SECOND ORDER NECESSARY OPTIMALITY CONDITIONS FOR SOME STATE-CONSTRAINED CONTROL PROBLEMS OF SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract. In this paper we are concerned with some optimal control problems governed by semilinear elliptic equations. The case of a boundary control is studied. We consider pointwise constraints on the control and a finite number of equality and inequality constraints on the state. The goal is to derive first and second order optimality conditions satisfied by locally optimal solutions of the problem.

Key words. Boundary control, semilinear elliptic equations, optimality conditions, state constraints

AMS subject classifications. 49K20, 35J25

1. Introduction. In this paper we discuss first and second order necessary optimality conditions for a class of optimal boundary control problems governed by a linear elliptic partial differential equation with nonlinear boundary condition. Hereby, pointwise constraints on the control and finitely many state-constraints of equality and inequality type are given.

First order necessary optimality conditions are already well known for this type of problems (cf. Bonnans and Casas [3]), and we derive them only for convenience. In contrast to this, it seems to the authors that up to now only the paper [8] deals with second order conditions for elliptic control problems. In that work, *sufficient* optimality conditions were derived for the case without state constraints, which are in some sense arbitrarily close to the corresponding necessary ones. We aim to extend these conditions to problems with state-constraints. To do so, we have to deal first with necessary conditions. This gives the information we need to know how far the established sufficient conditions are from the necessary ones. We found out that the discussion of second order *sufficient* optimality conditions is surprisingly difficult for state-constrained problems and requires a very extensive analysis. Therefore, we have to report on this issue in a separate paper to be published elsewhere. On the other hand, the technique to derive *necessary* conditions differs essentially from that used in the case of sufficiency. This is another reason to confine ourselves here to necessary conditions.

For parabolic control problems we refer the reader to Goldberg and Tröltzsch [10], [11], who consider sufficient conditions, too.

It should be mentioned that second order optimality conditions have already been studied extensively for mathematical programming problems in general spaces, we refer only to Ioffe [12], Maurer and Zowe [14], Maurer [13] and to the survey given in Ben-Tal and Zowe [2]. Moreover, there are numerous applications in the control theory of ordinary differential equations.

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Although the general way to derive such results is more or less known (see, for instance, [2]), these results cannot be transferred directly to our problem. It is the question of regularity of different primal and adjoint partial differential equations and the adequate choice of function spaces making this problem interesting and delicate. Even the specification of a reasonable constraint qualification is not an easy task.

The paper is organized as follows: In section 2 we discuss the problem of existence, uniqueness and regularity of solutions to the nonlinear equation of state. We should underline that the well known standard Hilbert space approach for elliptic equations is not sufficient for our aims. The optimal control problem is set up in section 3, while the next section deals with first order necessary conditions. The main result on second order conditions is established in section 5.

Before finishing this section, let us remark that the elliptic operator $Ay = -\Delta y + y$ is considered along this paper, but there is not difficulty to extend the results to more general semilinear monotone elliptic operators. The only reason of our particular choice is to avoid a heavy notation, making easier to the reader the understanding of the main ideas and results presented in this paper.

2. State Equation. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Given a function $u \in L^\infty(\Gamma)$, we consider the following boundary value problem

$$\begin{cases} -\Delta y(x) + y(x) = 0 \text{ in } \Omega \\ \partial_\nu y(x) = b(x, y(x), u(x)) \text{ on } \Gamma, \end{cases} \tag{2.1}$$

where $\partial_\nu y$ denotes the normal derivative of y and $b : \Gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function measurable w.r.t. the first variable and of class C^2 w.r.t. the others and satisfying

$$\begin{cases} \dfrac{\partial b}{\partial y}(x, y, u) \le 0 \quad \text{a.e. } x \in \Gamma, \ \forall (y, u) \in \mathbb{R}^2; \\ \forall M > 0 \ \exists \psi_M \in L^p(\Gamma) \ (p > n-1) \text{ and } C_M > 0 \text{ such that} \\ |b(x, 0, u)| \le \psi_M(x) \quad \text{a.e. } x \in \Gamma \ \text{ and } |u| \le M; \\ \displaystyle\sum_{1 \le i+j \le 2} \left| \frac{\partial^{i+j} b}{\partial y^i \partial u^j}(x, y, u) \right| \le C_M \quad \text{a.e. } x \in \Gamma, \ |y| \le M \text{ and } |u| \le M. \end{cases} \tag{2.2}$$

Under the previous assumptions we can prove the existence and uniqueness of a solution of (2.1). To do this, we first formulate a lemma that will be used several times along this paper; see,for instance, Troianiello [18] for the proof.

LEMMA 2.1. *Let $\beta_1 \in L^\infty(\Gamma)$, $\beta_1(x) \le 0$ on Γ, and $\beta_2 \in L^p(\Gamma)$, $p > n-1$. Then the problem*

$$\begin{cases} -\Delta y + y = 0 \quad \textit{in } \Omega \\ \partial_\nu y = \beta_1 y + \beta_2 \quad \textit{on } \Gamma, \end{cases} \tag{2.3}$$

has a unique solution $y \in H^1(\Omega) \cap C^\alpha(\bar\Omega)$ for some $\alpha \in (0, 1)$, and there exists a constant $C_{p,\alpha} > 0$ independent on β_1 and β_2 such that

$$\|y\|_{H^1(\Omega)} + \|y\|_{C^\alpha(\bar\Omega)} \le C_{p,\alpha} \|\beta_2\|_{L^p(\Gamma)}. \tag{2.4}$$

THEOREM 2.2. *Let $u \in L^\infty(\Gamma)$, then there exists a unique weak solution of* (2.1), $y \in H^1(\Omega) \cap C^\alpha(\bar\Omega)$, *for some* $\alpha \in (0,1)$. *Moreover*

$$\|y\|_{H^1(\Omega)} + \|y\|_{C^\alpha(\bar\Omega)} \le \eta\left(\|u\|_{L^\infty(\Gamma)}\right), \tag{2.5}$$

where $\eta : [0,+\infty) \longrightarrow [0,+\infty)$ *is a non decreasing function.*

Proof. First let us assume that b is a bounded function. Then an application of Schauder's fixed point theorem leads easily to the existence of a solution $y \in H^1(\Omega)$. The uniqueness is an immediate consequence of the monotonicity of b w.r.t. y. In the case where b is not bounded we define for every $k \in \mathbb{N}$

$$b_k(x,y,u(x)) = \begin{cases} b(x,y,u(x)) & \text{if } |b(x,y,u(x))| \le +k, \\ +k & \text{if } b(x,y,u(x)) > +k, \\ -k & \text{if } b(x,y,u(x)) < -k. \end{cases}$$

From the previous argumentation we know the existence of an element $y_k \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta y_k + y_k = 0 & \text{in } \Omega \\ \partial_\nu y_k = b_k(\cdot, y_k, u) & \text{on } \Gamma. \end{cases} \tag{2.6}$$

For each $j > 0$ we set $z_j(x) = \max\{y_k(x) - j, 0\}$. Then $z_j \in H^1(\Omega)$ and from the previous equation and (2.1) we deduce

$$\int_\Omega |\nabla z_j|^2 dx + \int_\Omega |z_j|^2 dx = \int_\Gamma b_k(x, y_k(x), u(x)) z_j(x) dS(x) =$$

$$\int_\Gamma b_k(x,0,u(x)) z_j(x) dS(x) + \int_\Gamma [b_k(x,y_k(x),u(x)) - b_k(x,0,u(x))] z_j(x) dS(x) \le$$

$$\|b_k(\cdot,0,u(\cdot))\|_{L^p(\Gamma)} \|z_j\|_{L^{p'}(\Gamma)} \le \|b(\cdot,0,u(\cdot))\|_{L^p(\Gamma)} \|z_j\|_{W^{1,s}(\Omega)},$$

with $p' = p/(p-1)$ and $s = np/(np-n+1) < n/(n-1)$; see Nečas [15, Theorem 4.2; pag. 84]. Then we can argue as in Stampacchia [17, Theorem 4.1] to deduce that $z_j(x) = 0$ if $j \ge j_+$ for some j_+ depending only on $\|b(\cdot,0,u(\cdot))\|_{L^p(\Gamma)}$ or equivalently $y_k(x) \le j_+$. Analogously, the inequality $y_k(x) \ge j_-$ is shown by taking $z_j(x) = \max\{0, -y_k(x) - j\}$ in the previous argumentation. Therefore $\{y_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Omega)$. Moreover, from (2.6) and assumptions (2.2) it is easy to deduce the boundedness of $\{y_k\}_{k=1}^\infty$ in $H^1(\Omega)$. By taking a subsequence, we obtain an element $y \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $y_k \to y$ weakly in $H^1(\Omega)$ and weakly* in $L^\infty(\Omega)$. Since the trace mapping from $H^1(\Omega)$ to $L^2(\Gamma)$ is compact, Nečas [15], then $y_k \to y$ strongly in $L^2(\Gamma)$. This fact, along with the boundedness of $\{y_k\}_{k=1}^\infty$ in $L^\infty(\Gamma)$ and (2.2) lead to $b_k(x, y_k(x), u(x)) \to b(x, y(x), u(x))$ in $L^p(\Gamma)$. Therefore Lemma 2.1 implies $y_k \to y$ in $C(\bar\Omega)$. Now we are able to pass to the limit in (2.6) and to deduce that y is a solution of (2.1) in $H^1(\Omega) \cap C(\bar\Omega)$. The uniqueness is again a consequence of the monotonicity of b w.r.t. y.

We have proved that the estimate of y in $L^\infty(\Omega)$ depends on $\|b(\cdot,0,u)\|_{L^p(\Gamma)}$.Then it is enough to take $\beta_1 = 0$ and $\beta_2 = b(\cdot,y,u)$ in (2.3), and to use (2.2) to deduce the Hölder regularity and the estimate (2.5) from Lemma 2.1. □

In our opinion, Theorem 2.2 is well known by people working in partial differential equations, but we do not know a precise reference where the proof is done. However the methods used in the proof are classical, so that we have avoided all the details and we only have provided a sketch of the proof.

Let us denote by $G : L^\infty(\Gamma) \longrightarrow H^1(\Omega) \cap C(\bar\Omega)$ the mapping associating to every function u the solution of (2.1). In what follows, we denote the first and second order Fréchet-derivative of G at a point u in the directions v or v_1, v_2 by $G'(u)v$ and $G''(u)[v_1, v_2]$, respectively. Moreover, we write for convenience $G''(u)v^2 := G''(u)[v, v]$.

The next theorem establishes the differentiability properties of G.

THEOREM 2.3. *G is of class C^2, i.e. twice continuously Fréchet differentiable. If $u, v \in L^\infty(\Gamma)$, $y = G(u)$ and $z_v = G'(u)v$, then z_v is the solution of*

$$\left\{\begin{array}{l} -\Delta z + z = 0 \quad \text{in } \Omega \\ \\ \partial_\nu z = \dfrac{\partial b}{\partial y}(\cdot, y, u)z + \dfrac{\partial b}{\partial u}(\cdot, y, u)v \quad \text{on } \Gamma. \end{array}\right. \tag{2.7}$$

If $v_1, v_2 \in L^\infty(\Gamma)$ and $z_{v_1v_2} = G''(u)[v_1, v_2]$, then $z_{v_1v_2}$ is the solution of

$$\left\{\begin{array}{l} -\Delta z + z = 0 \quad \text{in } \Omega \\ \\ \partial_\nu z = \dfrac{\partial b}{\partial y}(\cdot, y, u)z + \dfrac{\partial^2 b}{\partial y^2}(\cdot, y, u)z_{v_1}z_{v_2} + \\ \\ \dfrac{\partial^2 b}{\partial y \partial u}(\cdot, y, u)(z_{v_1}v_2 + z_{v_2}v_1) + \dfrac{\partial^2 b}{\partial u^2}(\cdot, y, u)v_1v_2 \quad \text{on } \Gamma. \end{array}\right. \tag{2.8}$$

Proof. Let us take

$$V(\Omega) = \left\{y \in H^1(\Omega) \cap C(\bar\Omega) : -\Delta y + y = 0\right\}$$

endowed with the norm

$$\|y\|_{V(\Omega)} = \|y\|_{H^1(\Omega)} + \|y\|_{C(\bar\Omega)}.$$

Then $V(\Omega)$ is obviously a Banach space and the mapping $\partial_\nu : V(\Omega) \longrightarrow H^{-1/2}(\Gamma)$ is linear and continuous; see, for instance, Casas and Fernández [7]. Let us take $X = \mathrm{Im}\partial_\nu$ with the norm

$$\|\partial_\nu y\|_X = \|y\|_{V(\Omega)}.$$

It is easy to check that X is a Banach space and $L^p(\Gamma) \subset X$ for every $p > n-1$, the inclusion being continuous; see Lemma 2.1. Now we define $L : V(\Omega) \times L^\infty(\Gamma) \longrightarrow X$ by $L(y, u) = \partial_\nu y - b(\cdot, y, u)$. It is an elementary exercise to prove that the mapping

$$(y, u) \in C(\Gamma) \times L^\infty(\Gamma) \mapsto b(\cdot, y, u) \in L^p(\Gamma)$$

is of class C^2, moreover $(\partial L/\partial y)(y, u)z = g$ if and only if

$$\left\{\begin{array}{l} -\Delta z + z = 0 \quad \text{in } \Omega \\ \\ \partial_\nu z = \dfrac{\partial b}{\partial y}(\cdot, y, u)z + g \quad \text{on } \Gamma. \end{array}\right.$$

It is clear that $(\partial L/\partial y) : V(\Omega) \longrightarrow X$ is linear and injective and we have

$$\|g\|_X \leq \left\|\frac{\partial b}{\partial y}(\cdot, y, u)z + g\right\|_X + \left\|\frac{\partial b}{\partial y}(\cdot, y, u)z\right\|_X = \|z\|_{V(\Omega)} + \left\|\frac{\partial b}{\partial y}(\cdot, y, u)z\right\|_X$$

$$\leq \|z\|_{V(\Omega)} + c_1 \left\|\frac{\partial b}{\partial y}(\cdot, y, u)z\right\|_{L^p(\Gamma)} \leq \|z\|_{V(\Omega)} + c_2\|z\|_{C(\Gamma)} \leq c_3\|z\|_{V(\Omega)},$$

which proves the continuity of $(\partial L/\partial y)(y, u)$.

Let us see that $(\partial L/\partial y)(y, u)$ is surjective. Given $g \in X$ we put $y_g \in V(\Omega)$ such that $\partial_\nu y_g = g$. Note that we do not necessarily have $g \in L^p(\Gamma)$. Then we have

$$\text{(2.9)} \qquad \left\{ \begin{array}{l} -\Delta y_g + y_g = 0 \quad \text{in } \Omega \\ \\ \partial_\nu y_g = \dfrac{\partial b}{\partial y}(\cdot, y, u)y_g + \tilde{g} \quad \text{on } \Gamma, \end{array} \right.$$

where $\tilde{g} = g - (\partial b/\partial y)(\cdot, y, u)y_g$. Let us consider now the problem

$$\text{(2.10)} \qquad \left\{ \begin{array}{l} -\Delta \tilde{y} + \tilde{y} = 0 \quad \text{in } \Omega \\ \\ \partial_\nu \tilde{y} = \dfrac{\partial b}{\partial y}(\cdot, y, u)\tilde{y} - \dfrac{\partial b}{\partial y}(\cdot, y, u)y_g \quad \text{on } \Gamma. \end{array} \right.$$

Then Lemma 2.1 implies that $\tilde{y} \in V(\Omega)$, consequently $z = y_g - \tilde{y} \in V(\Omega)$ and it satisfies

$$\left\{ \begin{array}{l} -\Delta z + z = 0 \quad \text{in } \Omega \\ \\ \partial_\nu z = \dfrac{\partial b}{\partial y}(\cdot, y, u)z + g \quad \text{on } \Gamma, \end{array} \right.$$

which means that $(\partial L/\partial y)(y, u)z = g$, and so $(\partial L/\partial y)(y, u)$ is surjective.

Now we can apply the implicit function theorem (cf. Cartan [4]) to deduce that G is of class C^2 and satisfies:

$$\text{(2.11)} \qquad L(G(u), u) = \partial_\nu G(u) - b(\cdot, G(u), u) = 0 \quad \forall u \in L^\infty(\Gamma).$$

Taking $y = G(u)$, $z = G'(u)v$ and differentiating (2.11) we find

$$\partial_\nu z = \frac{\partial b}{\partial y}(\cdot, y, u)z + \frac{\partial b}{\partial u}(\cdot, y, u)v,$$

which together with the fact that $z \in V(\Omega)$ implies (2.7).

On the other hand, if we take $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and $z_{v_1v_2} = G''(u)[v_1, v_2]$, we get by differentiating (2.11) twice

$$\partial_\nu(G'(u)v_1) = \frac{\partial b}{\partial y}(\cdot, G(u), u)G'(u)v_1 + \frac{\partial b}{\partial u}(\cdot, G(u), u)v_1$$

and $(\partial_\nu(G'(u)v_1))'v_2 = \partial_\nu(G''(u)[v_1, v_2]) = \partial_\nu z_{v_1v_2}$,

$$\partial_\nu z_{v_1v_2} = \frac{\partial b}{\partial y}(\cdot, y, u)z_{v_1v_2} + \frac{\partial^2 b}{\partial y^2}(\cdot, y, u)z_{v_1}z_{v_2} +$$

$$\frac{\partial^2 b}{\partial y \partial u}(\cdot, y, u)(z_{v_1}v_2 + z_{v_2}v_1) + \frac{\partial^2 b}{\partial u^2}(\cdot, y, u)v_1v_2,$$

which leads to (2.8). $\square$

3. The Control Problem. Let $J : L^\infty(\Gamma) \longrightarrow \mathbb{R}$ be defined by

$$J(u) = \int_\Omega f(x, y_u(x))dx + \int_\Gamma g(x, y_u(x), u(x))dS(x),$$

where $y_u = G(u)$ is the solution of (2.1) corresponding to u, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable w.r.t. the first variable and of class C^2 with respect to the second and $g : \Gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is also measurable in the first variable and of class C^2 w.r.t. the other two. Moreover we assume that

$$(3.1)\quad \left\{\begin{array}{l} f(\cdot, 0) \in L^1(\Omega); \\ \\ \forall M > 0\ \exists \psi^1_M \in L^1(\Omega) \text{ such that} \\ \\ \left|\dfrac{\partial f}{\partial y}(x,y)\right| + \left|\dfrac{\partial^2 f}{\partial y^2}(x,y)\right| \le \psi^1_M(x) \ \text{ for } |y| \le M \text{ and a.e. } x \in \Omega; \end{array}\right.$$

$$(3.2)\left\{\begin{array}{l} g(\cdot, 0, 0) \in L^1(\Gamma); \\ \\ \forall M > 0\ \exists \psi^2_M \in L^1(\Gamma) \text{ such that} \\ \\ \displaystyle\sum_{1 \le i+j \le 2} \left|\dfrac{\partial^{i+j} g}{\partial y^i \partial u^j}(x,y,u)\right| \le \psi^2_M(x) \text{ for } |y| \le M, |u| \le M \text{ and a.e. } x \in \Gamma. \end{array}\right.$$

Let us consider some functionals $F_j : C(\bar\Omega) \longrightarrow \mathbb{R}$ of class C^2, $1 \le j \le m$, and functions $u_a, u_b \in L^\infty(\Gamma)$, with $u_a(x) \le u_b(x)$ a.e. $x \in \Gamma$. The control problem is formulated as follows

$$(\mathrm{P})\left\{\begin{array}{l} \text{Minimize } J(u) \\ u_a(x) \le u(x) \le u_b(x) \ \text{ a.e. } x \in \Gamma \\ F_j(y_u) = 0, \ \ 1 \le j \le m_1 \\ F_j(y_u) \le 0, \ \ m_1 + 1 \le j \le m. \end{array}\right.$$

Let us show some examples of state constraints that fall into the previous abstract framework.

Example 1. For every $1 \le j \le m$ let $f_j : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function of class C^2 with respect to the second variable such that for each $M > 0$ there exists a function $\eta^j_M \in L^1(\Omega)$ satisfying

$$|f_j(x,0)| + \left|\frac{\partial f_j}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f_j}{\partial y^2}(x,y)\right| \le \eta^j_M(x) \ \ a.e.\ x \in \Omega,\ \forall |y| \le M.$$

Then the equality and inequality constraints defined by the functions

$$F_j(y_u) = \int_\Omega f_j(x, y_u(x))dx$$

are included in the formulation of (P).

Example 2. For every $1 \le j \le m$ let $f_j : \Gamma \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function of class C^2 with respect to the second variable such that for each $M > 0$ there exists a function $\eta^j_M \in L^1(\Gamma)$ satisfying

$$|f_j(x,0)| + \left|\frac{\partial f_j}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f_j}{\partial y^2}(x,y)\right| \le \eta^j_M(x) \ \ a.e.\ x \in \Gamma,\ \forall |y| \le M.$$

Then the functionals

$$F_j(y_u) = \int_\Gamma f_j(x, y_u(x))dS(x)$$

define some integral constraints included in the formulation of (P).

EXAMPLE 3. Given m functions $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2, a set of points $\{x_i\}_{i=1}^m \subset \bar{\Omega}$, and some integer m_1, $1 \leq m_1 \leq m$, the constraints

$$f_i(y_u(x_i)) = 0,\ 1 \leq i \leq m_1 \ \text{ and } \ f_i(y_u(x_i)) \leq 0,\ m_1 + 1 \leq i \leq m$$

can be written in the above framework by putting $F_i(y) = f_i(y(x_i))$.

The rest of this section will be devoted to the study of differentiability of the functions involved in the control problem. Regarding this question, let us start by making some observations about the Neumann problem with measures as boundary data. The reader is referred to Alibert and Raymond [1] and Casas [5] for the details; see also Casas [6] for the parabolic case.

Given two real Borel measures μ_Ω and μ_Γ in Ω and Γ respectively, and $\beta \in L^\infty(\Gamma)$, $\beta(x) \leq 0$ a.e. $x \in \Gamma$, we consider the boundary value problem

$$\left\{ \begin{array}{l} -\Delta\varphi + \varphi = \mu_\Omega \ \text{ in } \Omega \\ \partial_\nu \varphi = \beta\varphi + \mu_\Gamma \ \text{ on } \Gamma. \end{array} \right. \tag{3.3}$$

This problem has a unique solution $\varphi \in W^{1,s}(\Omega)$, for every $s < n/(n-1)$. Moreover it satisfies the integration by parts formula

$$\int_\Omega [-\Delta z + z]\varphi dx + \int_\Gamma (\partial_\nu z - \beta z)\varphi dS(x) = \int_\Omega z d\mu_\Omega + \int_\Gamma z d\mu_\Gamma \ \forall z \in Y_{q,p}, \tag{3.4}$$

where

$$Y_{q,p} = \left\{ z \in H^1(\Omega) : -\Delta z + z \in L^q(\Omega) \text{ and } \partial_\nu z - \beta z \in L^p(\Gamma) \right\},$$

with $q > n/2$ and $p > n - 1$. The partial differential equation equation of (3.3) is intended in the distribution sense and the Neumann condition in a trace sense, such as defined in [5]. We are using the fact that the differential operator is $-\Delta$. For more general elliptic operators with non continuous coefficients the uniqueness of a solution is only true in the class of functions of $W^{1,s}(\Omega)$ that satisfies (3.4); see Serrin [16] for an example of non uniqueness. It is important to remark here that the solutions satisfying (3.4) are the unique ones interesting in control theory, because (3.4) allows to make the integration by parts necessary to take advantage of the adjoint state.

We now study the differentiability of J.

THEOREM 3.1. *The functional J is of class C^2 and for every $\bar{u}, v \in L^\infty(\Gamma)$ we have*

$$J'(\bar{u})v = \int_\Gamma \left\{ \bar{\varphi}_0 \frac{\partial b}{\partial u}(\cdot, \bar{y}, \bar{u}) + \frac{\partial g}{\partial u}(\cdot, \bar{y}, \bar{u}) \right\} v dS(x) \tag{3.5}$$

and

$$J''(\bar{u})v^2 = \int_\Omega \frac{\partial^2 f}{\partial y^2}(\cdot, \bar{y}) z_v^2 dx + \int_\Gamma \left[\bar{\varphi}_0 \frac{\partial^2 b}{\partial y^2}(\cdot, \bar{y}, \bar{u}) + \frac{\partial^2 g}{\partial y^2}(\cdot, \bar{y}, \bar{u}) \right] z_v^2 dS(x) +$$

$$2\int_\Gamma\left[\bar\varphi_0\frac{\partial^2 b}{\partial y\partial u}(\cdot,\bar y,\bar u)+\frac{\partial^2 g}{\partial y\partial u}(\cdot,\bar y,\bar u)\right]z_v v\,dS(x)+ \tag{3.6}$$

$$\int_\Gamma\left[\bar\varphi_0\frac{\partial^2 b}{\partial u^2}(\cdot,\bar y,\bar u)+\frac{\partial^2 g}{\partial u^2}(\cdot,\bar y,\bar u)\right]v^2 dS(x),$$

where $\bar y = G(\bar u)$, $z_v \in H^1(\Omega)\cap C(\bar\Omega)$ *is the solution of* (2.7) *corresponding to* $(\bar y,\bar u)$, *i.e.* $z_v = G'(\bar u)v$, *and* $\bar\varphi_0 \in W^{1,s}(\Omega)$ *for every* $s < n/(n-1)$ *is the solution of*

$$\begin{cases} -\Delta\bar\varphi_0+\bar\varphi_0 = \dfrac{\partial f}{\partial y}(\cdot,\bar y) & in\ \Omega \\ \partial_\nu\bar\varphi_0 = \dfrac{\partial b}{\partial y}(\cdot,\bar y,\bar u)\bar\varphi_0+\dfrac{\partial g}{\partial y}(\cdot,\bar y,\bar u) & on\ \Gamma. \end{cases} \tag{3.7}$$

Proof. Let us consider the functional $F_0 : C(\bar\Omega)\times L^\infty(\Gamma)\longrightarrow\mathbb{R}$ defined by

$$F_0(y,u)=\int_\Omega f(x,y(x))dx+\int_\Gamma g(x,y(x),u(x))dS(x).$$

By the assumptions of f and g it is easy to prove that F_0 is of class C^2. Applying now the chain rule to $J(u) = F_0(G(u),u)$ and using Theorem 2.3 we get that J is of class C^2 and

$$J'(\bar u)v = \frac{\partial F_0}{\partial y}(G(\bar u),\bar u)G'(\bar u)v+\frac{\partial F_0}{\partial u}(G(\bar u),\bar u)v =$$

$$\int_\Omega\frac{\partial f}{\partial y}(\cdot,\bar y)z_v\,dx+\int_\Gamma\left\{\frac{\partial g}{\partial y}(\cdot,\bar y,\bar u)z_v+\frac{\partial g}{\partial u}(\cdot,\bar y,\bar u)v\right\}dS(x),$$

where $z_v = G'(\bar u)v$. Taking $\bar\varphi_0$ as solution of (3.7), we deduce (3.5) from the previous identity and (3.4). Indeed assumptions (3.1) and (3.2) imply $(\partial f/\partial y)(\cdot,\bar y)\in L^1(\Omega)$ and $(\partial g/\partial y)(\cdot,\bar y,\bar u)\in L^1(\Gamma)$, therefore the formula of integration by parts (3.4) can be used, replacing equation (3.3) by (3.7) (let us remind that $L^1(\Omega)$ (respect. $L^1(\Gamma)$) can be considered as a subspace of the space of Borel measures on Ω (respect. Γ)).

On the other hand, setting $z_{vv} = G''(\bar u)v^2$ we have

$$J''(\bar u)v^2 = \frac{\partial F_0}{\partial y}(G(\bar u),\bar u)G''(\bar u)v^2+$$

$$\frac{\partial^2 F_0}{\partial y^2}(G(\bar u),\bar u)(G'(\bar u)v)^2+2\frac{\partial^2 F_0}{\partial y\partial u}(G(\bar u),\bar u)(G'(\bar u)v)v+\frac{\partial^2 F_0}{\partial u^2}(G(\bar u),\bar u)v^2 =$$

$$\int_\Omega\frac{\partial f}{\partial y}(\cdot,\bar y)z_{vv}dx+\int_\Gamma\frac{\partial g}{\partial y}(\cdot,\bar y,\bar u)z_{vv}dS(x)+\int_\Omega\frac{\partial^2 f}{\partial y^2}(\cdot,\bar y)z_v^2dx+$$

$$\int_\Gamma\left\{\frac{\partial^2 g}{\partial y^2}(\cdot,\bar y,\bar u)z_v^2+2\frac{\partial^2 g}{\partial y\partial u}(\cdot,\bar y,\bar u)z_v v+\frac{\partial^2 g}{\partial u^2}(\cdot,\bar y,\bar u)v^2\right\}dS(x).$$

Taking into account that z_{vv} satisfies (2.8) and using again (3.4) and (3.7), we deduce (3.6). □

In the sequel we will denote by G_j, $1 \le j \le m$, the composite functionals $G_j : L^\infty(\Gamma) \longrightarrow \mathbb{R}$ defined by $G_j(u) = F_j(G(u))$. The next theorem provides formulas for the derivatives of these functionals.

THEOREM 3.2. *The functionals G_j, $1 \le j \le m$, are of class C^2 and for every $\bar{u}, v \in L^\infty(\Gamma)$ we have*

$$G_j'(\bar{u})v = \int_\Gamma \bar{\varphi}_j \frac{\partial b}{\partial u}(\cdot, \bar{y}, \bar{u}) v \, dS(x), \tag{3.8}$$

where $\bar{y} = G(\bar{u})$ and $\{\bar{\varphi}_j\}_{j=1}^m \subset W^{1,s}(\Omega)$ for every $s < n/(n-1)$ satisfy

$$\begin{cases} -\Delta \bar{\varphi}_j + \bar{\varphi}_j = F_j'(\bar{y})|_\Omega & \text{in } \Omega \\ \partial_\nu \bar{\varphi}_j = \dfrac{\partial b}{\partial y}(\cdot, \bar{y}, \bar{u})\bar{\varphi}_j + F_j'(\bar{y})|_\Gamma & \text{on } \Gamma. \end{cases} \tag{3.9}$$

Moreover

$$G_j''(\bar{u})v^2 = F_j''(\bar{y})z_v^2 +$$

$$\int_\Gamma \bar{\varphi}_j \left\{ \frac{\partial^2 b}{\partial y^2}(\cdot, \bar{y}, \bar{u}) z_v^2 + 2\frac{\partial^2 b}{\partial y \partial u}(\cdot, \bar{y}, \bar{u}) z_v v + \frac{\partial^2 b}{\partial u^2}(\cdot, \bar{y}, \bar{u}) v^2 \right\} dS(x), \tag{3.10}$$

where $z_v = G'(\bar{u})v$.

Let us remark that $F_j'(\bar{y}) : C(\bar{\Omega}) \longrightarrow \mathbb{R}$ is a linear and continuous functional, thus it can be identified with a real Borel measure in $\bar{\Omega}$. Therefore we can decompose $F_j'(\bar{y}) = F_j'(\bar{y})|_\Omega + F_j'(\bar{y})|_\Gamma$, where $F_j'(\bar{y})|_\Omega$ (resp. $F_j'(\bar{y})|_\Gamma$) denotes the restriction of $F_j'(\bar{y})$ to Ω (resp. to Γ).

Proof. Once again Theorem 2.3 along with the chain rule implies that G_j is of class C^2 and thanks to (3.4) and (3.9) we get

$$G_j'(\bar{u})v = F_j'(G(\bar{u}))G'(\bar{u})v = F_j'(\bar{y})z_v =$$

$$\langle F_j'(\bar{y})|_\Omega, z_v \rangle + \langle F_j'(\bar{y})|_\Gamma, z_v \rangle = \int_\Gamma \bar{\varphi}_j \frac{\partial b}{\partial u}(\cdot, \bar{y}, \bar{u}) v \, dS(x).$$

On the other hand,

$$G_j''(\bar{u})v^2 = F_j''(G(\bar{u}))(G_j'(\bar{u})v)^2 + F_j'(G(\bar{u}))G''(\bar{u})v^2 =$$

$$F_j''(\bar{y})z_v^2 + F_j'(\bar{y})z_{vv} = F_j''(\bar{y})z_v^2 + \langle F_j'(\bar{y}|_\Omega, z_{vv}\rangle + \langle F_j'(\bar{y}|_\Gamma, z_{vv}\rangle.$$

Now it is enough to use the equations satisfied by $\bar{\varphi}_j$ and $z_{vv} = G''(\bar{u})v^2$ to obtain the desired result. □

4. First Order Necessary Optimality Conditions. In this section we will assume that $\bar{u}$ is a local solution for problem (P). A function $\bar{u} \in U_{ad}$ is said to be a *local solution* or *locally optimal control*, if a $\delta > 0$ exists such that $J(u) \ge J(\bar{u})$ holds for all $u \in U_{ad}$ satisfying with their associated state $y = G(u)$ the state-constraints and $\|u - \bar{u}\|_{L^\infty(\Gamma)} < \delta$. We introduce by $I_0 = \{j \le m \mid F_j(\bar{y}) = 0\}$ and $I_- = \{j \le m \mid F_j(\bar{y}) < 0\}$ the sets of indices of active and inactive inequality constraints, respectively, where $\bar y = G(\bar u)$ is the associated state to $\bar u$. It is obvious that $\{1,\ldots,m_1\} \subset I_0$. We also follow the notation

$$\Gamma_\epsilon = \{x \in \Gamma : u_a(x) + \epsilon \le \bar u(x) \le u_b(x) - \epsilon\}, \quad \text{for } \epsilon \ge 0.$$

We rely on the following regularity assumption

$$(4.1) \qquad \left\{ \begin{array}{l} \exists \epsilon_{\bar u} > 0 \text{ and } \{h_j\}_{j\in I_0} \subset L^\infty(\Gamma), \text{ with } \operatorname{supp} h_j \subset \Gamma_{\epsilon_{\bar u}}, \text{ such that} \\ G_i'(\bar u)h_j = \delta_{ij}, \quad i,j \in I_0. \end{array} \right.$$

Obviously, our assumption is equivalent to the independence of the gradients $\{G_j'(\bar u)\}_{j\in I_0}$ in $L^\infty(\Gamma_{\epsilon_{\bar u}})$. Using Theorem 3.2 we can write the previous assumption in the following way

$$(4.2) \qquad \int_\Gamma \bar\varphi_i \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u) h_j \, dS(x) = \delta_{ij}, \; i,j \in I_0.$$

For finite dimensional constrained optimization problems, the usual regularity assumptions made to derive the optimality conditions involve the optimal solution, and it is not easy to check wether the assumption is satisfied or not. We can not do better in infinite dimension than in finite dimension. We cannot either avoid to include the the set of active constraints, which is related here to the choice of Γ_ϵ. These conditions are crucial in the proofs.
Now we establish the first order necessary conditions for optimality satisfied by $\bar u$.

THEOREM 4.1. *Let us assume that* (4.1) *holds. Then there exist real numbers* $\{\bar\lambda_j\}_{j=1}^m \subset \mathbb{R}^m$ *and functions* $\bar y \in H^1(\Omega) \cap C^\alpha(\bar\Omega)$, *for some* $\alpha \in (0,1)$, *and* $\bar\varphi \in W^{1,s}(\Omega)$ *for all* $s < n/(n-1)$ *such that*

$$(4.3) \qquad \bar\lambda_j \ge 0, \quad m_1 \le j \le m, \; \bar\lambda_j = 0 \textit{ if } j \in I_-;$$

$$(4.4) \qquad \left\{ \begin{array}{l} -\Delta \bar y + \bar y = 0 \quad \textit{in } \Omega \\ \partial_\nu \bar y = b(\cdot,\bar y,\bar u) \quad \textit{on } \Gamma, \end{array} \right.$$

$$(4.5) \qquad \left\{ \begin{array}{l} -\Delta \bar\varphi + \bar\varphi = \dfrac{\partial f}{\partial y}(\cdot,\bar y) + \displaystyle\sum_{j=1}^m \bar\lambda_j F_j'(\bar y)|_\Omega \quad \textit{in } \Omega \\ \partial_\nu \bar\varphi = \dfrac{\partial b}{\partial y}(\cdot,\bar y,\bar u)\bar\varphi + \dfrac{\partial g}{\partial y}(\cdot,\bar y,\bar u) + \displaystyle\sum_{j=1}^m \bar\lambda_j F_j'(\bar y)|_\Gamma \quad \textit{on } \Gamma. \end{array} \right.$$

$$(4.6) \qquad \int_\Gamma \left[\bar\varphi \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u) + \frac{\partial g}{\partial u}(\cdot,\bar y,\bar u) \right] (u - \bar u) \, dS(x) \ge 0 \quad \textit{for all } u_a \le u \le u_b.$$

Moreover, if $\bar\varphi_0$ *is the solution of* (3.7) *and* $\bar\varphi_j$ *is the solution of* (3.8), $1 \le j \le m$, *then*

$$(4.7) \qquad \bar\varphi = \bar\varphi_0 + \sum_{j=1}^m \bar\lambda_j \bar\varphi_j.$$

Proof. The control problem (P) can be written in the following way

$$\begin{cases} \text{Minimize } J(u) \\ u \in \mathcal{U}_{ad}, \\ G_j(u) = 0, \quad 1 \le j \le m_1, \\ G_j(u) \le 0, \quad m_1 + 1 \le j \le m, \end{cases}$$

where

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma) : u_a(x) \le u(x) \le u_b(x) \;\text{ a.e. } x \in \Gamma\}.$$

Then thanks to the regularity assumption on the constraints, we deduce (see, for instance, Bonnans and Casas [3] or Clarke [9]) the existence of Lagrange multipliers $\{\bar\lambda_j\}_{j=1}^m$ satisfying (4.3) and

$$\langle J'(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j'(\bar u), u - \bar u\rangle \ge 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{4.8}$$

Now defining $\bar\varphi$ by the expression (4.7), (4.5) follows from (3.7) and (3.9). Finally (4.6) follows from (3.5), (3.8), (4.7) and (4.8). ☐

5. Second Order Necessary Optimality Conditions. As in §4, $\bar u$ will denote a local solution of (P) and $\bar y$ the associated state. The goal of this section is to derive second order conditions for optimality satisfied by $\bar u$. As a first step we need the following Lemma.

LEMMA 5.1. *Let us assume that* (4.1) *holds and let* $h \in L^\infty(\Gamma)$ *such that* $G_j'(\bar u)h = 0$ *for every* $j \in I$, *with* $I \subset I_0$. *Then there exist a number* $\epsilon_h > 0$ *and* C^2*-functions* $\gamma_j : (-\epsilon_h, +\epsilon_h) \longrightarrow \mathbb{R}$, $j \in I$, *such that*

$$\begin{cases} G_j(u_t) = 0 \; j \in I, \; \textit{and } G_j(u_t) < 0 \; j \notin I_0, \quad \forall |t| \le \epsilon_h; \\ \gamma_j(0) = \gamma_j'(0) = 0, \; j \in I, \end{cases} \tag{5.1}$$

with

$$u_t = \bar u + th + \sum_{j\in I} \gamma_j(t) h_j,$$

$\{h_j\}_{j\in I}$ *given by* (4.1).

Proof. Let k be the cardinal number of I and let us define $\omega : \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ by

$$\omega(t, \rho) = (G_j(\bar u + th + \sum_{i\in I} \rho_i h_i))_{j\in I}.$$

Then ω is of class C^2 and

$$\frac{\partial \omega}{\partial t}(0,0) = (G_j'(\bar u)h)_{j\in I} = 0 \;\text{ and }\; \frac{\partial \omega}{\partial \rho}(0,0) = (G_j'(\bar u)h_i)_{i,j\in I} = \text{Identity}.$$

Therefore we can apply the implicit function theorem and deduce the existence of $\epsilon > 0$ and functions $\gamma_j : (-\epsilon, +\epsilon) \longrightarrow \mathbb{R}$ of class C^2, $j \in I$, such that

$$\omega(t, \gamma(t)) = \omega(0,0) = 0 \;\forall t \in (-\epsilon, +\epsilon) \;\text{ and }\; \gamma(0) = 0,$$

where $\gamma(t) = (\gamma_j(t))_{j\in I}$. Furthermore, by differentiation in the previous identity we get

$$\frac{\partial\omega}{\partial t}(0,0) + \frac{\partial\omega}{\partial\rho}(0,0)\gamma'(0) = 0 \Longrightarrow \gamma'(0) = 0.$$

Taking into account the continuity of γ and G_j and that $\gamma(0) = 0$, we deduce the existence of $\epsilon_h \le \epsilon$ such that (5.1) holds for every $t \in (-\epsilon_h, +\epsilon_h)$. □

Now we can prove the second order necessary optimality conditions.

THEOREM 5.2. *Let $\bar u$ be a local solution of* (P) *and $\bar y$, $\bar\varphi$, $\{\bar\varphi_j\}_{j=0}^m$ and $\{\bar\lambda_j\}_{j=1}^m$ given by Theorems 3.1, 3.2 and 4.1. Let us assume that the regularity hypothesis* (4.1) *holds. Suppose that $h \in L^\infty(\Gamma)$ is any direction satisfying*

$$\int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u) h\, dS(x) = 0 \text{ if } (j \le m_1) \text{ or } (j > m_1,\ F_j(\bar y) = 0 \text{ and } \bar\lambda_j > 0);$$

$$\int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u) h\, dS(x) \le 0 \quad \text{if } j > m_1,\ F_j(\bar y) = 0 \text{ and } \bar\lambda_j = 0; \tag{5.2}$$

$$h(x) = \begin{cases} 0 & \text{if } \bar u(x) = u_a(x) \text{ or } \bar u(x) = u_b(x) \text{ and } d(x) \ne 0;\\ \ge 0 & \text{if } \bar u(x) = u_a(x) \text{ and } d(x) = 0;\\ \le 0 & \text{if } \bar u(x) = u_b(x) \text{ and } d(x) = 0;\end{cases}$$

with

$$d(x) = \bar\varphi(x)\frac{\partial b}{\partial u}(x,\bar y(x),\bar u(x)) + \frac{\partial g}{\partial u}(x,\bar y(x),\bar u(x)).$$

Then the following inequality is satisfied

$$\int_\Omega \frac{\partial^2 f}{\partial y^2}(\cdot,\bar y) z_h^2\, dx + \int_\Gamma \left[\bar\varphi\frac{\partial^2 b}{\partial y^2}(\cdot,\bar y,\bar u) + \frac{\partial^2 g}{\partial y^2}(\cdot,\bar y,\bar u)\right] z_h^2\, dS(x) +$$

$$2\int_\Gamma \left[\bar\varphi\frac{\partial^2 b}{\partial y\partial u}(\cdot,\bar y,\bar u) + \frac{\partial^2 g}{\partial y\partial u}(\cdot,\bar y,\bar u)\right] z_h h\, dS(x) + \tag{5.3}$$

$$\int_\Gamma \left[\bar\varphi\frac{\partial^2 b}{\partial u^2}(\cdot,\bar y,\bar u) + \frac{\partial^2 g}{\partial u^2}(\cdot,\bar y,\bar u)\right] h^2\, dS(x) + \sum_{j=1}^m \bar\lambda_j F_j''(\bar y) z_h^2 \ge 0,$$

where $z_h \in C(\bar\Omega)\cap H^1(\Omega)$ is the solution of

$$\begin{cases} -\Delta z_h + z_h = 0 & \text{in } \Omega\\ \partial_\nu z_h = \dfrac{\partial b}{\partial y}(\cdot,\bar y,\bar u) z_h + \dfrac{\partial b}{\partial u}(\cdot,\bar y,\bar u) h & \text{on } \Gamma.\end{cases} \tag{5.4}$$

The conditions imposed on h can are motivated as follows: To show the theorem, we shall compare $\bar u$ with $u_t = \bar u + th + \sum_{j\in I}\gamma_j(t)h_j$. First of all, u_t has to satisfy the pointwise control constraints. If $\bar u(x)$ is on the boundary $\{u_a(x), u_b(x)\}$, then $h(x) \ge 0$ or $h(x) \le 0$ and $h_j(x) = 0, j \in I_0$, ensure feasibility. In the other cases, h should be arbitrary. However, second order conditions have to be imposed only at points, where the gradient of the Lagrange function with respect to u is vanishing, i.e., where $d(x) = 0$. This explains the pointwise conditions on h. The associated state has to satisfy also the state-constraints. Therefore, we require

$$G_j'(\bar u)h = \int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot, \bar y, \bar u) h dS(x) = 0$$

for the equality constraints and $G_j'(\bar u)h \le 0$ for the inequality constraints. The additional requirement $G_j'(\bar u)h = 0$ for the strongly active inequality constraints $(\bar\lambda_j > 0)$ is connected with our estimation in the proof.

Proof. Let us take $h \in L^\infty(\Gamma)$ satisfying (5.2) and the additional condition

$$h(x) = 0 \quad \text{if } u_a(x) < \bar u(x) < u_a(x) + \epsilon \;\text{ or }\; u_b(x) - \epsilon < \bar u(x) < u_b(x) \tag{5.5}$$

for some $\epsilon \in (0, \epsilon_{\bar u}]$, although h should allowed to be arbitrary in these points x. However, $\bar u(x)$ can be arbitrarily close to the boundary there. To overcome this problem, (5.5) is imposed in a first step. We introduce

$$I = \{1, \ldots, m_1\} \cup \{j : m_1 + 1 \le j \le m,\; G_j(\bar u) = 0 \text{ and } G_j'(\bar u)h = 0\}. \tag{5.6}$$

I includes all equality constraints, all strongly active inequality constraints and, depending on h, possibly some of the weakly active inequality constraints. From Theorem 3.2 and relations (5.2) we get

$$G_j'(\bar u)h = \int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot, \bar y, \bar u) h dS(x) = 0 \;\forall j \in I.$$

Then we are under the assumptions of Lemma 5.1. Let us set

$$u_t = \bar u + th + \sum_{j \in I} \gamma_j(t) h_j, \quad t \in (-\epsilon_h, \epsilon_h).$$

Now we define the function $\phi : [0, +\epsilon_h) \longrightarrow \mathbb{R}$ by

$$\phi(t) = J(u_t) + \sum_{j=1}^m \bar\lambda_j G_j(u_t).$$

From Lemma 5.1 we know that ϕ is of class C^2, $G_j(u_t) = 0$ if $j \in I$ and $G_j(u_t) < 0$ if $j \notin I_0$, provided that $t \in (-\epsilon_h, +\epsilon_h)$. From (5.2) we deduce that $G_j(\bar u) = 0$ and $G_j'(\bar u)h < 0$ for $j \in I_0 \setminus I$. Therefore we have that $G_j(u_t) < 0$ for every $j \notin I$ and $t \in (0, \epsilon_0)$, for some $\epsilon_0 > 0$ small. On the other hand, the assumptions on h along with the additional condition (5.5) and the fact that $\operatorname{supp} h_j \subset \Gamma_{\epsilon_{\bar u}}$ imply that $u_a(x) \le u_t(x) \le u_b(x)$ for $t \ge 0$ small enough. Consequently, by taking $\epsilon_0 > 0$ sufficiently small, we get that u_t is a feasible control for (P) for every $t \in [0, \epsilon_0)$. Now we know $G_j(u_t) = 0$ for $j \in I$ and $\bar\lambda_j = 0$ for $j \notin I_0$ (cf. (4.3)). According to (5.2) we require $G_j'(\bar u)h = 0$ for active inequalities with $\bar\lambda_j > 0$, hence these indices belong to I and $\bar\lambda_j = 0$ for $j \in I_0 \setminus I$. This leads to

$$\sum_{j=1}^m \bar\lambda_j G_j(u_t) = 0 \;\forall t \in [0, \epsilon_0).$$

This is the point, where we need $G_j'(\bar u)h = 0$ for the strongly active inequality constraints. Thus we have that ϕ has a local minimum at 0. Moreover

$$\phi'(0) = (J'(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j'(\bar u))(h + \sum_{j\in I}\gamma_j'(0)h_j) = (J'(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j'(\bar u))h =$$

$$\int_\Gamma \left[\bar\varphi \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u) + \frac{\partial g}{\partial u}(\cdot,\bar y,\bar u)\right] h\,dS(x) = \int_\Gamma d(x)h(x)dS(x) = 0.$$

Indeed (4.6) implies that $d(x) = 0$ for a.e. x with $u_a(x) < \bar u(x) < u_b(x)$. On the other hand, if $u_a(x) = \bar u(x)$ or $u_b(x) = \bar u(x)$, then either $h(x) = 0$ or $d(x) = 0$; see (5.2).

Since the first derivative of ϕ is zero we have the following second order necessary optimality condition

$$0 \le \phi''(0) = [J''(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j''(\bar u)]h^2 +$$

$$[J'(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j'(\bar u)](\sum_{i\in I}\gamma_i''(0)h_i) = [J''(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j''(\bar u)]h^2 +$$

$$\sum_{i\in I}\gamma_i''(0)\int_\Gamma d(x)h_i(x)dS(x) = [J''(\bar u) + \sum_{j=1}^m \bar\lambda_j G_j''(\bar u)]h^2.$$

The last integrals being zero because $\operatorname{supp} h_j \subset \Gamma_{\epsilon_{\bar u}}$ and $d(x) = 0$ on $\Gamma_{\epsilon_{\bar u}}$. To check the previous identities it is enough to apply Theorems 3.1 and 3.2, as well as (4.7) and the definition of d. Finally, (5.3) is an immediate consequence of the inequality obtained above, (3.6) and (3.10).

Now let us consider $h \in L^\infty(\Gamma)$ satisfying (5.2), but not (5.5). Then for every $\epsilon > 0$, we define

$$h_\epsilon = h\chi_{\Gamma_\epsilon} + \sum_{i\in I}\left(\int_{\Gamma\setminus\Gamma_\epsilon} \bar\varphi_i \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h\,dS(x)\right)h_i,$$

where χ_{Γ_ϵ} is the characteristic function of Γ_ϵ and I is given by (5.6). Thus for every $j \in I$, using (4.2) and taking $0 < \epsilon < \epsilon_{\bar u}$, we have

$$\int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h_\epsilon dS(x) = \int_{\Gamma_\epsilon} \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h\,dS(x) +$$

$$\sum_{i\in I}\left(\int_{\Gamma\setminus\Gamma_\epsilon} \bar\varphi_i \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h\,dS(\cdot)\right)\int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h_i dS(x) =$$

$$\int_\Gamma \bar\varphi_j \frac{\partial b}{\partial u}(\cdot,\bar y,\bar u)h\,dS(x) = 0.$$

In case of $j \in I_0 \setminus I$, the following inequality holds

$$\int_\Gamma \bar{\varphi}_j \frac{\partial b}{\partial u}(\cdot, \bar{y}, \bar{u}) h dS(x) < 0.$$

Then it is enough to take ϵ sufficiently small to get

$$\int_\Gamma \bar{\varphi}_j \frac{\partial b}{\partial u}(\cdot, \bar{y}, \bar{u}) h_\epsilon dS(x) < 0.$$

Thus, reminding that $\operatorname{supp} h_j \subset \Gamma_{\epsilon_{\bar{u}}}$, we have that h_ϵ satisfies the conditions (5.2) and (5.5), therefore (5.3) holds for each h_ϵ, $\epsilon > 0$ small enough.

Finally, it is clear that $h_\epsilon \to h$ in $L^p(\Gamma)$ for every $p < +\infty$, as $\epsilon \to 0$. Then (5.4) and Lemma 2.1 imply that $z_{h_\epsilon} \to z_h$ in $C(\bar{\Omega})$. Therefore, with the help of (2.2), (3.1) and (3.2), it is easy to pass to the limit in the second order optimality conditions satisfied for every h_ϵ and to conclude (5.3) □

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