Ψ, too.

DEFINITION **2.1** For any $u \in L^\infty(\Gamma)$ a function $y \in W^{1,2}(\Omega)$ is said to be a weak solution of (2.2), if for all $z \in W^{1,2}(\Omega)$ the following equation holds:

$$\int_\Omega \left(\Sigma_{i=1}^n \frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_i} + yz \right) dx = \int_\Gamma b(y,u) z dS_x. \tag{2.4}$$

◇

By the following lemma, proved in the Appendix, problem (O_h) is well posed.

LEMMA **2.2** *If* **(A1)** *and* **(A2)** *hold, then for any $u \in \mathcal{U}^{ad}$ and any $h \in G$ there exists a unique weak solution $y(u,h) \in W^{1,2}(\Omega) \cap C(\bar\Omega)$ of (2.2). Moreover, there exists $c > 0$ such that*

$$||y(u', h') - y(u'', h'')||_{C(\bar\Omega)} \le c(||u' - u''||_{L^\infty(\Gamma)} + ||h' - h''||_H). \tag{2.5}$$

◇

Proof: For all $u \in U_{ad}$ and $h \in G$, the existence of a unique solution y of (2.2) follows from [ ], Thm . In particular, we use that, for fixed h, the function $b = b(y, u, h)$ is a real function satisfying all assumptions stated in [ ]. There is a constant $K > 0$ such that $|y(x)| \le K$ on $\bar\Omega$, independently of the concrete choice of u and h. To see this, we write

$$b(y, u, h) = b(0, u, h) + D_y b(y^\vartheta, u, h) y,$$

where $y^\vartheta = \vartheta y, 0 < \vartheta < 1, \vartheta = \vartheta(x)$. Hence

$$\partial_\nu y + \beta y = b(0, u, h),$$

where $\beta \in L^\infty(\Gamma), \beta \geq 0$. The right hand side $b(0, u, h)$ is uniformly bounded by (A2), (A3). Lemma 2.3 applies to obtain a uniform bound K for y.

Let now $(u_i, h_i) \in U_{ad} \times G$ be given, $i = 1, 2$, with associated states y_i. Then

$$b(y_1, u_1, h_1) - b(y_2, u_2, h_2) = D_y b(y^\vartheta, u_1, h_1)(y_1 - y_2) + b(y_2, u_1, h_1) - b(y_2, u_2, h_2),$$

where now $y^\vartheta = y_1 + \vartheta(y_2 - y_1)$. Put $\beta = -D_y b(y^\vartheta, u_1, h_1)$ and $z = y_1 - y_2$. Then z solves

$$-\Delta z + z = 0, \partial_\nu z = f_2,$$

where

$$\begin{aligned} f_2 &= b(y_2, u_1, h_1) - b(y_2, u_2, h_2) \\ &= b(y_2, u_1, h_1) - b(y_2, u_2, h_1) + b(y_2, u_2, h_1) - b(y_2, u_2, h_2). \end{aligned}$$

According to (A2) and (A3), $|f_2(x)| \leq c(|u_1(x) - u_2(x)| + ||h_1 - h_2||_H)$. Lemma 2.3 yields $||z||_{C(\bar\Omega)} \leq c(||u_1 - u_2||_{L^\infty(\Gamma)} + ||h_1 - h_2||_H)$. ◇

We will also need the following standard regularity result for linear elliptic equations (see, eg., Casas [1]).

LEMMA **2.3** *Consider the following linear equation*

$$\begin{aligned} -\Delta z(x) + z(x) &= f_1(x) \qquad \text{in } \Omega, \\ \partial_\nu z(x) + \beta(x) z(x) &= f_2(x) \qquad \text{on } \Gamma. \end{aligned} \tag{2.6}$$

*If $f_1 \in L^q(\Omega)$, $q > n/2$, $f_2 \in L^p(\Gamma)$, $p > n-1$, and $\beta \in L^\infty(\Gamma)$ **is nonnegative**, then (2.6) has a unique weak solution $z \in W^{1,2}(\Omega) \cap C(\bar\Omega)$. There is a constant $c > 0$, **independent of β,** such that*

$$\|z\|_{W^{1,2}(\Omega)} + \|z\|_{C(\bar\Omega)} \leq c\,(\|f_1\|_{L^q(\Omega)} + \|f_2\|_{L^p(\Gamma)}). \tag{2.7}$$

◇

Let $h_0 \in G$ be a given reference value of the parameter. We assume:

(A4) There exists a local solution (y_0, u_0) of (O_{h_0}).

Let us denote by $s_h(\cdot) : L^\infty(\Gamma) \mapsto C(\bar\Omega)$ the mapping which, for a fixed $h \in G$, to a given control u assigns the weak solution of the state equation (2.2). Then problem (O_h) can be reformulated as the following problem of optimization with respect to the control alone:

$$(O'_h) \qquad \text{Find } u_h \in \mathcal{U}^{ad} \text{ such that}$$
$$\mathcal{F}(u_h, h) = \min_{u \in \mathcal{U}^{ad}} \mathcal{F}(u, h),$$

where

$$\mathcal{F}(u,h) = F(s_h(u),u,h)), \tag{2.8}$$

$\mathcal{F}:\ L^\infty(\Gamma)\times G \to I\!R$. In the sequel, we will use both equivalent formulations (O_h) and (O'_h) of the problem. A standard first order necessary optimality condition for (O'_{h_0}) is given in the form of the following variational inequality:

$$(D_u\mathcal{F}(u_0,h_0), u-u_0) \geq 0 \quad \text{for all } u \in \mathcal{U}^{ad}, \tag{2.9}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Gamma)$.

Denote by $\mathcal{S}_0 := (\mathcal{S}_0^\Omega, \mathcal{S}_0^\Gamma) : L^2(\Gamma) \mapsto L^2(\Omega)\times L^2(\Gamma)$ the mapping given by $\mathcal{S}_0 :\ v \mapsto (z_0, z_0|_\Gamma)$, where $z_0 = z_0(v)$ is the weak solution of the linearized boundary value problem:

$$\begin{aligned} -\Delta z(x) + z(x) &= 0 && \text{in } \Omega,\\ \partial_\nu z(x) &= D_y b(y_0,u_0,h_0)z(x) + D_u b(y_0,u_0,h_0)v(x) && \text{on } \Gamma. \end{aligned} \tag{2.10}$$

Standard calculations show that the adjoint mapping $\mathcal{S}_0^* : L^2(\Omega)\times L^2(\Gamma) \to L^2(\Gamma)$ is given by

$$\mathcal{S}_0^* \begin{pmatrix} r \\ s \end{pmatrix} = (\mathcal{S}_0^\Omega)^* r + (\mathcal{S}_0^\Gamma)^* s = D_u b(y_0,u_0,h_0)p(r,s)|_\Gamma, \tag{2.11}$$

where $p(r,s)$ is the weak solution of the adjoint equation

$$\begin{aligned} -\Delta p(x) + p(x) &= r(x) && \text{in } \Omega,\\ \partial_\nu z(x) &= D_y b(y_0,u_0,h_0)z(x) + s(x) && \text{on } \Gamma. \end{aligned} \tag{2.12}$$

Let us define the following Hamiltonian: $\mathcal{H} : I\!R^3 \times G \to I\!R$:

$$\mathcal{H}(y,u,p,h) := \psi(y,u,h) + p\, b(y,u,h). \tag{2.13}$$

Using (2.10)-(2.12) as well as (2.1) and (2.13), we get

$$\begin{aligned} D_u\mathcal{F}(u_0,h_0) &= \mathcal{S}_0^* D_y F(y_0,u_0,h_0) + D_u F(y_0,u_0,h_0)\\ &= (\mathcal{S}_0^\Omega)^* D_y\varphi(y_0,h_0) + (\mathcal{S}_0^\Gamma)^* D_y\psi(y_0,u_0,h_0) + D_u\psi(y_0,u_0,h_0)\\ &= D_u\mathcal{H}(y_0,u_0,p_0,h_0) \in L^\infty(\Gamma). \end{aligned} \tag{2.14}$$

where p_0 is the solution of the following adjoint equation

$$\begin{aligned} -\Delta p_0(x) + p_0(x) &= D_y\varphi(y_0,h_0) && \text{in } \Omega,\\ \partial_\nu p_0(x) &= D_y b(y_0,u_0,h_0)p_0 + D_y\psi(y_0,u_0,h_0) && \text{on } \Gamma. \end{aligned} \tag{2.15}$$

Hence we have

$$(D_u\mathcal{F}(u_0,h_0), u) = \int_\Gamma D_u\mathcal{H}(y_0,u_0,p_0,h_0)u\, dS_x. \tag{2.16}$$

Here, we have applied Lemma 2.3 to (2.15) to obtain the regularity $p_0 \in C(\bar{\Omega})$. By (2.16), optimality condition (2.9) takes the form

$$\int_\Gamma D_u \mathcal{H}(y_0, u_0, p_0, h_0)(u - u_0) dS_x \geq 0 \quad \text{for all } u \in \mathcal{U}^{ad}. \tag{2.17}$$

On the hand, since $D_u \mathcal{H}(y_0, u_0, p_0, h_0) \in L^\infty(\Gamma)$, then by (2.16) we can treat $D_u \mathcal{F}(u_0, h_0)$ as an element of $L^\infty(\Gamma)$ and rewrite condition (2.9) in the form of the following *generalized equation*:

$$0 \in D_u \mathcal{F}(u_0, h_0) + \mathcal{N}(u_0), \tag{2.18}$$

where

$$\mathcal{N}(u) := \begin{cases} \lambda \in \{L^\infty(\Gamma) \mid \int_\Gamma \lambda(v - u) dS_x \leq 0 \; \forall v \in \mathcal{U}^{ad}\} & \text{if } u \in \mathcal{U}^{ad}, \\ \emptyset & \text{if } u \notin \mathcal{U}^{ad} \end{cases}$$

is a multivalued mapping with closed graph.

3 Application of an abstract implicit function theorem

We are interested in the following problem:

> *Find conditions under which a neighborhood $G_0 \subset H$ of h_0 exists such that for all $h \in G_0$ there exists a locally unique solution (y_h, u_h) of (O_h), which is a Lipschitz continuous function of h.*

First, we will find conditions of existence, local uniqueness and Lipschitz continuity of stationary points of (O'_h), i.e., of the solutions to the generalized equation

$$0 \in D_u \mathcal{F}(u_h, h) + \mathcal{N}(u_h) \tag{3.19}$$

analogous to (2.18). Afterwards we will show that these stationary points correspond to the solutions of (O'_h). In a standard way (see, e.g., [4, 5, 9, 10]) we apply to (3.19) the implicit function theorem for generalized equations. To this end, along with (3.19) we consider the following generalized equation obtained by linearization and perturbation of (3.19) at the reference point:

$$\delta \in D_u \mathcal{F}(u_0, h_0) + D^2_{uu} \mathcal{F}(u_0, h_0)(v - u_0) + \mathcal{N}(v), \tag{3.20}$$

where $\delta \in L^\infty(\Gamma)$ is a perturbation.

We shall explain later that $D^2_{uu} \mathcal{F}(u_0, h_0)(v - u_0)$ can be identified with a measurable and essentially bounded function. In the sequel by

$$\mathcal{B}^X_\rho(x_0) = \{x \in X \mid \|x - x_0\|_X \leq \rho\}$$

we will denote the closed ball of radius ρ around $x_0 \in X$. Moreover, to simplify the notation, we will write $U = L^\infty(\Gamma)$.

Our sufficiency analysis is based on the following Robinson's abstract implicit function theorem (see [11]: Theorem 2.1 and Corollary 2.2).

THEOREM **3.1** *Suppose that $D_u\mathcal{F}(u,\cdot)$ is Lipschitz continuous in h, uniformly with respect to u in a neighborhood of u_0. If there is a constant $\widehat{\ell}$ such that*

- **(i)** *there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that, for each $\delta \in \mathcal{B}^U_{\rho_1}(0)$ there is a unique in $\mathcal{B}^U_{\rho_2}(v_0)$ solution to the linearized generalized equation (3.20), which is Lipschitz continuous in δ with any constant $\ell < \widehat{\ell}$,*

then

- **(ii)** *there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ such that, for each $h \in \mathcal{B}^H_{\sigma_1}(h_0)$ there is a unique in $\mathcal{B}^U_{\sigma_2}(u_0)$ solution to the nonlinear generalized equation (2.18), which is Lipschitz continuous in h with any constant $\ell < \widehat{\ell}$.*

◇

In verifying the necessity of the derived sufficient conditions of Lipschitz stability, we will consider a special situation, where the dependence of data upon the parameter is *strong* in the following sense:

$$\begin{aligned} &H = H^0 \times L^\infty(\Gamma), \text{ where } H^0 \text{ is an arbitrary Banach space, and} \\ &D_u\mathcal{F}(u,h) = D_u\mathcal{F}^0(u,h^0) + h^1, \text{ where } h^0 \in H^0 \text{ and } h^1 \in L^\infty(\Gamma). \end{aligned} \tag{3.21}$$

*Condition (3.21) requires that $D_u\mathcal{F}(u,y)$ is an *additive* function of the component $h^1 \in L^\infty(\Gamma)$ of $h = (h^0, h^1)$. In addition to that, $D_u\mathcal{F}(u,y)$ can be a function of another component h^0, of the funtional parameter h, which may belong to an arbitrary Banach space H^0.*

The next theorem follows from Theorem 3 in [3].

THEOREM **3.2** *If (3.21) holds, then* **(ii)** *implies* **(i)**. ◇

Theorem 3.1 allows to deduce existence, local uniqueness and Lipschitz continuity of solutions to (2.18) from the same properties of the solutions to the linear generalized equation (3.20). Usually, this last problem is much easier than the original one.

In order to apply Theorem 3.1 we have to find the concrete expressions of the derivatives in the linear generalized equation (3.20). Using (2.14) together with (2.15) and (2.10), we find the following form of (3.20).

$$\delta \in D^2_{uu}\mathcal{F}(u_0,h_0)v + \delta_0 + \mathcal{N}(v), \tag{3.22}$$

where

$$D^2_{uu}\mathcal{F}(u_0,h_0)\,v = \mathcal{K}(u_0,h_0)\,v + D^2_{uu}\mathcal{H}(y_0,u_0,p_0,h_0)\,v, \tag{3.23}$$

with

$$\begin{aligned}\mathcal{K}(u_0,h_0) &= (\mathcal{S}_0^\Omega)^* D_{yy}^2\varphi(y_0,h_0)\mathcal{S}_0^\Omega + (\mathcal{S}_0^\Gamma)^* D_{yy}^2\mathcal{H}(y_0,u_0,p_0,h_0)\mathcal{S}_0^\Gamma + \\ &+(\mathcal{S}_0^\Gamma)^* D_{yu}^2\mathcal{H}(y_0,u_0,p_0,h_0) + D_{uy}^2\mathcal{H}(y_0,u_0,p_0,h_0)\mathcal{S}_0^\Gamma,\end{aligned} \tag{3.24}$$

and

$$\delta_0 = D_u\mathcal{F}(u_0,h_0) - D_{uu}^2\mathcal{F}(u_0,h_0)u_0. \tag{3.25}$$

** By Lemma 2.3 we find that, for fixed $v \in L^\infty(\Gamma)$, $D_{uu}^2\mathcal{F}(u_0,h_0)\,v \in L^\infty(\Gamma)$.** Certainly, $v_0 = u_0$ is a solution of (3.22) for $\delta = 0$.

The generalized equation (3.22) constitutes the first order optimality condition for the following linear-quadratic problem.

(LO'_δ) Find $v_\delta \in \mathcal{U}^{ad}$ such that

$$\begin{aligned}&\tfrac{1}{2}(v_\delta, D_{uu}^2\mathcal{F}(u_0,h_0)v_\delta) + (\delta_0 - \delta, v_\delta) = \\ &= \min_{v\in\mathcal{U}^{ad}}\left\{\tfrac{1}{2}(v, D_{uu}^2\mathcal{F}(u_0,h_0)v) + (\delta_0-\delta, v)\right\}.\end{aligned}$$

In view of (3.23) and (3.24), problem (LO'_δ) is equivalent to the following linear-quadratic optimal control problem.

(LO_δ) Find $(z_\delta, v_\delta) \in Z^\infty$ such that

$\mathcal{I}(z_\delta, v_\delta, \delta) = \min \mathcal{I}(z, v, \delta)$

subject to

$$\begin{aligned}-\Delta z(x) + z(x) &= 0 && \text{in } \Omega \\ \partial_\nu z(x) &= D_y b(y_0(x), u_0(x), h_0) z(x) + \\ &\quad + D_u b(y_0(x), u_0(x), h_0) v(x) && \text{on } \Gamma,\end{aligned} \tag{3.26}$$

$v \in \mathcal{U}^{ad}$,

where

$$\mathcal{I}(z,v,\delta) := \frac{1}{2}\left((z,v), \mathcal{J}_0(z,v)\right) + \int_\Gamma (\delta_0(x) - \delta(x))v(x)dS_x$$

with the quadratic form

$$\begin{aligned}&((z,v), \mathcal{J}_0(z,v)) = \textstyle\int_\Omega D_{yy}^2\varphi(y_0,h_0)z^2dx + \\ &+ \textstyle\int_\Gamma [z,v]\begin{bmatrix} D_{yy}^2\mathcal{H}(y_0,u_0,p_0,h_0) & D_{yu}^2\mathcal{H}(y_0,u_0,p_0,h_0) \\ D_{uy}^2\mathcal{H}(y_0,u_0,p_0,h_0) & D_{uu}^2\mathcal{H}(y_0,u_0,p_0,h_0)\end{bmatrix}\begin{bmatrix} z \\ v\end{bmatrix} dS_x.\end{aligned} \tag{3.27}$$

To verify assumption **(i)** of Theorem 3.1, we have to show that, for all sufficiently small perturbations δ, problem (LO_δ) (or equivalently (LO'_δ)) has a locally unique stationary point, which is Lipschitz continuous in δ. For this purpose, we will need a coercivity assumption. To introduce it, for any $\alpha \geq 0$ define the sets

$$\begin{aligned}I^\alpha &= \{x \in \Gamma \mid D_u\mathcal{H}(y_0,u_0,p_0,h_0)(x) > \alpha\}, \\ J^\alpha &= \{x \in \Gamma \mid -D_u\mathcal{H}(y_0,u_0,p_0,h_0)(x) > \alpha\}.\end{aligned} \tag{3.28}$$

We assume:

(AC) *(coercivity)* There exist $\alpha > 0$ and $\gamma > 0$ such that

$$((z,v), \mathcal{J}_0(z,v)) \geq \gamma \|v\|^2_{L^2(\Gamma)} \tag{3.29}$$

for all pairs (z,v) satisfying (3.26) and such that $v \in V^2_\alpha$, where

$$V^q_\alpha := \{v \in L^q(\Gamma) \mid v(x) = 0 \text{ for a.a. } x \in I^\alpha \cup J^\alpha\}, \quad q \in [2,\infty]. \tag{3.30}$$

In view of definitions (3.23) and (3.27), condition (3.29) is equivalent to

$$\left(v, D^2_{uu}\mathcal{F}(u_0,h_0)v\right) \geq \gamma \|v\|^2_{L^2(\Gamma)} \quad \text{for all } v \in V^2_\alpha. \tag{3.31}$$

By Satz 18 in [12] we have

Lemma **3.3** *If* **(AC)** *holds, then there exist constants $\rho_1 > 0$ and $\rho_2 > 0$ such that, for all $\delta \in \mathcal{B}^U_{\rho_1}(0)$ there is a unique solution (z_δ, v_δ) of* (LO_δ) *such that $v_\delta \in \mathcal{B}^U_{\rho_2}(0)$. Moreover, there exists a constant $\ell > 0$ such that*

$$\|z_{\delta'} - z_{\delta''}\|_{C(\bar\Omega)}, \|v_{\delta'} - v_{\delta''}\|_{L^\infty(\Gamma)} \leq \ell\,\|\delta' - \delta''\|_{L^\infty(\Gamma)} \quad \textit{for all } \delta', \delta'' \in \mathcal{B}^U_{\rho_1}(0).$$

◇

Theorem 3.1 and Lemma 3.3 imply

Proposition **3.4** *If conditions* **(A1)-(A4)** *and* **(AC)** *hold, then there exist constants $\sigma_1 > 0$, $\sigma_2 > 0$ and $\ell > 0$ such that, for each $h \in \mathcal{B}^H_{\sigma_1}(h_0)$ there exists a unique in $\mathcal{B}^U_{\sigma_2}(u_0)$ solution u_h of (2.18) and*

$$\|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \ell \|h' - h''\|_H \quad \textit{for all } h', h'' \in \mathcal{B}^H_{\sigma_1}(h_0). \tag{3.32}$$

◇

It follows from (2.5) and (3.32) that there exists a constant $\ell' > 0$ such that

$$\|y_{h'} - y_{h''}\|_{C(\bar\Omega)} \leq \ell' \|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}^H_{\sigma_1}(h_0), \tag{3.33}$$

where y_h is the solution to the state equation (2.2) corresponding to the control u_h. A similar estimate follows in turn for p,

$$\|p_{h'} - p_{h''}\|_{C(\bar\Omega)} \leq \ell' \|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}^H_{\sigma_1}(h_0), \tag{3.34}$$

In view of Proposition 3.4, for $h \in \mathcal{B}^H_{\sigma_1}(h_0)$, we can define the maps $\mathcal{S}^\Omega_h$: $v \mapsto z_h(v)$, $\mathcal{S}^\Gamma_h$: $v \mapsto z_h(v)|_\Gamma$, where $z_h(v)$ is the solution of the linearized equation (2.10), with $D_y b$ and $D_u b$ evaluated at (y_h, u_h, h). Analogously to the definition of S_0, we put $S_h = (\mathcal{S}^\Omega_h, \mathcal{S}^\Gamma_h)$ and view S_h as an operator from $L^2(\Gamma)$ to $L^2(\Omega) \times L^2(\Gamma)$. Its adjoint operator $S^*_h : L^2(\Omega) \times L^2(\Gamma) \to L^2(\Gamma)$ is given by (2.11), (2.12), with the subscript h substituted for 0. The following lemma summarizes regularity properties of the mappings $\mathcal{S}_h$ and $\mathcal{S}^*_h$, which will be used in the sequel.

LEMMA **3.5** *The operators S_h and S_h^* are compact in the L^2-spaces defined above. Moreover, there is a constant $\tau > 0$ such that they are continuous between the following spaces: For all $p \geq 2$ and all $s < \infty$,*

$$S_h: \quad L^r(\Gamma) \to \begin{cases} L^s(\Omega) \times L^s(\Gamma) & \text{if } n = 2 \\ L^{r+\tau}(\Omega) \times L^{r+\tau}(\Gamma) & \text{if } n > 2 \end{cases} \tag{3.35}$$

$$S_h^*: \quad L^r(\Omega) \times L^r(\Gamma) \to \begin{cases} L^s(\Gamma) & \text{if } n = 2 \\ L^{r+\tau}(\Gamma) & \text{if } n > 2 \end{cases} \tag{3.36}$$

◇

Proof Define Λ_h by Λ_h: $(u, v) \mapsto (z, z|_\Gamma)$, where

$$\begin{aligned} -\Delta z + z &= u \\ \partial_\nu z &= D_y b(y_h, u_h, h)\, z + D_u b(y_h, u_h, h)\, v. \end{aligned}$$

Write for short $L^p = L^p(\Omega) \times L^p(\Gamma)$. We show that Λ_h is continuous from L^r to $L^{r+\tau}$ (resp. L^r to L^s for all s). This includes the statement of the theorem. Let us discuss the case $n > 2$, since the simpler case $n = 2$ follows by an obvious modification. It is known that $\Lambda_h : L^2 \to H^1(\Omega) \times H^{1/2}(\Gamma)$, continuously. The embeddings $H^1(\Omega) \subset L^\alpha(\Omega)$ and $H^{1/2}(\Gamma) \subset L^\beta(\Gamma)$ are compact for all $\alpha < 2n/(n-2)$ and all $\beta < 2(n-1)/(n-2)$. Therefore,

$$\Lambda_h : L^2 \to L^\beta \text{ is compact for } \beta < \frac{2(n-1)}{n-2} = 2 + \frac{2}{n-2}. \tag{3.37}$$

In particular, Λ_h is compact in L^2, which implies in turn the compactness of S_h and S_h^* stated in the Lemma. On the other hand, Lemma 2.3 shows that $\Lambda_h : L^q(\Omega) \times L^p(\Gamma) \to L^\infty$, if $q > n/2$ and $p > n-1$. Since $n - 1 \geq n/2$ for $n \geq 2$,

$$\Lambda_h : L^p \to L^\infty, \tag{3.38}$$

continuously. We fix $p > n - 1$. By interpolation,

$$\Lambda_h : L^{r_\theta} \to L^{s_\theta}, \tag{3.39}$$

continuously, where $0 \leq \theta \leq 1$, and r_θ, s_θ are defined by

$$\frac{1}{r_\theta} = \frac{1-\theta}{2} + \frac{\theta}{p}, \qquad \frac{1}{s_\theta} = \frac{1-\theta}{\beta} + \frac{\theta}{\infty} = \frac{1-\theta}{\beta}.$$

If θ runs from 0 to 1, then $r = r_\theta$ varies from 2 to p. We shall estimate below that $s_\theta - r_\theta \geq 2/(n-2) - \varepsilon$ for all $\varepsilon > 0$. Therefore, our statement is true for arbitrary $0 < \tau < 2/(n-2)$ and all $2 \leq r \leq p$. For $r > p$, the statement holds true by (3.38). Let us finally estimate $s_\theta - r_\theta$: We obtain

$$s_\theta - r_\theta = 2\left\{\frac{n-1}{(1-\theta)(n-2)} - \frac{p}{p-(p-2)\theta}\right\} - \delta,$$

where $\delta > 0$ can be taken arbitarily small. It is easy to verify that $2\{...\} \geq 2/(n-2)$ (multiply by $(1-\theta)(n-2)/2$ to see the equivalence to $n-1 \geq (1-\theta)(p(n-2)/(p-(p-2)\theta)+1)$. Therefore, $s_\theta - r_\theta > \tau$ holds for arbitrary $0 < \tau < 2/(n-2)$. ◇

The following theorem is our principal *sufficiency* result.

THEOREM **3.6** *If conditions* **(A1)-(A4)** *and* **(AC)** *hold, then there exist constants $\sigma_1 > 0$, $\sigma_2 > 0$ and $\lambda > 0$ such that, for each $h \in \mathcal{B}^H_{\sigma_1}(h_0)$ there exists a unique in $\mathcal{B}^{Z^\infty}_{\sigma_2}(y_0, u_0)$ solution (y_h, u_h) of* (O_h) *and*

$$\|y_{h'} - y_{h''}\|_{C(\bar\Omega)}, \|p_{h'} - p_{h''}\|_{C(\bar\Omega)}, \|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \lambda \|h' - h''\|_H$$

for all $h', h'' \in \mathcal{B}^H_{\sigma_1}(h_0)$. ◇

Proof In view of (3.32) and (3.33), to prove the theorem, it is enough to show that u_h satisfying (2.18) is a locally unique solution to (O'_h). **We have already pointed out that by regularity of u_h, for $h \in \mathcal{B}^H_{\sigma_1}(h_0)$, the derivatives $D_u\mathcal{F}(u_h, h)$ and $D^2_{uu}\mathcal{F}(u_h, h)$ can be represented as in (2.14) and (3.23), with all terms evaluated at h, rather than at h_0. Thus $D_u\mathcal{F}(u_h, h)$ can be treated as an element of $L^\infty(\Gamma)$. In view of Theorem 3.6 we find that the mapping

$$h \mapsto D_u\mathcal{F}(u_h, h) : \mathcal{B}^H_{\sigma_1}(h_0) \to L^\infty(\Gamma) \tag{3.40}$$

is continuous. Moreover, the quadratic form $D^2_{uu}\mathcal{F}(u_h, h)$ depends continuously h, since u_h y_h, and p_h depend continuously on h and therefore the estimate

$$(u_1, [D^2_{uu}\mathcal{F}(u_{h'}, h') - D^2_{uu}\mathcal{F}(u_{h''}, h'')]\ u_2) \leq c\ \|h' - h''\|_H\ \|u_1\|_{L^2(\Gamma)}\|u_2\|_{L^2(\Gamma)} \tag{3.41}$$

can be derived with some effort. From the continuity of $D_u\mathcal{F}(u_h, h) = D_u\mathcal{H}(y_h, u_h, p_h, h)$ and from (3.28) it follows that, for $\sigma_1 > 0$ sufficiently small, we have

$$|D_u\mathcal{F}(u_h, h)(x)| \geq \frac{\alpha}{2} \quad \text{for a.a. } x \in I^\alpha \cup J^\alpha \text{ and all } h \in \mathcal{B}^H_{\sigma_1}(h_0). \tag{3.42}$$

On the other hand, using the continuity of the mapping $h \mapsto D^2_{uu}\mathcal{F}(u_h, h)$ expressed by (3.41), and shrinking $\sigma_1 > 0$ if necessary, we obtain from (3.31) that

$$(v, D^2_{uu}\mathcal{F}(u_h, h)v) \geq \frac{\gamma}{2}\|v\|^2_{L^2(\Gamma)} \text{ for all } v \in V^2_\alpha \text{ and all } h \in \mathcal{B}^H_{\sigma_1}(h_0). \tag{3.43}$$

The estimates (3.42) and (3.43) constitute sufficient optimality conditions for (O_h). Indeed, expanding $\mathcal{F}(u, h)$ into Taylor series at u_h we get

$$\begin{aligned} &\mathcal{F}(u, h) - \mathcal{F}(u_h, h) = (D_u\mathcal{F}(u_h, h), u - u_h) + \\ &+ \tfrac{1}{2}((u - u_h), D^2_{uu}\mathcal{F}(u_h, h)(u - u_h)) + r(u - u_h), \end{aligned} \tag{3.44}$$

where

$$\frac{|r(v)|}{\|v\|_{L^2(\Gamma)}} \to 0 \quad \text{as } \|v\|_{L^\infty(\Gamma)} \to 0. \tag{3.45}$$

In view of (2.16), the optimality condition (2.9) evaluated at h yields

$$D_u\mathcal{F}(u_h, h)(x)(u(x) - u_h(x)) \geq 0 \ \text{ for a.a. } x \in \Gamma \text{ and all } u \in \mathcal{U}^{ad}. \tag{3.46}$$

On the other hand, in view of (3.41), there exists a constant $c > 0$ such that

$$(v, D^2_{uu}\mathcal{F}(u_h, h)v) \leq c\|v\|^2_{L^2(\Gamma)}. \tag{3.47}$$

For any $u \in L^2(\Gamma)$, let us denote by u^1 and u^2 the projections onto V^2_α and onto its orthogonal complement, respectively, i.e.,

$$u^1 = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ u(x) & \text{on } \Gamma \setminus I^\alpha \cup J^\alpha, \end{cases} \quad u^2 = \begin{cases} u(x) & \text{on } I^\alpha \cup J^\alpha, \\ 0 & \text{on } \Gamma \setminus I^\alpha \cup J^\alpha. \end{cases}$$

Using (3.43), (3.47) and Young's inequality we obtain

$$\begin{aligned} &((u - u_h), D^2_{uu}\mathcal{F}(u_h, h)(u - u_h)) = \\ &= ((u^1 - u^1_h), D^2_{uu}\mathcal{F}(u_h, h)(u^1 - u^1_h)) + 2((u^1 - u^1_h), D^2_{uu}\mathcal{F}(u_h, h)(u^2 - u^2_h)) + \\ &+ ((u^2 - u^2_h), D^2_{uu}\mathcal{F}(u_h, h)(u^2 - u^2_h)) \geq \\ &\geq \tfrac{\gamma}{2}\|u^1 - u^1_h\|^2_{L^2(\Gamma)} - 2c\|u^1 - u^1_h\|_{L^2(\Gamma)}\|u^2 - u^2_h\|_{L^2(\Gamma)} - c\|u^2 - u^2_h\|^2_{L^2(\Gamma)} \geq \\ &\geq \tfrac{\gamma}{4}\|u^1 - u^1_h\|^2_{L^2(\Gamma)} - c(1 + \tfrac{8c}{\gamma})\|u^2 - u^2_h\|^2_{L^2(\Gamma)} = \\ &= \tfrac{\gamma}{4}\int_{\Gamma\setminus(I^\alpha \cup J^\alpha)} (u^1(x) - u^1_h(x))^2 dS_x - c_1 \int_{(I^\alpha \cup J^\alpha)} (u^2(x) - u^2_h(x))^2 dS_x ** \\ &\text{for all } h \in \mathcal{B}^H_{\sigma_1}(h_0), \end{aligned} \tag{3.48}$$

where $c_1 = c(1+\frac{8c}{\gamma})$. Combining (3.46) through (3.48) and using (3.42) together with (3.44) we obtain

$$\begin{aligned} &\mathcal{F}(u, h) - \mathcal{F}(u_h, h) \geq \int_{I^\alpha \cup J^\alpha} \left(\tfrac{\alpha}{2}|u(x) - u_h(x)| - \tfrac{c_1}{2}|u(x) - u_h(x)|^2\right) dS_x + \\ &+ \tfrac{\gamma}{8}\int_{\Gamma\setminus(I^\alpha \cup J^\alpha)} |u(x) - u_h(x)|^2 dS_x + r(u - u_h) \quad \text{for all } u \in \mathcal{U}^{ad}. \end{aligned} \tag{3.49}$$

Choosing $\sigma = \frac{4\alpha}{\gamma + 4c_1}$ we get

$$\mathcal{F}(u, h) - \mathcal{F}(u_h, h) \geq \frac{\gamma}{16}\|u - u_h\|^2_{L^2(\Gamma)} + r(u - u_h) \quad \text{for all } u \in \mathcal{U}^{ad} \cap \mathcal{B}^U_\sigma(u_h).$$

By (3.45), for $\sigma > 0$ sufficiently small, we obtain

$$\mathcal{F}(u, h) - \mathcal{F}(u_h, h) \geq \frac{\gamma}{32}\|u - u_h\|^2_{L^2(\Gamma)} \quad \text{for all } u \in \mathcal{U}^{ad} \cap \mathcal{B}^U_\sigma(u_h),$$

i.e., u_h is a solution of (O_h) unique in $\mathcal{B}^U_\sigma(u_h)$. Choosing $\sigma_2 = \frac{\sigma}{2}$ and $\sigma_1 = \frac{\sigma}{2\ell}$ we complete the proof of the theorem. □

4 Lipschitz stability: necessity

In this section we are going to show that **(AC)** is not only a sufficient, but also a *necessary* condition of local Lipschitz stability of the solutions to (O_h), provided that the dependence of data upon the parameter h is *sufficiently strong* in the sense that (3.21) holds. In view of definition (2.8), condition(3.21) is satisfied if

(SD) (*strong dependence*)

$$F(y,u,h) = \int_\Omega \varphi(y(x),h)dx + \int_\Gamma [\psi^0(y(x),u(x),h^0) + h^1(x)u(x)]dS_x,$$
$$b(y,u,h) = b(y,u,h^0), \tag{4.50}$$

where $h^0 \in H^0$ and $h^1 \in L^\infty(\Gamma)$.

Note that if **(SD)** holds, then the Hamiltonian (2.18) takes the form

$$\mathcal{H}(y,u,p,h) = \mathcal{H}^0(y,u,p,h^0) + h^1 u, \tag{4.51}$$

where !?$\mathcal{H}^0(y,u,h^0) = \psi(y,u,h^0) + p\ b(y,u,h^0)$.!?

We assume that condition **(ii)** in Theorem 3.1 is satisfied and we will show that, if (4.50) holds, then **(AC)** is satisfied with some $\alpha > 0$ and $\gamma > 0$. The proof is based on the same idea as in [6] and in [10], but technically it is simplified. Namely, we introduce small variations $(\widehat{h}, u_{\widehat{h}})$ of the reference values, such that in a neighborhood of $(\widehat{h}, u_{\widehat{h}})$, problems (O'_h) with *inequality* constraints can be locally treated as problems with *equality* constraints, which are much easier to analyze. First, we derive necessary coercivity conditions for $(\mathrm{O}'_{\widehat{h}})$ and then deduce similar conditions for the original problem (O'_{h_0}). To this end, let us choose any $\alpha < \frac{\sigma_1}{2}$ and $\epsilon < \min\{\sigma_2, \frac{1}{2}(r-q)\}$, where q and r are the bounds in the inequality constraints (2.3), while σ_1 and σ_2 are given in Theorem 3.1/**(ii)**.

Define the set

$$K = \{x \in \Gamma \mid u_0(x) \le \frac{1}{2}(q+r)\}$$

and introduce the following increment Δu of the reference values u_0:

$$\Delta u(x) = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ +\epsilon & \text{on } K \setminus I^\alpha, \\ -\epsilon & \text{on } [\Gamma \setminus K] \setminus J^\alpha. \end{cases} \tag{4.52}$$

Define $u_{\widehat{h}} = u_0 + \Delta u$. It follows from (4.52) that , for $u_{\widehat{h}}$, the control constraints (2.3) are active on the set $I^\alpha \cup J^\alpha$ and *non-active with the margin* $\epsilon > 0$ on the complement of this set:

$$u_{\widehat{h}}(x) \begin{cases} = q & \text{on } I^\alpha, \\ = r & \text{on } J^\alpha, \\ \in [q+\epsilon, r-\epsilon] & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha] \end{cases} \tag{4.53}$$

(see, Fig.1). Let $y_{\widehat{h}}$ denote the solution of the state equartion (2.2) for $h = h_0$ and $u = u_{\widehat{h}}$. Similarly, let $p_{\widehat{h}}$ be the solution of the adjoint equation (2.15) corresponding to $h = h_0, u = u_{\widehat{h}}$ and $y = y_{\widehat{h}}$. Note that in view of (4.50) and (4.51), $y_{\widehat{h}}$ and $p_{\widehat{h}}$ *do not depend* on the component h_0^1 of $h_0 = (h_0^0, h_0^1)$. Using this fact we introduce such a variation $\widehat{h}^1$ of h_0^1, that $(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}})$ is a stationary point of $(\mathrm{O}_{\widehat{h}})$, where $\widehat{h} = (h_0^0, \widehat{h}^1)$. Namely, we put $\widehat{h}^1 = h_0^1 + \Delta h^1$, where

$$\Delta h^1(x) = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ -D_u\mathcal{H}^0(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, h_0^0)(x) - h_0^1(x) & \text{on } \Gamma \setminus (I^\alpha \cup J^\alpha). \end{cases} \tag{4.54}$$

(4.55)

Note that !? by **(ii)**!?,

$$\|D_u\mathcal{H}^0(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, h_0^0) - D_u\mathcal{H}^0(y_0, u_0, p_0, h_0^0)\|_{L^\infty(\Gamma)} \to 0 \quad \text{as } \|\Delta u\|_{L^\infty(\Gamma)} \to 0. \tag{4.56}$$

Using (3.28), (4.52) and (4.56), we find that, for $\epsilon > 0$ sufficiently small

$$D_u\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})(x) \begin{cases} > 0 & \text{on } I^\alpha, \\ < 0 & \text{on } J^\alpha, \\ = 0 & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha], \end{cases} \tag{4.57}$$

which, together with (4.53), shows that the variational inequality (2.17) is satisfied at $(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})$. On the other hand, for $x \in \Gamma \setminus (I^\alpha \cup J^\alpha)$ we have

$$\begin{aligned} &|D_u\mathcal{H}^0(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, h_0^0)(x) + h_0^1(x)| \le |D_u\mathcal{H}^0(y_0, u_0, p_0, h_0^0)(x)| + \\ &+ |D_u\mathcal{H}^0(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, h_0^0)(x) - D_u\mathcal{H}^0(y_0, u_0, p_0, h_0^0)(x)| \le \\ &\le \alpha + |D_u\mathcal{H}^0(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, h_0^0)(x) - D_u\mathcal{H}^0(y_0, u_0, p_0, h_0^0)(x)|. \end{aligned} \tag{4.58}$$

In view of (4.52), (4.54), (4.56) and (4.58), for $\alpha > 0$ and $\epsilon > 0$ small enough, we obtain $\|\Delta h^1\|_{L^\infty(\Gamma)} < \sigma_1$, i.e., $\widehat{h} \in \mathcal{B}_{\sigma_1}^H(h_0)$. Hence, by our assumption, $u_{\widehat{h}}$ is a unique in $\mathcal{B}_{\sigma_2}^U(u_0)$ solution to (2.9).

LEMMA **4.1** *For $\alpha > 0$ and $\epsilon > 0$ sufficiently small*

$$(u, D_{uu}^2\mathcal{F}(u_{\widehat{h}}, \widehat{h})u) \ge 0 \quad \textit{for all } v \in V_\alpha^2. \tag{4.59}$$

◇

Proof In view of (4.53) and (4.57), locally around $u_{\widehat{h}}$, the *inequality* constraints (2.3) can be treated as *equalities*:

$$u_{\widehat{h}}(x) \begin{cases} = q & \text{on } I^\alpha, \\ = r & \text{on } J^\alpha, \\ \text{free} & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha] \end{cases} \tag{4.60}$$

Indeed, for any feasible variation $u_{\widehat{h}} + \Delta u$ satisfying the first order optimality condition at $\widehat{h}$, we must have $\Delta u = 0$ on $I^\alpha \cup J^\alpha$. On the other hand, any variation $u_{\widehat{h}} + \Delta u$ such that

$$|\Delta u(x)| \begin{cases} = 0 & \text{on } I^\alpha \cup J^\alpha, \\ \leq \epsilon & \text{on } \Gamma \setminus (I^\alpha \cup J^\alpha), \end{cases}$$

is feasible for $(\mathrm{O}_{\widehat{h}})$. Hence, the first order optimality condition (2.18) at $(\widehat{h}, u_{\widehat{h}})$ reduces to the equation

$$(D_u \mathcal{F}(u_{\widehat{h}}, \widehat{h}), u) = 0 \qquad \text{for all } u \in V_\alpha^\infty. \tag{4.61}$$

Expanding $\mathcal{F}(\cdot, \widehat{h})$ into a Taylor series at $u_{\widehat{h}}$ and using (4.61), we obtain

$$0 \leq \mathcal{F}(u, \widehat{h}) - \mathcal{F}(u_{\widehat{h}}, \widehat{h}) = \tfrac{1}{2}((u - u_{\widehat{h}}), D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})(u - u_{\widehat{h}})) + r(u - u_{\widehat{h}})$$
$$\text{for all } u \in V_\alpha^\infty.$$

Dividing by $\|u - u_{\widehat{h}}\|^2_{L^2(\Gamma)}$ and passing to the limit with $\|u - u_{\widehat{h}}\|_{L^\infty(\Gamma)} \to 0$ and using (3.45) we obtain

$$(u, D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})u) \geq 0 \qquad \text{for all } u \in V_\alpha^\infty.$$

In view of the linearity of V_α^∞ as well as of the continuity and density of the embedding $V_\alpha^\infty \subset V_\alpha^2$, we arrive at (4.59). □

We are going to show now that the quadratic form (4.59) is actually *coercive*, with a constant independent of $\widehat{h}$. To this end we will use Theorem 3.2.

LEMMA **4.2** *If* **(i)** *holds, then*

$$\|D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})u\|_{L^\infty(\Gamma)} \geq \ell^{-1}\|u\|_{L^\infty(\Gamma)} \quad \textit{for all } u \in V_\alpha^\infty. \tag{4.62}$$

◇

Proof Let us introduce the linear generalized equation analogous to (3.20) but evaluated at $(u_{\widehat{h}}, \widehat{h})$:

$$\delta \in D_u\mathcal{F}(u_{\widehat{h}}, \widehat{h}) + D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})(u - u_{\widehat{h}}) + \mathcal{N}(u). \tag{4.63}$$

By Theorem 3.2, for δ sufficiently small, (4.63) has a locally unique solution v_δ, which is locally Lipschitz in δ, with the Lipschitz constant $\ell > 0$ independent of the choice of $\widehat{h} \in \mathcal{B}^H_{\sigma_1}(h_0)$. Note that $v_0 = u_{\widehat{h}}$ is the solution of (4.63) for $\delta = 0$. Hence, in the same way as in (4.61), we deduce from the construction of $\widehat{h}$ and $u_{\widehat{h}}$, that for all δ, in a small neighborhood of zero, (4.63) reduces to the equation

$$(D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})v_\delta + (D_u\mathcal{F}(u_{\widehat{h}}, \widehat{h}) - D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})u_{\widehat{h}}) - \delta, u) = 0 \quad \text{for all } u \in V_\alpha^\infty.$$

In view of (4.61), choosing $\delta \in V_\alpha^\infty$ we obtain from the above equation

$$D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})(v_\delta - v_0) + \delta = 0.$$

Since, for any sufficiently small $\delta \in V_\alpha^\infty$, this equation must have a unique solution $w_\delta := v_\delta - v_0 \in V_\alpha^\infty$, which is Lipschitz in δ, with Lipschitz constant $\ell > 0$, we arrive at (4.62). □

To show the coercivity of the quadratic form (4.59), note that, as in (3.23) $D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h}) = \mathcal{K}(u_{\widehat{h}}, \widehat{h}) + D_{uu}^2 \mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})$, where $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$ is given in (3.24) with indices 0 substituted by $\widehat{h}$. By (3.24) and (3.35), $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$ is continuous from $L^p(\Gamma)$ into $C(\Gamma)$, for $p > n-1$. Hence, by a known argument (see, e.g., Lemma 6.3 in [10]), (3.23) and (4.62) imply

$$D_{uu}^2 \mathcal{H}(y_{\widehat{h}}(x), u_{\widehat{h}}(x), p_{\widehat{h}}(x), \widehat{h}) \geq \ell^{-1} \quad \text{for a.a. } x \in \Gamma \setminus (I^\alpha \cup J^\alpha). \tag{4.64}$$

On the other hand, we obtain from Lemma 3.5 that $\mathcal{S}_h : L^2(\Gamma) \to L^2(\Omega) \times L^2(\Gamma)$ is compact. Therefore, it follows from (3.24) that $\mathcal{K}(u_{\widehat{h}}, \widehat{h}) : L^2(\Gamma) \to L^2(\Gamma)$ is *compact.*

LEMMA **4.3** *If (4.62) holds, then*

$$(u, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u) \geq \ell^{-1} \|u\|_{L^2(\Gamma)}^2 \quad \textit{for all } v \in V_\alpha^2. \tag{4.65}$$

◇

Proof Denote by $\mathcal{P} : L^2(\Gamma) \to V_\alpha^2$ the orthogonal projection in $L^2(\Gamma)$ onto the closed subspace V_α^2. Then we have

$$(u, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u) = (u, \mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u)_{V_\alpha^2} \quad \text{for all } u \in V_\alpha^2.$$

By a well known property of the spectrum of self-adjoint operators in a Hilbert space (see, e.g., Theorem 2, p.320 in [13]) we have

$$\begin{aligned} &\min\{\mu \in \mathbb{R} \mid \mu \in \sigma\} = \\ &= \inf\{(u, \mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u)_{V_\alpha^2} \mid u \in V_\alpha^2 \text{ with } \|u\|_{V_\alpha^2} = 1\}, \end{aligned}$$

where σ is the spectrum of

$$\mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h}) := \mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) + \mathcal{P} D_{uu}^2 \mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) : V_\alpha^2 \to V_\alpha^2.$$

Hence, in view of (4.59), condition (4.65) will be satisfied if the operator

$$\begin{aligned} &\mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) + (\mathcal{P} D_{uu}^2 \mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu)\mathcal{J} : V_\alpha^2 \to V_\alpha^2 \\ &\text{is invertible for all } \mu \in [0, \ell^{-1}), \end{aligned} \tag{4.66}$$

where $\mathcal{J}$ denotes the identity in V_α^2.

Note that by (4.64) the real function

$$(\mathcal{P}D^2_{uu}\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})(x) - \mu)^{-1}$$

is bounded, measurable and positive on $\Gamma \setminus (I^\alpha \cup J^\alpha)$ for any $\mu \in [0, \ell^{-1})$. Define the operator

$$\mathcal{R}_{\widehat{h}} := (\mathcal{P}D^2_{uu}\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) + \mu)^{-1}\mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) : V^2_\alpha \to V^2_\alpha \tag{4.67}$$

where $\mu \in [0, \ell^{-1})$. In view of compactness of $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$, the operator $\mathcal{R}_{\widehat{h}}$ is compact. To prove (4.66), it is enough to show that the operator $\mathcal{R}_{\widehat{h}} + \mathcal{J}$ is invertible. It follows from the definition of $\mathcal{R}_{\widehat{h}}$ and from (3.35), (3.36) that there exists $\tau > 0$ such that

$$\begin{array}{lll} & \mathcal{R}_{\widehat{h}} V^r_\alpha \subset V^{r+\tau}_\alpha & \text{for all } r \geq 2, \\ \text{and by Lemma 2.3} & \mathcal{R}_{\widehat{h}} **V^r_\alpha \subset V^\infty_\alpha & \text{for all } r > n-1. \end{array} \tag{4.68}$$

Consider the homogeneous equation

$$(\mathcal{R}_{\widehat{h}} + \mathcal{J})u = 0, \quad u \in V^2_\alpha. \tag{4.69}$$

Let us apply in (4.69) a bootstrapping procedure. Starting with $r = 2$ and using (4.68), after a finite number of steps, we find that $u = -\mathcal{R}_{\widehat{h}} u \in V^\infty_\alpha$, which in view of (4.62), shows that $u = 0$ is the only solution to (4.69). By a known property of compact operators (see, e.g., Theorem 2, Chapter XIII, Sec.1 in [8]) the uniqueness of the solution of the homogeneous equation (4.69) implies that the operator $(\mathcal{R}_{\widehat{h}} + \mathcal{J}) : V^2_\alpha ** \to V^2_\alpha$ has a bounded inverse. !? Hence (4.66) holds and the proof of the lemma is completed. □

We can formulate now the principal result of this paper, i.e., a characterization of the Lipschitz stability property for the solutions to problems (O_h).

THEOREM **4.4** *If conditions* **(A1)**-**(A4)** *hold, then* **(AC)** *is a sufficient condition in order that*

(LC) *There exist constants $\sigma_1 > 0$, $\sigma_2 > 0$ and $\lambda > 0$ such that for each $h \in \mathcal{B}^H_{\sigma_1}(h_0)$ there exists a unique in $\mathcal{B}^{Z^\infty}_{\sigma_2}(y_0, u_0)$ solution (y_h, u_h) of (O_h) and*

$$\|y_{h'} - y_{h''}\|_{C(\bar{\Omega})}, \|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \lambda\|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}^H_{\sigma_1}(h_0). \tag{4.70}$$

If in addition, condition **(SD)** *holds, then* **(AC)** *is also necessary for* **(LC)** *to be satisfied.* □

Proof Sufficiency is given in Theorem 3.6. To prove necessity, note that, from (4.65) we have

$$\begin{aligned} &(u, D^2_{uu}\mathcal{F}(u_0, h_0)u) = (u, D^2_{uu}\mathcal{F}(u_{\widehat{h}}, \widehat{h})u) + \\ &+(u, [D^2_{uu}\mathcal{F}(u_0, h_0) - D^2_{uu}\mathcal{F}(u_{\widehat{h}}, h)]u) \geq \ell^{-1}\|u\|^2_{L^2(\Gamma)} + \\ &+(u, [D^2_{uu}\mathcal{F}(u_0, h_0) - D^2_{uu}\mathcal{F}(u_{\widehat{h}}, h)]u) \quad \text{for all } u \in V^2_\alpha. \end{aligned} \tag{4.71}$$

By (3.41) and (4.70), choosing $\alpha > 0$ and $\epsilon > 0$ sufficiently small, we get

$$|(u, [D^2_{uu}\mathcal{F}(u_0, h_0) - D^2_{uu}\mathcal{F}(u_{\widehat{h}}, h)]u)| \le \frac{\ell^{-1}}{2}\|u\|^2_{L^2(\Gamma)}. \tag{4.72}$$

Conditions (4.71) and (4.72) show that **(AC)** is satisfied with $\gamma = \frac{\ell^{-1}}{2}$. □

References

[1] E. CASAS, Boundary control of semilinear elliptic equations with pointwise state-constraints, *SIAM J. Control Optim.*, **31** (1993), 993–1006.

[2] E. CASAS, F. TRÖLTZSCH, Second order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations. *Applied Math. Optimization*, **39** (1999), 211–228.

[3] A.L.DONTCHEV, Characterization of Lipschitz stability in optimization, in *Recent Developments in Well-Posed Variational Problems*, R.LUCCHETTI, J.REVALSKI Eds., Kluwer, 1995, pp. 95-116.

[4] A. L. DONTCHEV, W. W. HAGER, Lipschitzian stability in nonlinear control and optimization, *SIAM J. Control Optim.*, **31** (1993), 569–603.

[5] A.L.DONTCHEV, W.W.HAGER, A.B.POORE AND B.YANG, Optimality, stability and convergence in nonlinear control, *Appl. Math. Optim.* **31** (1995) 297-326.

[6] A. L. DONTCHEV, K. MALANOWSKI, A characterization of Lipschitzian stability in optimal control, *Proc. Conf. Calculus of Variations,* Haifa, March 1998, to appear.

[7] A. L. DONTCHEV, R. T. ROCKAFELLAR, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, *SIAM J. Optim.,* **6** (1996), 1087–1105.

[8] L.V. KANTOROVICH, G.P. AKILOV, *Functional Analysis,* Nauka, Moscow, 1977 (in Russian), English edition: Pergamon Press, Oxford, 1982.

[9] K. MALANOWSKI, Second order conditions and constraints qualifications in stability and sensitivity analysis of solutions to optimization problems in Hilbert spaces, *Appl. Math. Optim.* **25** (1992) 51-79.

[10] K. MALANOWSKI, F.TRÖLTZSCH, Lipschitz stability of solutions to parametric optimal control problems for parabolic equations,*Zeitschrift Analysis Anwend.* **18** (1999), 469-489

[11] S. M. ROBINSON, Strongly regular generalized equations, *Math. Oper. Res.*, **5** (1980), 43–62.

[12] A. UNGER, *Hinreichende Optimalitätsbedingungen 2. Ordnung und Konvergenz des SQP-Verfahrens für semilineare elliptische Randsteuerprobleme* Thesis, Chemnitz, 1997.

[13] K. YOSIDA, *Functional Analysis*, Springer, Berlin, 1980.