SIAM J. CONTROL OPTIM.
Vol. 49, No. 4, pp. 1707–1736

OPTIMAL CONTROL OF THREE-DIMENSIONAL STATE-CONSTRAINED INDUCTION HEATING PROBLEMS WITH NONLOCAL RADIATION EFFECTS*

PIERRE-ÉTIENNE DRUET†, OLAF KLEIN†, JÜRGEN SPREKELS†, FREDI TRÖLTZSCH‡, AND IRWIN YOUSEPT‡

Abstract. The paper is concerned with a class of optimal heating problems in semiconductor single crystal growth processes. To model the heating process, time-harmonic Maxwell equations are considered in the system of the state. Due to the high temperatures characterizing crystal growth, it is necessary to include nonlocal radiation boundary conditions and a temperature-dependent heat conductivity in the description of the heat transfer process. The first goal of this paper is to prove existence and uniqueness of the state. The regularity analysis associated with the time-harmonic Maxwell equations is also studied. In the second part of the paper, existence and uniqueness of the solution of the corresponding linearized equation are shown. With this result at hand, the differentiability of the control-to-state operator is derived. Finally, based on the theoretical results, first order necessary optimality conditions for an associated optimal control problem are established.

Key words. optimal control, induction heating, time-harmonic Maxwell equations, quasi-linear elliptic equations, nonlocal radiation boundary conditions, control-state constraints, existence and regularity, linearized system, optimality conditions

AMS subject classifications. 49K20, 35Q61

DOI. 10.1137/090760544

1. Introduction.

In this paper, a class of optimal control problems arising in the context of crystal growth of semiconductor single crystals is studied. Heat transfer problems in crystal growth are mathematically challenging. Optimizing the temperature—the state of the system—is one of the important goals in crystal growth. Due to the high temperatures and the complex geometries involved, heat radiation has to be included in the model. This leads to a class of nonlinear and nonlocal boundary conditions (cf. [Voi01, Tii97, KPS04]). Such problems have not yet been widely studied from a mathematical point of view. Moreover, as inductive heating is used in crystal growth, Maxwell's equations have to be taken into account.

In a fairly simplified setting, the study of optimal control problems involving nonlocal boundary conditions was initiated in [MPT06]. Further contributions toward similar models including pointwise control and state constraints were made in [MY09a, MY09b]. In the aforementioned articles Maxwell's equations were not considered in the system of the state. In addition, the temperature dependence of the heat conductivity that becomes significant at high temperatures was not included in the model. The present paper is aimed at the analysis of a more realistic model: First, Maxwell's equations are included. Second, we consider a temperature-dependent heat conductivity, and so the temperature distribution is governed by a *quasi-linear elliptic equation.*

*Received by the editors June 1, 2009; accepted for publication (in revised form) April 26, 2011; published electronically August 4, 2011. This work was supported by DFG Research Center "Mathematics for Key Technologies" Matheon (FZT 86) in Berlin.
http://www.siam.org/journals/sicon/49-4/76054.html
†Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstraße 39, D-10117 Berlin, Germany (druet@wias-berlin.de, klein@wias-berlin.de, sprekels@wias-berlin.de).
‡Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany (troeltzsch@math.tu-berlin.de, yousept@math.tu-berlin.de).

Our focus is set on a quasi-static description of induction heating [KPS04]. The model is based on the assumption that all electromagnetic quantities are harmonic in time and given as the imaginary part of a *complex extension*, according to the usual ansatz

$$H(x,t) = \mathrm{Im}(H_{\mathrm{comp}}(x)\ \exp(i\,\omega\,t))\,, \quad E(x,t) = \mathrm{Im}(E_{\mathrm{comp}}(x)\ \exp(i\,\omega\,t)),$$

where H denotes the magnetic field intensity, and E the electric field strength. Similar representations are assumed for the remaining electromagnetic fields. Notice that H_{comp}, E_{comp} denote the complex-valued amplitude of the complex extension of the vector fields H, E with a fixed angular frequency $\omega > 0$. We assume that the period $2\,\pi/\omega$ of oscillation of the electromagnetic fields is much smaller than the time for heat diffusion. In this way, the Joule heat source density can be approximated by its averaged value over a period according to

$$f(x,t) \approx \frac{\omega}{2\,\pi} \int_0^{\frac{2\,\pi}{\omega}} f(\cdot,t)\,dt\,.$$

Thus, assuming a stationary temperature distribution in the furnace, we attain a time-independent description of the problem, which allows us to work with the complex amplitudes instead of the electromagnetic fields themselves.

Problem formulation. Let $\Omega \subseteq \mathbb{R}^3$ with $\Gamma := \partial\Omega$ be a bounded domain representing a crystal growth furnace. The global temperature distribution in Ω is governed by the following stationary heat equation with radiation boundary conditions:

$$\text{(1.1)} \qquad \begin{cases} -\mathrm{div}(\kappa(x,y)\,\nabla y) = \dfrac{1}{2\mathfrak{s}}|\mathrm{curl}\ H|^2 & \text{in } \Omega\,, \\ \left[-\kappa(x,y)\dfrac{\partial y}{\partial \vec{n}}\right] = G(\sigma\,|y|^3 y) & \text{on } \Sigma\,, \\ [y] = 0 & \text{on } \Sigma\,, \\ \kappa(x,y)\dfrac{\partial y}{\partial \vec{n}} + \varepsilon\sigma\,|y|^3 y = \varepsilon\sigma\,y_0^4 & \text{on } \Gamma\,, \end{cases}$$

where y denotes the absolute temperature and y_0 is a given external temperature. Further, σ denotes the Boltzmann radiation constant, ε the emissivity, κ the thermal conductivity, and $\mathfrak{s}$ the electrical conductivity. The jump of a quantity across boundaries is denoted by $[\cdot]$, and $\vec{n}$ is the outward unit normal to the corresponding surface.

The surface Σ and the nonlocal radiation operator G are related to the modeling of the radiative heat transfer. Heat radiation is incoming and outgoing at the surface of each body located next to a transparent medium. To describe this phenomenon, we assume that a part of the region $\Omega_{\mathrm{transparent}} \subset \Omega$ is occupied by transparent materials. We set $\Sigma := \partial\Omega_{\mathrm{transparent}}$. A schematic geometrical example is given in Figure 1.1. The operator G in (1.1) is a linear and continuous operator (see, e.g., [LT01, KPS04] for in-depth discussions on G and its physical background). For the convenience of the reader, we recall the definition of G and its essential properties in Appendix B.

We assume that the domain Ω is convex. In this case, the *local* Stefan–Boltzmann radiation condition assumed on the outer boundary Γ is an exact model of the real situation. It should be emphasized that the convexity assumption on Ω is related to modeling issues, and *not* to the regularity results on the state and the magnetic

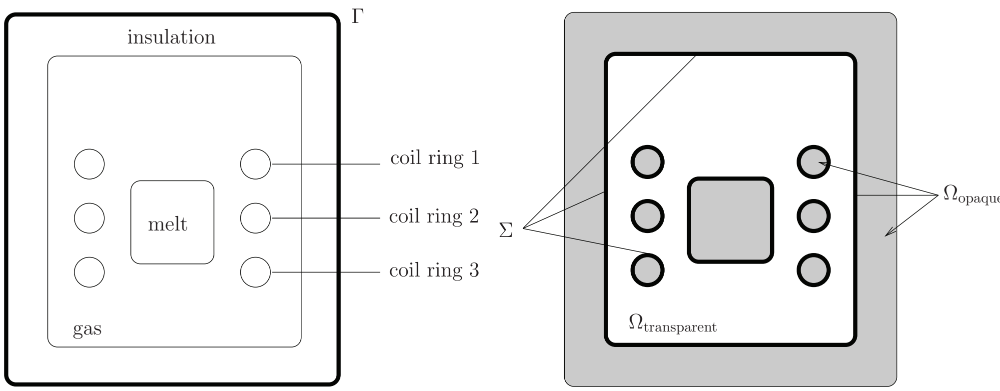

FIG. 1.1. *Two-dimensional schematic cut of the domain Ω. Left: The furnace components and the outer boundary Γ (thick black line). Right: Description of Ω from the point of view of heat radiation, with the transparent cavity $\Omega_{transparent}$ (white), its boundary Σ (thick black line), and the opaque materials Ω_{opaque} (gray).*

field. If Ω were nonconvex, then a *nonlocal* Stefan–Boltzmann condition would be more adequate to model the radiation at the outer boundary Γ. This is due to the fact that, in the nonconvex case, different parts of Γ can interact with radiation. In particular, this is important for a high boundary temperature. Details on the modeling of such situations can be found, e.g., in [KPS04] or in section 2.3 of [KP05]. The mathematical analysis for the model including nonlocal radiation boundary condition on Γ is very similar to our case, where nonlocal radiation is only considered on Σ.

It is not to be expected that the electromagnetic fields generated to heat the region Ω will be confined to it. We therefore introduce a bounded "hold all domain" $O \subset \mathbb{R}^3$, which contains Ω and is typically much larger,[1] to represent the region in which the electromagnetic fields are acting. To adequately describe the electromagnetic phenomena taking place in the larger region O, we denote by $O_c \subseteq O$ the region occupied by electrically conducting materials. We set $O_{nc} := O \setminus O_c$ for the nonconductors.

The complex-valued magnetic field intensity H appearing in (1.1) is given by the solution to a time-harmonic Maxwell system posed in O:

$$(1.2)\qquad \begin{cases} i\,\omega\,B + \operatorname{curl}\,E = 0 & \text{in } O,\\ \operatorname{curl}\,H = J & \text{in } O,\\ J = \mathfrak{s}E + \chi_{O_{c_0}}\,j_g & \text{in } O_c,\\ \operatorname{div}\,D = 0 & \text{in } O_{nc},\\ B = \mu H, \quad D = \mathfrak{e}E, \quad \operatorname{div}\,B = 0 & \text{in } O,\\ B\cdot\vec{n} = 0, \quad E\times\vec{n} = 0 & \text{on } \partial O,\\ [H\times\vec{n}]_{i,j} = 0, \quad [B\cdot\vec{n}]_{i,j} = 0, \quad [E\times\vec{n}]_{i,j} = 0 & \text{on } \partial O_i\cap\partial O_j\,. \end{cases}$$

Here B denotes the magnetic induction, D the electric displacement, and J the current density. All of these three-dimensional vector functions are complex valued. The real-valued functions $\mathfrak{e}$, μ, and $\mathfrak{s}$ denote the electric permittivity, the magnetic permeability, and the electrical conductivity, respectively. We assume the decomposition

[1]In fact, the electromagnetic fields extend to the entire space, but at a certain distance from the region of interest, they become negligible. The auxiliary domain O has to reflect this property.

$\overline{O} := \bigcup_{i=0}^{m} \overline{O_i}$ with disjoint domains O_i representing different *material subdomains* filling the region O.[2] As before, $[\cdot]_{i,j}$ denotes the jump of a quantity across the interface $\partial O_i \cap \partial O_j$, $i, j = 0, \dots, m$, $i \neq j$.

The three-dimensional vector function j_g is an applied current generated by some voltages acting in a set $O_{c_0} \subseteq O_c$ (see (M3) below). Typically O_{c_0} represents an induction coil or a system of coils (see the examples in [KLDP+]).

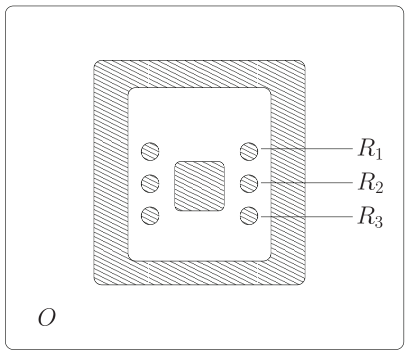

FIG. 1.2. *Two-dimensional cut of the Maxwell "hold all domain" O, with the electrical conductors O_c (gray), the nonconductors O_{nc} (white), and the coil rings $R_1, \dots, R_3$ indicated by arrows.*

Optimal control problem. It is not realistic to assume that we can control the density of the current at each point of space. Therefore, we consider the following assumptions:

(M1) We make the customary idealization that the coil system O_{c_0} can be represented as $O_{c_0} = \bigcup_{j=1}^{n} R_j$ $(n \geq 1)$, where $R_1, \dots, R_n$, are disjoint bounded domains a positive distance from each other. For every $j = 1, \dots, n$, R_j is assumed to be a ring (see Figure 1.2).[3] Here each ring represents one coil.[4]

(M2) The controlled voltages $u_j \in \mathbb{R}^+$ in each coil ring $R_j \subseteq O_{c_0}$ $(j = 1, \dots, n)$ can be maintained constant. In other words, coupling effects on the voltages u_j can be neglected. Notice that we do not directly control the total current density J in (1.2), since J depends also on the term $\mathfrak{s}E$ that is added to j_g in O_c. This term models coupling effects on the current.

(M3) The applied current j_g in each induction coil R_j $(j = 1, \dots, n)$ is obtained from Ohm's law by the associated electrical resistance as the voltage u_j is applied.

Due to the hypotheses (M2) and (M3), the current j_g is given by the ansatz $j_g = \sum_{j=1}^{n} u_j\, v_j$, where $u \in \mathbb{R}^n$, and $\{v_1, \dots, v_n\}$ is a given system of vector fields such

[2]This means that for each $i = 0, \dots, m$, the set O_i is a domain (a connected open set), in which the material properties are uniformly continuous. Observe that in this way, several domains O_i can consist of the same material, provided that they are a positive distance from each other.

[3]For each $j \in \{1, \dots, n\}$, there exist numbers $r_{j,1} > r_{j,2} > 0$ and a fixed vector $z_j \in \mathbb{R}^3$, such that the set R_j is the torus

$$R_j = \left\{ z_j + \begin{pmatrix} (r_{j,1} + s\, \cos\phi)\, \cos\theta \\ (r_{j,1} + s\, \cos\phi)\, \sin\theta \\ s\, \sin\phi \end{pmatrix} : s \in [0,\, r_{j,2}],\ \phi,\, \theta \in]0,\, 2\pi] \right\}.$$

[4]This is different from [KPS04], since therein each ring represents just one loop of the considered single induction coil.

that $v_j : O \to \mathbb{R}^3$ and $v_j = 0$ on $O \setminus R_j$. Notice that since the density j_g in the conductor O_{c_0} represents a current, we have to make the consistency assumption

$$\text{div}\, j_g = 0 \;\text{ in } O_{c_0}, \qquad j_g \cdot \vec{n} = 0 \text{ on } \partial O_{c_0}. \tag{1.3}$$

Thus, the vector fields $\{v_j\}_{j=1,\dots,n}$ in turn must satisfy

$$\text{div}\, v_j = 0 \text{ in } R_j, \qquad v_j \cdot \vec{n} = 0 \text{ on } \partial R_j \qquad \text{for } j = 1, \dots, n.$$

Under the simplifying assumption (M1), and assuming a constant electrical conductivity in R_j, we can set

$$v_j = \mathfrak{s} \begin{pmatrix} -x_2/(x_1^2 + x_2^2) \\ x_1/(x_1^2 + x_2^2) \\ 0 \end{pmatrix}. \tag{1.4}$$

For more general forms of the inductor O_{c_0}, we construct a particular system $\{v_j\}_{j=1,\dots,n}$ in Remark 2.2 below.

Given fixed data $z \in L^2(\Omega)^3$, $H_d \in L^2(O;\mathbb{C})^3$, $\rho \geq 0$, and $\beta > 0$, we focus on the following optimal control problem:

$$\text{(P)} \qquad \text{minimize } J(u, H, y) := \frac{1}{2}\int_\Omega |\nabla y - z|^2 + \frac{\rho}{2}\int_O |H - H_d|^2 + \frac{\beta}{2}|u|^2 ,$$

where (H, y, u) solves (1.1)–(1.2). In addition, the optimization problem (P) is subject to the following state and control constraints:

$$\begin{aligned} y_a(x) \leq y(x) &\leq y_b(x) && \text{for a.a. } x \in \Omega, \\ u_a \leq u_j &\leq u_b && \text{for all } j \in \{1, \dots, n\}. \end{aligned} \tag{1.5}$$

Notice that including the state constraints (1.5) into the model is necessary. They are assigned to avoid melting of the apparatus and to keep the crystallization process within a desirable temperature range. Let us also remark that H_d is a desired magnetic field which is included in the objective functional of (P) for mathematical generalization. The term $\int_O |H - H_d|^2$ can be dropped by setting $\rho = 0$. However, it can also be important in certain applications such as the control of MHD (cf. Griesse and Kunisch [GK06]).

The analysis of the control problem (P) turns out to be delicate in some aspects. First, we are confronted here with a state equation of quasi-linear type with source terms generated by the Maxwell equations. Second, the pointwise state constraints in the set of explicit constraints (1.5) considerably complicate the analysis. In addition, the nonlocal radiation operator G and the heat conductivity κ in the state equation (1.1) are not monotone with respect to y so that the existence and uniqueness theory based on monotone operators is not applicable.

The first contribution of the present paper is the regularity analysis for the state of the system (1.1)–(1.2). The regularity result relies on recent advances in regularity theory [ERS07, HDMR08] and may interest the reader in its own right. The second part of the paper is concerned with the linearized equation of (1.1)–(1.2). Our main goal is to prove the existence and uniqueness result of the corresponding linearized system which leads mainly to the differentiability of the control-to-state operator associated with (1.1)–(1.2). To the best of our knowledge, no study on these topics has been carried out so far. The optimization theory for (P), on the other hand, is devised based on the mentioned theoretical results and provides a basis for our forthcoming paper on the numerical computation of (P).

The rest of the paper is organized as follows: We begin by introducing our main assumptions and notation. In the section 3, we conduct a study concerning existence and uniqueness of the weak solution to (1.1)–(1.2). Higher regularity of the solution will also be discussed. Section 4 is devoted to the linearized equation of (1.1)–(1.2). Based on the theoretical results in sections 3 and 4, we derive the first order optimality conditions for (P) in section 5.

2. General assumptions and notation.

2.1. Notation. We first introduce some spaces that will be needed for the analysis of the state equation. For $1 < q < \infty$, we denote by $q' := q/(q-1)$ the conjugated exponent to q. We define

$$L^q_{\text{curl}}(O) := \left\{ \psi \in [L^q(O)]^3 \,\middle|\, \text{curl}\,\psi \in [L^q(O)]^3 \right\},$$

$$L^q_{\text{div}}(O) := \left\{ \psi \in [L^q(O)]^3 \,\middle|\, \text{div}\,\psi \in L^q(O) \right\},$$

where the differential operators curl and div are intended in the weak (distributional) sense. The spaces $L^q_{\text{curl}}(O)$ and $L^q_{\text{div}}(O)$ are Banach spaces with respect to the graph norm. The linear operator $\gamma_n : L^q_{\text{div}}(O) \to W^{1,q'}(O)^*$, given by

$$\langle \gamma_n(\psi),\, \phi \rangle := \int_O \text{div}\,\psi\,\phi + \int_O \psi \cdot \nabla\phi \quad \forall\, \phi \in W^{1,q'}(O), \tag{2.1}$$

is a generalization of the trace $\psi \cdot \vec{n}$ ($\vec{n}$ = outward unit normal to ∂O), which is well defined for $\psi \in L^q_{\text{div}}(O)$. Analogously, the linear operator $\gamma_t : L^q_{\text{curl}}(O) \to L^{q'}_{\text{curl}}(O)^*$, given by

$$\langle \gamma_t(\psi),\, \phi \rangle := \int_O \psi \cdot \text{curl}\,\phi - \int_O \phi \cdot \text{curl}\,\psi \quad \forall\, \phi \in L^{q'}_{\text{curl}}(O), \tag{2.2}$$

generalizes the trace $-\psi \times \vec{n}$ for $\psi \in L^q_{\text{curl}}(O)$.

In order to represent current vectors, we need the space

$$\mathcal{H}^q(O) := \left\{ H \in L^q_{\text{curl}}(O) \,\middle|\, \text{curl}\,H = 0 \text{ in } O_{nc} \right\}, \tag{2.3}$$

and we set $\mathcal{H}(O) := \mathcal{H}^2(O)$.

The spaces of complex-valued vector fields associated with $L^q_{\text{curl}}(O)$ and $\mathcal{H}^q(O)$ are denoted by $L^q_{\text{curl}}(O; \mathbb{C}^3)$ and $\mathcal{H}^q(O; \mathbb{C}^3)$, respectively. The linear constraints characterizing these spaces are then intended to hold for both real and imaginary parts of the vector field.

The inner product on the Hilbert space $L^2_{\text{curl}}(O; \mathbb{C}^3)$ is given by

$$(H_1\,,\, H_2)_{L^2_{\text{curl}}(O)} := \int_O \left(\text{curl}\, H_1 \cdot \text{curl}\,\overline{H_2} + H_1 \cdot \overline{H_2} \right), \tag{2.4}$$

where $\overline{a}$ denotes the complex conjugate of $a \in \mathbb{C}^3$.

2.2. Main assumptions on the data. The data of the problem are the geometry, the coefficients κ, ε, μ, $\mathfrak{c}$, $\mathfrak{s}$, the vector fields v_j, and the external temperature y_0. We summarize the corresponding assumptions in the following.

(A1) Assumption on the geometry. In order to describe complex electromagnetic and thermodynamical phenomena, we have to account for the multimaterial structure of the domains O and Ω: A decomposition $\overline{O} := \bigcup_{i=0}^m \overline{O_i}$ is assumed with disjoint open sets O_i that represent the different material subdomains (see footnote 2) that fill the "hold all" region O. We define $\Omega_i := O_i \cap \Omega$ such that $\overline{\Omega} := \bigcup_{i=0}^m \overline{\Omega_i}$. Here each Ω_i represents a different material subdomain that fills the region Ω (see Figures 1.1 and 1.2).

For simplicity, we assume that there is only one connected transparent cavity in Ω which is denoted by Ω_0. Therefore, the boundary Σ of the transparent materials is simply given by $\Sigma := \partial\Omega_0$. The *enclosure property* has to be satisfied:

$$\text{Every } x \in \Sigma \text{ is an interior point of } \Omega\,. \tag{2.5}$$

In other words, the cavity Ω_0 is enclosed by the remaining (opaque) materials ($\overline{\Omega_{\text{opaque}}} := \bigcup_{i=1}^m \overline{\Omega_i}$). We further assume that the domain O is simply connected and Lipschitz. In order to obtain regular magnetic fields, the main geometrical restriction considered throughout the paper is the following:

$$\partial O_i \in \mathcal{C}^1 \quad \text{for } i = 0, \dots, m\,, \quad \partial O \in \mathcal{C}^{0,1}\,. \tag{2.6}$$

From (2.6), it also follows that

$$\partial\Omega_i \in \mathcal{C}^1 \quad \text{for } i = 0, \dots, m\,, \tag{2.7}$$

since $\Omega_i = O_i \cap \Omega$. The assumption (2.7) is important in order to obtain a temperature field in $W^{1,q}(\Omega)$ for some $q > 3$. Finally, for simplicity (cf. Remark 2.1), we make the assumption that each conductor is isolated:

$$\text{dist}(O_i, O_j) > 0 \quad \forall O_i, O_j \subseteq O_c, \text{ with } j \neq i\,. \tag{2.8}$$

(A2) Assumption on the source fields and coefficients. As mentioned in the introduction, the applied current j_g is given by the ansatz

$$j_g = \sum_{j=1}^n u_j\, v_j. \tag{2.9}$$

Further we assume that there exists a real number $\bar{q} > 3$ such that $v_j \in [L^{\bar{q}}(O)]^3$ and

$$v_j = 0 \text{ in } O \setminus R_j, \qquad \text{div } v_j = 0 \text{ in } R_j, \quad v_j \cdot \vec{n} = 0 \text{ on } \partial R_j \quad \text{for } j = 1, \dots, n\,. \tag{2.10}$$

Throughout the paper, we assume that there exist positive constants $\mathfrak{s}_l$, $\mathfrak{s}_u$, μ_l, μ_u such that

$$0 < \mathfrak{s}_l \leq \mathfrak{s} \leq \mathfrak{s}_u < +\infty \quad \text{a.e. in } O_c\,, \quad 0 < \mu_l \leq \mu \leq \mu_u < +\infty \quad \text{a.e. in } O\,. \tag{2.11}$$

Note that, since O_{nc} is nonconducting, we have $\mathfrak{s} = 0$ in O_{nc}. For the boundary data y_0, we assume that

$$y_0 \in L^\infty(\Gamma), \quad \operatorname*{ess\,inf}_{\Gamma} y_0 > 0\,. \tag{2.12}$$

We recall that the surface $\Sigma \cup \Gamma$ is an interface between transparent and opaque material: Σ is the boundary of a transparent cavity located in the furnace Ω, whereas

Γ denotes the boundary of Ω, which is surrounded by air in the "hold all" region O. Thus, heat radiation has to be modeled at the surface $\Sigma\cup\Gamma$, and we have to introduce the emissivity parameter on $\Sigma\cup\Gamma$. The emissivity denoted by ε is a function of the position. We assume that $\varepsilon:\ \Sigma\cup\Gamma\longrightarrow\mathbb{R}$ is measurable and satisfies

$$\exists\varepsilon_l\in\mathbb{R} \text{ such that } 0<\varepsilon_l\le\varepsilon_i\le 1 \ \text{ on } \partial\Omega_i\cap\Sigma \qquad \text{for } i=1,\dots,m\,. \tag{2.13}$$

The above condition ensures in particular that the operator G is well defined (see Appendix B).

(A3) Assumption on the continuity. There exist continuous functions $\varepsilon_i\in\mathcal{C}(\partial\Omega_i\cap\Sigma)$ such that $\varepsilon=\varepsilon_i$ in $\partial\Omega_i\cap\Sigma$ for all $i=1,\dots,m$. In addition, we require the continuity of the coefficients in each material,

$$\mathfrak{s}_i,\ \mu_i\in\mathcal{C}(\overline{O_i})\,, \tag{2.14}$$

where $\mathfrak{s}_i$, μ_i are the restrictions of $\mathfrak{s}$, μ to the set O_i. We now formulate assumptions for the heat conductivity: Let $\kappa:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be measurable and satisfy

$$\begin{cases}\kappa=\kappa_i \ \text{ in } \Omega_i\times\mathbb{R},\ \text{with continuous functions } \kappa_i:\overline{\Omega}_i\times\mathbb{R}\to\mathbb{R}\ \forall\, i=0,\dots,m,\\ \exists\kappa_l,\kappa_u\in\mathbb{R} \ \text{ with } 0<\kappa_l<\kappa_u \ \text{ such that } \kappa_l\le\kappa(x,y)\le\kappa_u \text{ for a.a. } x\in\overline{\Omega} \text{ and } \forall\, y\in\mathbb{R},\\ \forall\, M>0,\ \exists\, C_M>0 \text{ such that } |\kappa_i(x,y_1)-\kappa_i(x,y_2)|\le C_M\,|y_1-y_2|,\\ \forall\, y_1,y_2\in[-M,M],\forall\, i\in\{0,\dots,m\},\ \text{and for a.a. } x\in\Omega_i.\end{cases} \tag{2.15}$$

Remark 2.1. The geometrical assumption (2.6) is too restrictive when dealing with realistic geometries in industrial crystal growth. As a matter of fact, jumps of the material properties are allowed only between at most two materials. In order to deal with more general junctions, we rely on continuous approximations of the material parameters. It is therefore particularly important to consider space-dependent coefficients $\mathfrak{s}$, κ, μ (cf. the assumptions (2.14), (2.15)). The simplifying assumption (2.8) is also to be understood in this context: we could allow for the junction of two conductors, provided that one of them is embedded in the second and has a $\mathcal{C}^1$ boundary. This would, however, increase the technicality without being an essential progress.

Remark 2.2. If R_j is an arbitrary 2-connected Lipschitz domain, and the electrical conductivity $\mathfrak{s}$ is constant in R_j, the field v_j can also be computed in advance and satisfy (2.10). Denote by $P\subset R_j$ a hypersurface that cuts the ring R_j transversally, such that the domain $\tilde{R}_j:=R_j\setminus P$ is simply connected.[5] Under the assumptions

[5]To help the representation, let us note that if R_j is the torus characterized by the radii $r_{j,1}>r_{j_2}$, then the surface P is any of the disks

$$P=z_j+\left\{\begin{pmatrix}(r_{j,1}+s\,\cos\phi_0)\,\cos\theta\\ (r_{j,1}+s\,\cos\phi_0)\,\sin\theta\\ s\,\sin\phi\end{pmatrix}:s\in[0,\,r_{j,2}],\ \theta\in]0,\,2\pi]\right\},$$

with $\phi_0\in]0,\,2\pi]$ and $z_j\in\mathbb{R}^3$.

(M2) and (M3), we have $v_j = \mathfrak{s}\,\nabla\tilde{p}_j$ in $\tilde{R}_j$, where $\tilde{p}_j$ is the solution to the problem

$$\left\{\begin{aligned} \triangle\tilde{p}_j &= 0 \quad \text{in } \tilde{R}_j,\\ \tfrac{\partial\tilde{p}_j}{\partial\vec{n}} &= 0 \quad \text{on } \partial\tilde{R}_j\setminus P,\\ \left[\tfrac{\partial\tilde{p}_j}{\partial\vec{n}}\right] &= 0 \quad \text{on } P,\\ [\tilde{p}_j] &= 1 \quad \text{on } P\,, \end{aligned}\right. \tag{2.16}$$

where $[\cdot]$ denotes the jump of a quantity across the surface P. It is well known (cf. [FT78]) that (2.16) admits a unique solution $\tilde{p}_j \in W^{1,2}(\tilde{R}_j)$, and $v_j = \mathfrak{s}\,\nabla\tilde{p}_j$ satisfies

$$\text{div } v_j = 0 \text{ in } R_j, \qquad v_j\cdot\vec{n} = 0 \text{ on } \partial R_j.$$

Furthermore, $v_j = \mathfrak{s}\,\nabla\tilde{p}_j$ belongs to $[L^{\bar{q}}(\tilde{R}_j)]^3$ for some $\bar{q} > 3$ (see [Mon03, Theorem 3.50]).

3. State equation. Let $\mathfrak{r}$ refer to the function of electric resistivity in the conducting material O_c, i.e., $\mathfrak{r} := 1/\mathfrak{s}$ in O_c. In order to improve the readability, we write all integrals related to (1.2) as integrals over the whole domain O. For this purpose, we define

$$r := \begin{cases}\mathfrak{r} & \text{on } O_c,\\ 1 & \text{on } O_{nc},\end{cases} \qquad r_l := \mathfrak{s}_u^{-1}, \quad r_u := \mathfrak{s}_l^{-1}\,, \tag{3.1}$$

with $\mathfrak{s}_l$, $\mathfrak{s}_u$ from (2.11). In the upcoming definition, we introduce the notion of weak solution to the system (1.1)–(1.2).

DEFINITION 3.1 (weak solution to (1.1)–(1.2)). *Let $3 < q < \infty$ and let q' be the conjugate index of q.*

(i) *We introduce an operator $A_q : \mathcal{H}^q(O;\mathbb{C}^3)\times W^{1,q}(\Omega) \to \mathcal{H}^{q'}(O;\mathbb{C}^3)^*\times W^{1,q'}(\Omega)^*$, defined by*

$$\begin{aligned}\langle A_q(H,y),(\psi,\xi)\rangle &:= i\int_O \omega\,\mu\,H\cdot\overline{\psi} + \int_O r \text{ curl } H\cdot\overline{\text{curl } \psi}\\ &\quad+\int_\Omega \kappa(\cdot,y)\nabla y\cdot\nabla\xi + \int_\Sigma G(\sigma|y|^3y)\xi + \int_\Gamma \varepsilon\sigma|y|^3y\xi - \frac{1}{2}\int_\Omega r|\text{curl } H|^2\,\xi\end{aligned}$$

for all $(\psi,\xi)\in\mathcal{H}^{q'}(O;\mathbb{C}^3)\times W^{1,q'}(\Omega)$.

(ii) *We further introduce an operator $E_q : \mathbb{R}^n \to \mathcal{H}^{q'}(O;\mathbb{C}^3)^*\times W^{1,q'}(\Omega)^*$, defined by*

$$\langle E_q u,(\psi,\xi)\rangle := \int_{O_{c_0}} r\sum_{j=1}^n u_j v_j\cdot\overline{\text{curl } \psi} + \int_\Gamma \varepsilon\sigma\,y_0^4\,\xi \quad \forall\,(\psi,\xi)\in\mathcal{H}^{q'}(O;\mathbb{C}^3)\times W^{1,q'}(\Omega).$$

(iii) *For given $u\in\mathbb{R}^n$, we call a pair $(H,y)\in\mathcal{H}^q(O;\mathbb{C}^3)\times W^{1,q}(\Omega)$ the* weak solution *to* (1.1)–(1.2) *if*

$$A_q(H,y) = E_q u \quad \text{in } \mathcal{H}^{q'}(O;\mathbb{C}^3)^*\times W^{1,q'}(\Omega)^*. \tag{3.2}$$

Remark 3.2.

(1) Let $(H,\, y)$ be a weak solution in the sense of Definition 3.1. For every $\phi \in W^{1,q'}(O;\mathbb{C}^3)$, the vector field $\nabla\phi$ belongs to $\mathcal{H}^{q'}(O;\mathbb{C}^3)$. Taking in (3.2) the pair $(\nabla\phi, 0)$ as a test function, and observing that curl $\nabla\phi = 0$, we have

$$i \int_O \mu\,\omega\, H \cdot \overline{\nabla\phi} = 0\,.$$

Therefore, every weak solution $H \in \mathcal{H}^q(O;\mathbb{C}^3)$ in the sense of Definition 3.1 satisfies the conditions $\operatorname{div}(\mu\, H) = 0$ and $\gamma_n(\mu\, H) = 0$ in the weak sense.

(2) Let $(H,\, y)$ be a weak solution in the sense of Definition 3.1. The continuous embedding $W^{1,q'}(\Omega) \hookrightarrow L^s(\Omega)$ is valid for all $1 \le s \le \frac{3q'}{3-q'}$. Since $\frac{q}{q-2} \le \frac{3q'}{3-q'}$, we can apply Hölder's inequality and Sobolev's embedding theorem to verify that

$$\begin{aligned}\left|\int_\Omega r\,|\text{curl } H|^2 \xi\right| &\le r_u\,\|\text{curl } H\|^2_{[L^q(\Omega;\mathbb{C})]^3}\,\|\xi\|_{L^{q/(q-2)}(\Omega)}\\ &\le r_u\, c_0\,\|\text{curl } H\|^2_{[L^q(O;\mathbb{C})]^3}\,\|\xi\|_{W^{1,q'}(\Omega)}\,.\end{aligned}$$

Analogously, since the embedding $W^{1,q}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ is continuous, we can verify that under the assumptions (2.11), (2.15), (2.13), and (3.1), the operator A_q is well defined. Due to the validity of (2.10) with $\bar q > 3$ and Hölder's inequality, the operator E_q is well defined for $q \le \bar q$.

(3) Let us also note that the embedding $W^{1,q}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ particularly implies the continuity of the state across the interface Σ.

THEOREM 3.3. *Let* (A1)–(A3) *be satisfied. Then there exists* $3 < q \le \bar q$ *such that, for all* $u \in \mathbb{R}^n$*, the problem* (1.1)–(1.2) *possesses a unique weak solution* $(H,\, y) \in \mathcal{H}^q(O;\mathbb{C}^3) \times W^{1,q}(\Omega)$ *satisfying* $y \ge \operatorname{ess\,inf}_\Gamma y_0 > 0$*. In particular, thanks to the embedding* $W^{1,q}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$*, the solution* y *is continuous.*

We split the proof of Theorem 3.3 into the following two lemmas.

LEMMA 3.4. *Let* $O \subset \mathbb{R}^3$ *satisfy* (A1)*. Assume further that the functions* $\mathfrak{s}_i,\, \mu_i \in \mathcal{C}(\overline{O_i})$ *satisfy* (2.11)*. Let* $j_g \in [L^{\bar q}(O_{c_0};\mathbb{C})]^3$ *with* $\bar q > 3$*. Then there exist a* $3 < q \le \bar q$ *and a unique* $H \in \mathcal{H}^q(O;\mathbb{C}^3)$ *such that* $\operatorname{div}(\mu\, H) = 0$*,* $\gamma_n(\mu\, H) = 0$ *in the weak sense, and*

$$i\,\omega \int_O \mu\, H \cdot \bar\psi + \int_O r\,\text{curl } H \cdot \text{curl } \bar\psi = \int_O r\, j_g \cdot \text{curl } \bar\psi \tag{3.3}$$

for all $\psi \in \mathcal{H}^{q'}(O;\mathbb{C}^3)$*.*

Proof. Introducing the abbreviations $H^{(1)} := \operatorname{Re} H$ and $H^{(2)} := \operatorname{Im} H$, we first observe that (3.3) is equivalent to the validity of the system

$$-\omega \int_O \mu\, H^{(2)} \cdot \psi + \int_O r\,\text{curl } H^{(1)} \cdot \text{curl } \psi = \int_{O_{c_0}} r\ \operatorname{Re} j_g \cdot \text{curl } \psi\,, \tag{3.4}$$

$$\omega \int_O \mu\, H^{(1)} \cdot \psi + \int_O r\,\text{curl } H^{(2)} \cdot \text{curl } \psi = \int_{O_{c_0}} r\ \operatorname{Im} j_g \cdot \text{curl } \psi \tag{3.5}$$

for all real-valued $\psi \in \mathcal{H}(O)$. To obtain (3.4) and (3.5), we simply have inserted the field $\psi + 0\, i$, $\psi \in \mathcal{H}(O)$, in the relation (3.3), and we have then equated the real and imaginary parts, respectively.

We consider the linear subspace of $\mathcal{H}(O)$

$$\mathcal{H}_\mu(O) := \left\{\psi \in \mathcal{H}(O) \,\middle|\, \operatorname{div}(\mu\,\psi) = 0,\, \gamma_n(\mu\,\psi) = 0\right\},$$

where the constraints on div and γ_n are intended in the weak sense of these operators, explained in section 2. The space $\mathcal{H}_\mu(O)$ is a Hilbert space if endowed with the inner product (2.4). Moreover, there exists a constant $C > 0$ such that for all $\psi \in \mathcal{H}_\mu(O)$ it holds that

$$\|\psi\|_{[L^2(O)]^3} \leq C\,\|\text{curl }\psi\|_{[L^2(O)]^3}\,,$$

so that the space $\mathcal{H}_\mu(O)$ is equivalently normed by the expression $\|\text{curl }\cdot\|_{[L^2(O)]^3}$. This fact is widely known, and a proof is given, for example, in [Dru07].

With the standard isomorphism $\mathbb{C} \cong \mathbb{R}^2$, we can identify $H \in \mathcal{H}_\mu(O;\mathbb{C}^3)$ with the pair $(H^{(1)}, H^{(2)}) \in \mathcal{H}_\mu(O) \times \mathcal{H}_\mu(O)$. On the Hilbert space $\mathcal{H}_\mu(O) \times \mathcal{H}_\mu(O)$, we introduce the bilinear form

$$\begin{aligned} a(H,\,\phi) := &-\omega\int_O \mu\,H^{(2)}\cdot\phi^{(1)} + \int_O r\,\text{curl }H^{(1)}\cdot\text{curl }\phi^{(1)} \\ &+\omega\int_O \mu\,H^{(1)}\cdot\phi^{(2)} + \int_O r\,\text{curl }H^{(2)}\cdot\text{curl }\phi^{(2)}\,, \end{aligned} \tag{3.6}$$

which is continuous and bounded in view of (2.11). On the other hand, the bilinear form a satisfies

$$a(H,\,H) = \int_O r\,(|\text{curl }H^{(1)}|^2 + |\text{curl }H^{(2)}|^2) \geq r_l\,\|H\|^2_{\mathcal{H}_\mu(O)\times\mathcal{H}_\mu(O)}\,.$$

The functional

$$F(\phi) := \int_O r\ \text{Re}\,j_g\cdot\text{curl }\phi^{(1)} + \int_O r\ \text{Im}\,j_g\cdot\text{curl }\phi^{(2)}$$

is clearly a well-defined element of $[\mathcal{H}_\mu(O)\times\mathcal{H}_\mu(O)]^*$, since $j_g \in [L^2(O_{c_0})]^3$. The Lax–Milgram lemma gives the existence of a unique $H \in \mathcal{H}_\mu(O)\times\mathcal{H}_\mu(O)$ such that $a(H,\,\phi) = F(\phi)$ for all $\phi \in \mathcal{H}_\mu(O)\times\mathcal{H}_\mu(O)$.

Taking in (3.6) $\phi^{(1)} = \psi$, with $\psi \in \mathcal{H}_\mu(O)$ arbitrary, and $\phi^{(2)} = 0$, we obtain (3.4). Taking $\phi^{(1)} = 0$, $\phi^{(2)} = \psi$, we obtain (3.5). We thus easily verify that (3.4) and (3.5) are valid for all $\psi \in \mathcal{H}_\mu(O)$, and so we have also proved that (3.3) is valid for all $\psi \in \mathcal{H}_\mu(O;\,\mathbb{C}^3)$.

At last, we verify that (3.3) is even valid for all $\psi \in \mathcal{H}(O;\mathbb{C}^3)$. As a matter of fact, if $\psi \in \mathcal{H}(O;\mathbb{C}^3)$, then $\psi_1 := \psi - \nabla\zeta \in \mathcal{H}_\mu(O;\mathbb{C}^3)$ if we take $\zeta \in W^{1,2}(O;\mathbb{C})$ as the weak solution to

$$\int_O \mu\,\nabla\zeta\cdot\nabla\phi = \int_O \mu\,\psi\cdot\nabla\phi$$

for all $\phi \in W^{1,2}(O;\mathbb{C})$. It follows that

$$\begin{aligned} i\,\omega\int_O \mu\,H\cdot\bar{\psi} + \int_O r\,\text{curl }H\cdot\text{curl }\bar{\psi} &= i\,\omega\int_O \mu\,H\cdot\overline{(\psi-\nabla\zeta)} + \int_O r\,\text{curl }H\cdot\text{curl }\overline{(\psi-\nabla\zeta)} \\ &\underset{(3.3)}{=} \int_O r\,j_g\cdot\text{curl }\overline{(\psi-\nabla\zeta)} = \int_O r\,j_g\cdot\text{curl }\bar{\psi}\,. \end{aligned}$$

Here, in the first line, we used $\operatorname{div}(\mu H) = 0$ and $\gamma_n(\mu H) = 0$, which implies that $\int_O \mu H \cdot \overline{\nabla\zeta} = 0$. In the second line, we used the validity of (3.3) for $\psi - \nabla\zeta \in \mathcal{H}_\mu(O;\mathbb{C}^3)$.

We now prove the existence of some $q > 3$ such that $H \in L^q_{\text{curl}}(O;\mathbb{C}^3)$. Applying at first the embedding result of Lemma A.2, it follows that $H \in [L^s(O;\mathbb{C})]^3$ for some $s > 3$, and that

$$\|H\|_{[L^s(O;\mathbb{C})]^3} \le \tilde{c}\,\|\text{curl } H\|_{[L^2(O;\mathbb{C})]^3} \le c\,\|j_g\|_{[L^2(O_{c_0};\mathbb{C})]^3}\,. \tag{3.7}$$

Next, we prove that curl $H^{(1)}$ and curl $H^{(2)}$ belong to $[L^q(O)]^3$ for $q := \min\{\bar{q},\, s\}$.

We consider an arbitrary $f \in [L^2(O)]^3$ with $f = 0$ a.e. in O_{nc}. According to Lemma A.4, we can decompose

$$f = \text{curl } A + \sum_{i\in I_c} \frac{1}{r}\nabla p_i\,\chi_{O_i},$$

where $A \in \mathcal{H}(O)$, $i \in I_c$ if O_i is a conductor, and $p_i \in W^{1,2}(O_i)$. Thanks to the equivalent formulation (3.4), we can write

$$\begin{aligned}\int_O r\,\text{curl } H^{(1)} \cdot f &= \int_O r\,\text{curl } H^{(1)} \cdot \text{curl } A + \sum_{i\in I_c}\int_{O_i} \text{curl } H^{(1)} \cdot \nabla p_i \\ &\underset{(3.4)}{=} \int_{O_{c_0}} r\ \text{Re}\,j_g \cdot \text{curl } A + \omega\int_O \mu H^{(2)} \cdot A\,. \end{aligned} \tag{3.8}$$

Here, we used the fact that $A \in \mathcal{H}(O)$ can be inserted in (3.4). In order to verify that the terms involving the p_i vanish, we have used Lemma A.3, which implies that

$$\int_{O_i} \text{curl } H^{(1)} \cdot \nabla p_i = \langle \gamma_n(\text{curl } H^{(1)}),\, p_i\rangle_{\partial O_i} = 0\,.$$

Due to (3.8) and the continuity estimate (A.6) associated with the decomposition of Lemma A.4, we then have

$$\begin{aligned}\left|\int_O \text{curl } H^{(1)} \cdot f\right| &\le c\,(\|\,\text{Re}\,j_g\|_{[L^q(O)]^3} + \|H^{(2)}\|_{[L^s(O)]^3})\,\|A\|_{L^{q'}_{\text{curl}}(O;\mathbb{C}^3)} \\ &\le c\,(\|\,\text{Re}\,j_g\|_{[L^q(O)]^3} + \|H^{(2)}\|_{[L^s(O)]^3})\,\|f\|_{[L^{q'}(O)]^3}\,.\end{aligned}$$

Consider now the functional

$$\tilde{F}(f) := \int_O r\,\text{curl } H^{(1)} \cdot f, \quad f \in [L^2(O)]^3,\, f = 0 \text{ a.e. in } O_{nc}\,.$$

With the Hahn–Banach theorem, we can extend the functional $\tilde{F}$ to the whole space $[L^{q'}(O)]^3$ by preserving its norm. Still denoting the extension $\tilde{F}$, we apply the well-known representation theorem for $L^{q'}(O)^*$, $q' > 1$, to find some $\Phi \in [L^q(O)]^3$ such that $\tilde{F}(f) = \int_O \Phi \cdot f$ for all $f \in [L^{q'}(O)]^3$. But then

$$\int_O (r\,\text{curl } H^{(1)} - \Phi) \cdot f = 0 \quad \forall\ f \in [L^2(O)]^3,\ f = 0 \text{ a.e. in } O_{nc}.$$

We conclude that $r \operatorname{curl} H^{(1)} - \Phi = 0$ a.e. in O_c. Thus, we see that $\operatorname{curl} H^{(1)} \in [L^q(O)]^3$, and due to (3.7), we have

$$\|\operatorname{curl} H^{(1)}\|_{[L^q(O)]^3} \leq c\,(\|j_g\|_{[L^{\bar{q}}(O;\mathbb{C})]^3} + \|j_g\|_{[L^2(O;\mathbb{C})]^3})\,.$$

We obtain the result for curl $H^{(2)}$ in exactly the same way. The lemma is proved. ∎

We now prove the second lemma. For the heat equation with radiation terms, we introduce the space

$$V^{2,5}(\Omega) := \left\{u \in W^{1,2}(\Omega) \,\middle|\, \tau_\Gamma u \in L^5(\Gamma),\ \tau_\Sigma u \in L^5(\Sigma)\right\},$$

where the operators τ_Γ and τ_Σ denote the trace operators on Γ and Σ, respectively.

LEMMA 3.5. *Let $H \in \mathcal{H}^q(O;\mathbb{C}^3)$ satisfy* (3.3) *with q given by Lemma* 3.4. *Then for some $\gamma > 0$, there exists a unique $y \in V^{2,5}(\Omega) \cap C^\gamma(\Omega)$ such that $y \geq \operatorname{ess\,inf}_\Gamma y_0$ a.e. in Ω and such that*

$$\int_\Omega \kappa(\cdot,\, y)\, \nabla y \cdot \nabla \xi + \int_\Gamma \varepsilon\, \sigma\, (y^4 - y_0^4)\, \xi + \int_\Sigma G(\sigma\, y^4)\, \xi = \int_\Omega r/2\, |\operatorname{curl} H|^2\, \xi \tag{3.9}$$

for all $\xi \in V^{2,5}(\Omega)$. Assuming that the domain Ω satisfies (2.7), *we even obtain that $y \in W^{1,q}(\Omega)$, with the q of Lemma* 3.4.

Proof. The existence of y in the class $V^{2,5}(\Omega) \cap L^\infty(\Omega)$ was proved in [LT01] for cavities with the smoothness $\Sigma \in \mathcal{C}^{1,\alpha}$, $\alpha > 0$. Notice that the boundedness has also been shown in [MPT06] by invoking the truncation method of Kinderlehrer and Stampacchia (see [KS80]). The existence result has been extended in [Dru09] to the case of a temperature-dependent heat conductivity and piecewise smooth surfaces. From the aforementioned references, we derive the estimate

$$\|y\|_{L^\infty(\Omega)} \leq \|y_0\|_{L^\infty(\Gamma)} + C\, \|\operatorname{curl} H\|^2_{[L^q(\Omega;\mathbb{C})]^3}\,, \tag{3.10}$$

where q is given by Lemma 3.4. The uniqueness has been proved in [LT01], using an interesting comparison principle for the case that κ_i is a positive constant in Ω_i for all $i \in \{0, \ldots, m\}$. Here we have to extend the result to the case of a temperature-dependent heat conductivity. This can be done with a *comparison technique* analogous to the one in the proof of Theorem 4.3.

Let us now justify that $y \in C(\overline{\Omega})$. Observe that under the assumption (2.15), the coefficient $\kappa = \kappa(x, y)$ belongs to $L^\infty(\Omega)$. The function y solves the problem

$$-\operatorname{div}(\kappa(\cdot,\, y)\, \nabla y) = F \quad \text{in } \Omega\,, \tag{3.11}$$

where F is the functional

$$F(\xi) := -\int_\Gamma \varepsilon\, \sigma\, (y^4 - y_0^4)\, \xi - \int_\Sigma G(\sigma\, y^4)\, \xi + \int_\Omega r/2\, |\operatorname{curl} H|^2\, \xi\,.$$

Let $q > 3$ be the exponent obtained in Lemma 3.4. Invoking Hölder's inequality, observe that

$$\left|\int_\Omega r/2\, |\operatorname{curl} H|^2\, \xi\right| \leq (r_u/2)\, \|\operatorname{curl} H\|^2_{L^q(\Omega;\mathbb{C}^3)}\, \|\xi\|_{[L^{q/(q-2)}(\Omega)]^3}\,. \tag{3.12}$$

We now look for the minimal $1 < p' < 3$ such that the continuous embedding $W^{1,p'}(\Omega) \hookrightarrow L^{q/(q-2)}(\Omega)$ is valid. Short computations give $p' := 3\,q/(4\,q - 6)$. Using in particular (3.12) combined with Sobolev's embedding theorems, we now obtain the estimate

$$|F(\xi)| \le c\,(\|y^4 - y_0^4\|_{L^\infty(\Gamma)} + \sigma\,\|y\|^4_{L^\infty(\Sigma)} + r_u\,\|\mathrm{curl}\ H\|^2_{L^q(\Omega;\mathbb{C}^3)})\,\|\xi\|_{W^{1,p'}(\Omega)}\,,$$

which proves that $F \in W^{1,p'}(\Omega)^*$.

The conjugate exponent to p' is $p := \frac{3q}{6-q}$, and we observe that $p > 3$, since $q > 3$. In view of Theorem 3.3 in [HDMR08], we thus obtain the Hölder continuity of y in Ω.

It remains to prove that $y \in W^{1,q}(\Omega)$ under the assumption (2.7). Thanks to the hypothesis (2.15) and to the fact that $y \in C(\overline{\Omega})$, we now see that $\kappa(\cdot,\, y) \in C(\overline{\Omega_i})$ for all subdomains Ω_i, $i = 0, \ldots, m$. Observe on the other hand that $\partial\Omega_i \in \mathcal{C}^1$, $i = 0, \ldots, m$, in view of (2.7). Thus, the coefficient κ is uniformly continuous on both sides of the surfaces $\partial\Omega_i$ for $i = 0, \ldots, m$.

Using again the fact that y is a solution to (3.11), we can apply Remark 3.18 of paper [ERS07] to find the existence of $q_1 > 3$ such that $y \in W^{1,\tilde{q}}(\Omega)$ for all $3 < \tilde{q} \le \min\{p, q_1\}$. Assuming without loss of generality that the exponent q given in Lemma 3.4 satisfies $q \le q_1$, and observing that $\min\{p, q_1\} \le \min\{q, q_1\}$, we find that the choice $\tilde{q} = q$ is possible. □

Theorem 3.3 is an immediate consequence of Lemmas 3.4 and 3.5.

Corollary 3.6 (control-to-state operator). *Let* (A1)–(A3) *be satisfied and let* $q > 3$ *as in Theorem* 3.3. *Then the solution operator*

$$\mathcal{S} : \mathbb{R}^n \to \mathcal{H}^q(O;\mathbb{C}^3) \times W^{1,q}(\Omega),$$

which assigns to every control $u \in \mathbb{R}^n$ *the weak solution* $(H, y) \in \mathcal{H}^q(O;\mathbb{C}^3) \times W^{1,q}(\Omega)$ *of* (1.1)–(1.2)*, is well defined and continuous.*

4. Linearized equation. Our goal in this section is to establish the differentiability of the control-to-state operator $\mathcal{S} : \mathbb{R}^n \to \mathcal{H}^q(O;\mathbb{C}^3) \times W^{1,q}(\Omega)$. For the remainder of the presentation, let $q \in \mathbb{R}$ with $3 < q \le \bar{q}$ be the exponent obtained in Theorem 3.3. We decompose the control-to-state operator into $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$, where

$$\begin{aligned} &\mathcal{S}_1 : \mathbb{R}^n \to \mathcal{H}^q(O;\, \mathbb{C}^3), \qquad \mathcal{S}_1 : u \mapsto H,\\ &\mathcal{S}_2 : \mathbb{R}^n \to W^{1,q}(\Omega), \qquad\quad \mathcal{S}_2 : u \mapsto y. \end{aligned} \tag{4.1}$$

Let us recall that $\mathcal{S}_1(u) = H \in \mathcal{H}^q(O;\, \mathbb{C}^3)$ is given by the unique solution to

(4.2)
$$i\,\omega \int_O \mu\, H \cdot \bar{\psi} + \int_O r\,\mathrm{curl}\ H \cdot \mathrm{curl}\ \bar{\psi} = \int_{O_{c_0}} r \sum_{j=1}^n u_j v_j \cdot \overline{\mathrm{curl}\ \psi} \quad \forall\, \psi \in \mathcal{H}^{q'}(O;\, \mathbb{C}^3).$$

Further, $\mathcal{S}_2(u) = y \in W^{1,q}(\Omega)$ is given by the unique solution to

$$\begin{aligned} &\int_\Omega \kappa(\cdot, y)\nabla y \cdot \nabla\xi + \int_\Sigma G(\sigma|y|^3 y)\xi + \int_\Gamma \varepsilon\sigma|y|^3 y\xi\\ &= \frac{1}{2}\int_\Omega r|\mathrm{curl}\ \mathcal{S}_1(u)|^2\,\xi + \int_\Gamma \varepsilon\sigma y_0^4\xi \quad \forall\,\xi \in W^{1,q'}(\Omega). \end{aligned}$$

Note that, thanks to the linearity of $\mathcal{S}_1$, we can simplify $\mathcal{S}_1$ by making use of the following vector fields.

DEFINITION 4.1. *For every $j = 1, \dots, n$, let $H_j \in \mathcal{H}^q(O; \mathbb{C}^3)$ be defined as the unique solution to*

$$i\omega \int_O \mu\, H_j \cdot \overline{\psi} + \int_O r \text{ curl } H_j \cdot \overline{\text{curl } \psi} = \int_{O_{c_0}} r\, v_j \cdot \overline{\text{curl } \psi} \quad \forall \psi \in \mathcal{H}^{q'}(O; \mathbb{C}^3). \tag{4.3}$$

According to Lemma 3.4, (4.3) for every $j \in \{1, \dots, n\}$ admits a unique solution $H_j \in \mathcal{H}^q(O; \mathbb{C}^3)$. Therefore, by a superposition principle,

$$\mathcal{S}_1(u) = \sum_{j=1}^{n} u_j H_j.$$

Consequently, for every $u \in \mathbb{R}^n$, $\mathcal{S}_2(u) = y$ is given by the unique solution to

(4.4)

$$\begin{aligned}\langle X_q(y), \xi \rangle_{W^{1,q'}(\Omega)^*, W^{1,q'}(\Omega)} &:= \int_\Omega \kappa(\cdot, y) \nabla y \cdot \nabla \xi + \int_\Sigma G(\sigma |y|^3 y) \xi + \int_\Gamma \varepsilon \sigma |y|^3 y \xi \\ &= \frac{1}{2} \int_\Omega r \left| \sum_{j=1}^{n} u_j \text{curl } H_j \right|^2 \xi + \int_\Gamma \varepsilon \sigma y_0^4 \xi \quad \forall \xi \in W^{1,q'}(\Omega).\end{aligned}$$

Note that $\mathcal{S}_1$ is a bounded linear operator, and hence it is continuously differentiable. Its derivative at an arbitrary point $u^* \in \mathbb{R}^n$ in an arbitrary direction $u \in \mathbb{R}^n$ is given by

$$\mathcal{S}_1'(u^*)u = \sum_{j=1}^{n} u_j H_j. \tag{4.5}$$

To show the continuous differentiability of $\mathcal{S}_2 : \mathbb{R}^n \to W^{1,q}(\Omega)$, we need to establish the differentiability of $X_q : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$. For this purpose, we impose further assumptions on the heat conductivity.

(A4) Assumption on the differentiability. The function $\kappa : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ with respect to the second variable. Further, for every positive real number K, there exists a constant $C_K > 0$ such that

$$\frac{\partial \kappa}{\partial y}(x, y) \le C_K$$

for a.a. $x \in \overline{\Omega}$ and all $y \in [-K, K]$.

Notice that the mapping $y \mapsto \sigma |y|^3 y$ is continuously differentiable from $L^\infty(\Gamma)$ to $L^\infty(\Gamma)$ (cf. [AZ90]). Since $G : L^\infty(\Sigma) \to L^\infty(\Sigma)$ is linear and continuous (Lemma B.2), a similar result applies also to the term containing the nonlocal radiation. Therefore, (A4) implies that the operator $X_q : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ is continuously differentiable. Its derivative at an arbitrary point $y^* \in W^{1,q}(\Omega)$ in an arbitrary direction $y \in W^{1,q}(\Omega)$ is given by

$$\begin{aligned}\langle X_q'(y^*) y, \xi \rangle &= \int_\Omega \kappa(\cdot, y^*)\, \nabla y \cdot \nabla \xi + \int_\Omega \frac{\partial \kappa}{\partial y}(\cdot, y^*)\, y\, \nabla y^* \cdot \nabla \xi \\ &\quad + 4 \int_\Sigma G(\sigma |y^*|^3\, y)\, \xi + 4 \int_\Gamma \varepsilon\, \sigma\, |y^*|^3\, y\, \xi \quad \forall \xi \in W^{1,q'}(\Omega).\end{aligned} \tag{4.6}$$

In the following, we prove that $X'_q(y^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ is an isomorphism. In other words, we should demonstrate that, for every given $\mathcal{F} \in W^{1,q'}(\Omega)^*$, the operator equation

$$X'_q(y^*)y = \mathcal{F} \quad \text{in } W^{1,q'}(\Omega)^* \tag{4.7}$$

admits a unique solution $y \in W^{1,q}(\Omega)$.

Remark 4.2. Notice that (4.7) corresponds to the following (strong) PDE formulation:

$$\begin{cases} -\operatorname{div}\left(\kappa(x,y^*)\,\nabla y + \dfrac{\partial\kappa}{\partial y}(x,y^*)y\cdot\nabla y^*\right) = \mathcal{F}_{|\Omega} \quad \text{in } \Omega\,, \\ \left[\kappa(x,y^*)\dfrac{\partial y}{\partial\vec{n}} + \dfrac{\partial\kappa}{\partial y}(x,y^*)y\dfrac{\partial y^*}{\partial\vec{n}}\right] + 4G(\sigma|y^*|^3 y) = \mathcal{F}_{|\Sigma} \quad \text{on } \Sigma, \\ \kappa(x,y^*)\dfrac{\partial y}{\partial\vec{n}} + \dfrac{\partial\kappa}{\partial y}(x,y^*)y\dfrac{\partial y^*}{\partial\vec{n}} + 4\varepsilon\sigma|y^*|^3 y = \mathcal{F}_{|\Gamma} \quad \text{on } \Gamma, \end{cases} \tag{4.8}$$

where $\mathcal{F}_{|\Omega}$, $\mathcal{F}_{|\Sigma}$, $\mathcal{F}_{|\Gamma}$ are the corresponding restriction of $\mathcal{F}$ to Ω, Σ, and Γ, respectively.

THEOREM 4.3. *Let* (A1)–(A4) *be satisfied. Suppose further that* $u^* \in \mathbb{R}^n$, *and let* $(H^*, y^*) = \mathcal{S}(u^*)$. *Then, for every* $\mathcal{F} \in W^{1,q'}(\Omega)^*$, *the variational problem*

$$\langle X'_q(y^*)y, \xi\rangle = \langle \mathcal{F}, \xi\rangle \quad \forall \xi \in W^{1,q'}(\Omega) \tag{4.9}$$

admits a unique solution $y \in W^{1,q}(\Omega)$. *Moreover, there is a constant* $c > 0$ *independent of* $\mathcal{F}$ *such that*

$$\|y\|_{W^{1,q}(\Omega)} \le c\|\mathcal{F}\|_{W^{1,q'}(\Omega)^*}\,. \tag{4.10}$$

Proof. First of all, let us introduce the following operators:

$$\begin{aligned} &B_q(y^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*, \ \langle B_q(y^*)z, \xi\rangle = \int_\Omega \kappa(x,y^*)\,\nabla z\cdot\nabla\xi + 4\int_\Gamma \varepsilon\sigma|y^*|^3 z\xi, \\ &Q_q(y^*) : L^\infty(\Omega) \to W^{1,q'}(\Omega)^*, \quad \langle Q_q(y^*)z, \xi\rangle = \int_\Omega \tfrac{\partial\kappa}{\partial y}(x,y^*)\,z\,\nabla y^*\cdot\nabla\xi, \\ &F_q(y^*) : L^\infty(\Sigma) \to W^{1,q'}(\Omega)^*, \quad \langle F_q(y^*)z, \xi\rangle = 4\int_\Sigma G(\sigma|y^*|^3\,z)\xi \end{aligned}$$

for all $\xi \in W^{1,q'}(\Omega)$. Recall that, by virtue of Theorem 3.3 and (2.12), we have

$$y^* \ge \operatorname*{ess\,inf}_\Gamma y_0 := \theta_0 > 0. \tag{4.11}$$

Therefore, as shown in [MY09a, Lemma 2.1], which is based on the result of [ERS07], there exists some $q_0 > 3$ such that, for all $\tilde{q} \in (3, q_0]$, the operator $B_{\tilde{q}}(y^*) : W^{1,\tilde{q}}(\Omega) \to W^{1,\tilde{q}'}(\Omega)^*$ is continuously invertible. Without loss of generality, we may assume that the exponent $q > 3$ of Theorem 3.3 satisfies $q \le q_0$.

Now, the operator $X'_q(y^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ as given in (4.6) can be decomposed into

$$X'_q(y^*) = B_q(y^*) + Q_q(y^*)E_{q,\infty} + F_q(y^*)\tau_\Sigma, \tag{4.12}$$

where the operator $E_{q,\infty}$ denotes the continuous injection $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and, as previously mentioned, the operator $\tau_\Sigma : W^{1,q}(\Omega) \to L^\infty(\Sigma)$ is the trace operator. Consequently, (4.9) can equivalently be written as the following operator equation:

$$B_q(y^*)y + Q_q(y^*)E_{q,\infty}\, y + F_q(y^*)\tau_\Sigma y = \mathcal{F} \quad \text{in } W^{1,q'}(\Omega)^*.$$

Thus, we arrive at

$$(I + B_q(y^*)^{-1}(Q_q(y^*)E_{q,\infty} + F_q(y^*)\tau_\Sigma))y = B_q(y^*)^{-1}\mathcal{F} \quad \text{in } W^{1,q}(\Omega).$$

Since $q > 3$, the embedding operator $E_{q,\infty} : W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and the trace operator $\tau_\Sigma : W^{1,q}(\Omega) \to L^\infty(\Sigma)$ are compact. Therefore, by Fredholm's theorem, the assertion will be proven once we are able to show that the equation

$$(I + B_q(y^*)^{-1}(Q_q(y^*)E_{q,\infty} + F_q(y^*)\tau_\Sigma))y = 0 \tag{4.13}$$

admits only the trivial solution $y = 0$. Let $y \in W^{1,q}(\Omega)$ be a solution to (4.13). Applying the operator $B_q(y^*)$ to (4.13) and taking (4.12) into consideration, we infer that y satisfies

$$X_q'(y^*)y = 0.$$

According to (4.6), the above equality is equivalent to

$$\int_\Omega \kappa(\cdot, y^*)\, \nabla y \cdot \nabla \xi + 4 \int_\Gamma \varepsilon\, \sigma\, |y^*|^3\, y\, \xi = -4 \int_\Sigma G(\sigma |y^*|^3\, y)\, \xi - \int_\Omega \frac{\partial \kappa}{\partial y}(\cdot, y^*)\, y\, \nabla y^* \cdot \nabla \xi \tag{4.14}$$

for all $\xi \in W^{1,q'}(\Omega)$.

We are now about to show that $y = 0$. To this aim, we follow the comparison principle of Casas and Tröltzsch [CT09], which is an extension result of Křížek and Liu [KL96]. In combination with this technique, we utilize some well-known properties of the nonlocal radiation operator G. For every $\delta \geq 0$, let us introduce the following sets:

$$\begin{aligned} &\Omega_\delta := \{x \in \Omega \mid y(x) > \delta\}, && \Omega_0 := \{x \in \Omega \mid y(x) > 0\}, \\ &\Sigma_\delta := \{x \in \Sigma \mid \big(\tau_\Sigma y\big)(x) > \delta\}, && \Sigma_0 := \{x \in \Sigma \mid \big(\tau_\Sigma y\big)(x) > 0\}. \end{aligned} \tag{4.15}$$

Notice that, in order to improve the readability, we will neglect the trace operator in the arguments of boundary integrals; i.e., we always write $\tau_\Sigma y = y$ on Σ. Further, we define the function

$$y_\delta := \min\{\delta, y^+\}, \tag{4.16}$$

where $y^+ = \max(0, y)$. For all $\delta \geq 0$, y_δ belongs to $W^{1,q}(\Omega)$. Further, notice that $\nabla y_\delta = 0$ a.e. in Ω_δ. Setting $\xi = y_\delta$ in (4.14) and then using the fact that G is self-adjoint (see Lemma B.2 in the appendices) leads to

$$\begin{aligned} &\int_\Omega \kappa(\cdot, y^*)\, \nabla y \cdot \nabla y_\delta + 4 \int_\Gamma \varepsilon\, \sigma\, |y^*|^3\, y\, y_\delta \\ &= -\int_{\Omega_0 \setminus \Omega_\delta} \frac{\partial \kappa}{\partial y}(\cdot, y^*)\, y\, \nabla y^* \cdot \nabla y_\delta - 4 \int_\Sigma G(\sigma |y^*|^3\, y)\, y_\delta \\ &= -\int_{\Omega_0 \setminus \Omega_\delta} \frac{\partial \kappa}{\partial y}(\cdot, y^*)\, y\, \nabla y^* \cdot \nabla y_\delta - 4 \int_\Sigma \sigma |y^*|^3\, y G(y_\delta). \end{aligned} \tag{4.17}$$

Let us investigate the second term on the right-hand side, which involves the nonlocal radiation operator G. To this aim, consider now the decomposition

$$\Sigma = \big(\Sigma_0 \setminus \Sigma_\delta\big) \cup \big(\Sigma \setminus \Sigma_0\big) \cup \Sigma_\delta.$$

The surface integral associated with the operator G is investigated in the following steps.

Step (i). Let us consider the set $\Sigma_0 \setminus \Sigma_\delta$. Since $0 < y \le \delta$ a.e. on $\Sigma_0 \setminus \Sigma_\delta$, Hölder's inequality implies that

$$\begin{aligned} -4\int_{\Sigma_0\setminus\Sigma_\delta} \sigma|y^*|^3 y\, G(y_\delta) &\le 4\;\delta\,\sigma\,\|y^*\|^3_{L^\infty(\Sigma)} \int_{\Sigma_0\setminus\Sigma_\delta} |G(y_\delta)| \\ &\le 4\,\delta\,\sigma\,\|y^*\|^3_{L^\infty(\Sigma)}\,\mathrm{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2}\|G(y_\delta)\|_{L^2(\Sigma)} \\ &\le c\,\delta\,\mathrm{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2}\,\|y_\delta\|_{H^1(\Omega)}, \end{aligned} \tag{4.18}$$

with a constant $c > 0$ independent of δ. Note that, in the latter inequality, we also made use of the continuity of $G : L^2(\Sigma) \to L^2(\Sigma)$ and the continuity of the trace operator from $H^1(\Omega)$ to $L^2(\Sigma)$.

Step (ii). Let us consider the set $\Sigma \setminus \Sigma_0$. According to Lemma B.2 (4), we can write $G = I - \mathbb{H}$ with a positive operator $\mathbb{H} : L^2(\Sigma) \to L^2(\Sigma)$ in the sense that if $v \ge 0$ a.e. on Σ, then $\mathbb{H}(v) \ge 0$ a.e. on Σ. Moreover, the operator $\mathbb{H}$ is self-adjoint. According to (4.15)–(4.16), it holds that $y_\delta \ge 0$ a.e. on Σ, $y_\delta = 0$ a.e. on $\Sigma \setminus \Sigma_0$, and $y \le 0$ a.e. on $\Sigma \setminus \Sigma_0$. These facts, along with the positivity of $\mathbb{H}$, lead to

$$-4\int_{\Sigma\setminus\Sigma_0} \sigma|y^*|^3 y\, G(y_\delta) = -4\int_{\Sigma\setminus\Sigma_0} \underbrace{\sigma|y^*|^3 y\, y_\delta}_{=0} + 4\int_{\Sigma\setminus\Sigma_0} \underbrace{\sigma|y^*|^3 y}_{\le 0}\, \underbrace{\mathbb{H}(y_\delta)}_{\ge 0} \le 0. \tag{4.19}$$

Step (iii). Finally, let us consider the set Σ_δ. By Lemma B.2, the operator $\mathbb{H}$ also belongs to $\mathcal{L}(L^\infty(\Sigma), L^\infty(\Sigma))$ and satisfies $\|\mathbb{H}\|_{\mathcal{L}(L^\infty(\Sigma),\, L^\infty(\Sigma))} \le 1$. Consequently

$$G(y_\delta) = y_\delta - \mathbb{H}(y_\delta) \ge y_\delta - \|\mathbb{H}\|_{\mathcal{L}(L^\infty(\Sigma),\, L^\infty(\Sigma))}\,\|y_\delta\|_{L^\infty(\Sigma)} \ge y_\delta - \delta = 0 \text{ on } \Sigma_\delta.$$

The above inequality, together with the fact that $y > \delta \ge 0$ a.e. on Σ_δ, implies immediately that

$$-4\int_{\Sigma_\delta} \sigma|y^*|^3 y\, G(y_\delta) \le 0. \tag{4.20}$$

Now applying the inequalities (4.18)–(4.20) to (4.17) yields

$$\begin{aligned} &\int_\Omega \kappa(\cdot, y^*)\,\nabla y \cdot \nabla y_\delta + 4\int_\Gamma \varepsilon\sigma|y^*|^3 y y_\delta \\ &\qquad \le -\int_{\Omega_0\setminus\Omega_\delta} \frac{\partial\kappa}{\partial y}(\cdot, y^*)\, y\,\nabla y^* \cdot \nabla y_\delta + c\,\delta\,\mathrm{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2}\,\|y_\delta\|_{H^1(\Omega)}. \end{aligned}$$

Since $yy_\delta \ge y_\delta^2$, $y\,\nabla y_\delta = y_\delta\,\nabla y_\delta$, and $\nabla y \cdot \nabla y_\delta = |\nabla y_\delta|^2$, it follows that

$$\begin{aligned} &\int_\Omega \kappa(\cdot, y^*)\,|\nabla y_\delta|^2 + 4\int_\Gamma \varepsilon\sigma|y^*|^3 y_\delta^2 \\ &\qquad \le -\int_{\Omega_0\setminus\Omega_\delta} \frac{\partial\kappa}{\partial y}(\cdot, y^*)\,\nabla y^* y_\delta \nabla y_\delta + c\,\delta\,\mathrm{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2}\,\|y_\delta\|_{H^1(\Omega)} \\ &\qquad \le c\,\delta(\|\nabla y^*\|_{L^2(\Omega_0\setminus\Omega_\delta)}\|\nabla y_\delta\|_{L^2(\Omega_0\setminus\Omega_\delta)} + \mathrm{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2}\|y_\delta\|_{H^1(\Omega)}), \end{aligned} \tag{4.21}$$

with a constant c independent of δ. Notice that, in the latter inequality, we have also used (A4) together with the facts that $y^* \in \mathcal{C}(\overline{\Omega})$ and $y_\delta \leq \delta$ (see (4.16)). Hence, along with (2.15)–(2.13) and (4.11), Friedrich's inequality applied to (4.21) yields that

$$\min\{\kappa_l, 4\,\varepsilon_l\,\sigma\,\theta_0^3\}\|y_\delta\|^2_{H^1(\Omega)} \leq c\,\delta\,(\|\nabla y^*\|_{L^2(\Omega_0\setminus\Omega_\delta)}\|\nabla y_\delta\|_{L^2(\Omega_0\setminus\Omega_\delta)} + \text{meas}(\Sigma_0 \setminus \Sigma_\delta)^{1/2}\|y_\delta\|_{H^1(\Omega)}),$$

with a constant $c > 0$ independent of δ. This implies that

$$\|y_\delta\|_{L^2(\Omega)} \leq c\,\delta\,(\|\nabla y^*\|_{L^2(\Omega_0\setminus\Omega_\delta)} + \text{meas}(\Sigma_0 \setminus \Sigma_\delta)^{1/2})$$

holds with a constant $c > 0$ independent of δ. Based on the latter estimate, we arrive at

$$\text{meas}(\Omega_\delta) = \frac{1}{\delta^2}\int_{\Omega_\delta}\delta^2 \leq \frac{1}{\delta^2}\int_\Omega y_\delta^2 \leq c\,(\|\nabla y^*\|_{L^2(\Omega_0\setminus\Omega_\delta)} + \text{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2})^2\,. \tag{4.22}$$

On the other hand, in view of (4.15),

$$\text{meas}(\Omega_0 \setminus \Omega_\delta) \to 0 \quad \text{and} \quad \text{meas}(\Sigma_0 \setminus \Sigma_\delta) \to 0, \qquad \text{as } \delta \to 0. \tag{4.23}$$

Thus, by (4.22)–(4.23), we conclude that

$$\text{meas}(\Omega_0) = \lim_{\delta\searrow 0}\text{meas}(\Omega_\delta) \leq \lim_{\delta\searrow 0} c(\|\nabla y^*\|_{L^2(\Omega_0\setminus\Omega_\delta)} + \text{meas}(\Sigma_0\setminus\Sigma_\delta)^{1/2})^2 = 0.$$

The latter equality implies that $y \leq 0$ holds a.e. in Ω. Applying the same procedure to the solution $-y$ of (4.13), we obtain $y \geq 0$. In conclusion $y = 0$. This completes the proof. ∎

THEOREM 4.4. *Let* (A1)–(A4) *be satisfied. Then the operator* $\mathcal{S} : \mathbb{R}^n \to \mathcal{H}^q(O; \mathbb{C}^3) \times W^{1,q}(\Omega)$ *is continuously differentiable. Its derivative at* $u^* \in \mathbb{R}^n$ *in an arbitrary direction* $u \in \mathbb{R}^n$ *is given by* $\mathcal{S}'(u^*)u = (\sum_{j=1}^n u_j H_j, \mathcal{S}_2'(u^*)u)$*, where* H_j *is as defined in Definition* 4.1 *and* $\mathcal{S}_2'(u^*)u = y \in W^{1,q}(\Omega)$ *is given by the unique solution to*

$$\begin{aligned}&\int_\Omega \kappa(\cdot, y^*)\,\nabla y\cdot\nabla\xi + \int_\Omega \frac{\partial\kappa}{\partial y}(\cdot, y^*)\,y\,\nabla y^*\cdot\nabla\xi + 4\int_\Sigma G(\sigma|y|^3 y)\xi + 4\int_\Gamma \varepsilon\sigma|y^*|^3 y\xi \\ &= \sum_{j=1}^n u_j\int_\Omega r(\text{Re curl } H^*\cdot\text{Re curl } H_j + \text{Im curl } H^*\cdot\text{Im curl } H_j)\xi \quad \forall\xi\in W^{1,q'}(\Omega),\end{aligned} \tag{4.24}$$

with $(H^*, y^*) := \mathcal{S}(u^*)$.

Proof. It suffices to prove that $\mathcal{S}_2 : \mathbb{R}^n \to W^{1,q}(\Omega)$ is continuously differentiable. Let us introduce the operator $T : W^{1,q}(\Omega)\times\mathbb{R}^n \to W^{1,q'}(\Omega)^*$ by

$$\langle T(y,u),\xi\rangle_{W^{1,q'}(\Omega)^*W^{1,q'}(\Omega)} := \langle X_q(y),\xi\rangle_{W^{1,q'}(\Omega)^*W^{1,q'}(\Omega)} - \frac{1}{2}\int_\Omega r\left|\sum_{j=1}^n u_j\text{curl } H_j\right|^2 \xi \quad \forall\xi\in W^{1,q'}(\Omega),$$

where X_q is as defined in (4.4). Note that T is well defined since $|\text{curl } H_j|^2 \in L^{\frac{q}{2}}(\Omega) \hookrightarrow W^{1,q'}(\Omega)^*$ (see Remark 3.2). For an arbitrarily fixed $u^* \in \mathbb{R}^n$, we set $y^* = \mathcal{S}_2(u^*)$, and hence $T(y^*,u^*) = 0$ (cf. (4.4)). Furthermore, T is continuously differentiable with $\partial_y T(y^*,u^*) = X_q'(y^*)$. Consequently, Theorem 4.3 implies that $\partial_y T(y^*,u^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ is an isomorphism. Thus, by the implicit function theorem, $\mathcal{S}_2$ is continuously differentiable at u^*, and its derivative is given by

$$\mathcal{S}_2'(u^*)u = -\partial_y T(y^*,u^*)^{-1}\partial_u T(y^*,u^*)u = -X_q'(y^*)^{-1}\partial_u T(y^*,u^*)u. \tag{4.25}$$

The derivative $\partial_u T(y^*,u^*)u \in W^{1,q'}(\Omega)^*$ on the other hand is given by

(4.26)
$$\langle \partial_u T(y^*,u^*)u, \xi\rangle_{W^{1,q'}(\Omega)^* W^{1,q'}(\Omega)} = -\sum_{j=1}^n u_j \int_\Omega r\Bigg(\text{Re curl}\left(\sum_{j=1}^n u_j^* H_j\right)\cdot \text{Re curl } H_j + \text{Im curl}\left(\sum_{j=1}^n u_j^* H_j\right)\cdot \text{Im curl } H_j\Bigg)\xi \quad \forall \xi \in W^{1,q'}(\Omega).$$

Using the expression $H^* = \mathcal{S}_1(u^*) = \sum_{j=1}^n u_j^* H_j$, (4.25)–(4.26) immediately imply that $\mathcal{S}_2'(u^*)u = y$ is given by the unique solution to (4.24). Hence, the assertion is valid. □

Remark 4.5. Note that, since $X_q'(y^*)$ is an isomorphism (Theorem 4.3), we also conclude from (4.24) that

$$\mathcal{S}_2'(u^*)u = \sum_{j=1}^n u_j X_q'(y^*)^{-1} K_j(H^*), \tag{4.27}$$

where $K_j(H^*) \in W^{1,q'}(\Omega)^*$, $j = 1, \ldots, n$, is defined by

(4.28)
$$\langle K_j(H^*), \xi\rangle_{W^{1,q'}(\Omega)^*, W^{1,q'}(\Omega)} := \int_\Omega r(\text{Re curl } H^* \cdot \text{Re curl } H_j + \text{Im curl } H^* \cdot \text{Im curl } H_j)\xi \quad \forall \xi \in W^{1,q'}(\Omega).$$

5. Optimal control problem. We now focus on the control of the solution to (1.1)–(1.2), which shall be established based on the theoretical results presented in the previous sections. Given fixed data $z \in L^2(\Omega)^3$, $H_d \in L^2(O;\mathbb{C})^3$, $\rho \geq 0$, and $\beta > 0$, we look for solutions of the following control problem:

$$\text{(P)} \qquad \text{minimize } J(u,H,y) := \frac{1}{2}\int_\Omega |\nabla y - z|^2 + \frac{\rho}{2}\int_O |H - H_d|^2 + \frac{\beta}{2}|u|^2$$

subject to

$$A_q(H,y) = E_q u \tag{5.1}$$

and

$$\begin{aligned} y_a(x) \leq y(x) &\leq y_b(x) && \text{for a.a. } x \in \Omega, \\ u_a \leq u_j &\leq u_b && \text{for all } j \in \{1,\ldots,n\}. \end{aligned} \tag{5.2}$$

Through the use of the control-to-state operator $\mathcal{S}$, the control problem (P) can be reduced to

$$\text{(P)} \qquad \begin{cases} \displaystyle\min_{u\in\mathcal{U}_{ad}} & f(u) := J(u, \mathcal{S}_1(u), \mathcal{S}_2(u)) \\ \text{subject to} & y_a(x) \leq (\mathcal{S}_2(u))(x) \leq y_b(x) \text{ for a.a. } x \in \Omega, \end{cases}$$

where the admissible set is defined by

$$\mathcal{U}_{ad} = \{u \in \mathbb{R}^n \mid u_a \leq u_j \leq u_b \ \forall j \in \{1, \ldots, n\}\}.$$

In what follows, a control $u \in \mathbb{R}^n$ is said to be feasible if and only if $u \in \mathcal{U}_{ad}$ and $y_a(x) \leq (\mathcal{S}_2(u))(x) \leq y_b(x)$ holds for a.a. $x \in \Omega$.

THEOREM 5.1. *Let* (A1)–(A3) *be satisfied and assume that there exists a feasible control of* (P). *Then the optimal control problem* (P) *admits a solution.*

Proof. The assertion follows from the Weierstrass theorem since the set of all feasible controls is compact and the objective functional f is continuous. ∎

Notice that the solution to (P) is not necessarily unique due to the nonlinearities involved in the state equation. We therefore concentrate in our analysis on local solutions in the following sense: A feasible control $u^* \in \mathbb{R}^n$ is called a local solution to (P) with respect to the $\mathbb{R}^n$-topology if there exists some $r > 0$ such that

$$f(u^*) \leq f(u)$$

holds for all feasible controls u satisfying $|u - u^*| \leq r$. Next, by $\mathcal{M}(\overline{\Omega})$, we denote the space of all regular Borel measures on the compact set $\overline{\Omega}$. According to the Riesz–Radon theorem, the space $\mathcal{M}(\overline{\Omega})$ can be isometrically identified with the dual space $\mathcal{C}(\overline{\Omega})^*$ with respect to the duality pairing

$$\langle \mu, \varphi \rangle_{\mathcal{C}(\overline{\Omega})^*, \mathcal{C}(\overline{\Omega})} := \int_{\overline{\Omega}} \varphi \, d\mu, \quad \varphi \in \mathcal{C}(\overline{\Omega}), \, \mu \in \mathcal{M}(\overline{\Omega}).$$

Let us now introduce the notion of the Lagrange functional associated with (P).

DEFINITION 5.2 (Lagrange functional associated with (P)). *The Lagrange functional associated with* (P) *is defined by* $\mathscr{L} : \mathbb{R}^n \times \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega}) \to \mathbb{R}$,

$$\mathscr{L}(u, \mu_a, \mu_b) = f(u) + \int_{\overline{\Omega}} \big(y_a - \mathcal{S}_2(u)\big) \, d\mu_a + \int_{\overline{\Omega}} \big(\mathcal{S}_2(u) - y_b\big) \, d\mu_b.$$

In what follows, let u^* stand for a local solution to (P) and $y^* = \mathcal{S}_2(u^*)$. Thanks to the continuous differentiability of the solution operator $\mathcal{S}$, the objective functional $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(u) = \frac{1}{2} \int_\Omega |\nabla \mathcal{S}_2(u) - z|^2 + \frac{\rho}{2} \int_O |\mathcal{S}_1(u) - H_d|^2 + \frac{\beta}{2} |u|^2$$

is continuously Fréchet differentiable. Its first derivative at u^* in the direction $u \in \mathbb{R}^n$ is given by

$$\begin{aligned} f'(u^*) \, u = & \int_\Omega (\nabla y^* - z) \cdot \nabla(\mathcal{S}_2'(u^*)u) + \rho \int_O \mathrm{Re}\,(H^* - H_d) \cdot \mathrm{Re}\,\big(\mathcal{S}_1'(u^*)u\big) \\ & + \rho \int_O \mathrm{Im}\,(H^* - H_d) \cdot \mathrm{Im}\,\big(\mathcal{S}_1'(u^*)u\big) + \beta u^* \cdot u, \end{aligned}$$

where $H^* = \mathcal{S}_1(u^*)$ and $y^* = \mathcal{S}_2(u^*)$. We now recall from (4.5) that $\mathcal{S}_1'(u^*)u = \sum_{j=1}^n u_j H_j$, where the vector fields $H_j \in \mathcal{H}^q(O; \mathbb{C}^3)$ are as defined in Definition 4.1.

Consequently

$$
\begin{aligned}
f'(u^*)\,u &= \int_\Omega (\nabla y^* - z)\cdot\nabla(\mathcal{S}_2'(u^*)u) \\
&\quad + \rho\sum_{j=1}^n u_j\left(\int_O \mathrm{Re}\,(H^*-H_d)\cdot\mathrm{Re}\,H_j + \int_O \mathrm{Im}\,(H^*-H_d)\cdot\mathrm{Im}\,H_j\right) + \beta u^*\cdot u \\
&= \int_\Omega (\nabla y^* - z)\cdot\nabla(\mathcal{S}_2'(u^*)u) + (\rho h^* + \beta u^*)\cdot u,
\end{aligned}
$$

with

$$
h_j^* := \int_O \mathrm{Re}\,(H^*-H_d)\cdot\mathrm{Re}\,H_j + \mathrm{Im}\,(H^*-H_d)\cdot\mathrm{Im}\,H_j, \qquad j=1,\ldots,n.
$$

Further, let us introduce the linear and continuous operator $L : W^{1,q}(\Omega)\to W^{1,q}(\Omega)^*$ defined by

$$
\langle Ly, v\rangle_{W^{1,q}(\Omega)^*,W^{1,q}(\Omega)} := \int_\Omega (\nabla y - z)\cdot\nabla v \quad \forall v\in W^{1,q}(\Omega). \tag{5.3}
$$

Using this operator, we arrive at

$$
f'(u^*)\,u = \langle Ly^*, \mathcal{S}_2'(u^*)\,u\rangle_{W^{1,q}(\Omega)^*,W^{1,q}(\Omega)} + (\rho h^* + \beta u^*)\cdot u. \tag{5.4}
$$

Notice that, since f and $\mathcal{S}_2$ are continuously Fréchet differentiable, $\mathscr{L}$ is continuously Fréchet differentiable, so that the following definition makes sense.

Definition 5.3 (Lagrange multiplier associated with (P)). *Let $u^*\in\mathbb{R}^n$ be a local solution to* (P)*. Then $(\mu_a,\mu_b)\in\mathcal{M}(\overline{\Omega})\times\mathcal{M}(\overline{\Omega})$ is said to be a pair of Lagrange multipliers associated with the state constraints of* (P) *if and only if*

$$
\partial_u\mathscr{L}(u^*,\mu_a,\mu_b)(u-u^*)\ge 0 \quad \forall\, u\in\mathcal{U}_{ad}, \tag{5.5}
$$

$$
\mu_a\ge 0, \quad \mu_b\ge 0, \tag{5.6}
$$

$$
\int_{\overline{\Omega}} \big(y_a - \mathcal{S}_2(u^*)\big)\, d\mu_a = \int_{\overline{\Omega}} \big(\mathcal{S}_2(u^*) - y_b\big)\, d\mu_b = 0. \tag{5.7}
$$

To establish the existence of Lagrange multipliers, we apply the Karush–Kuhn–Tucker theorem (cf. Zowe and Kurcyusz [ZK79]). More precisely, we rely on a Slater-type constraint qualification with respect to the state constraints in (P). This assumption is referred to as the *linearized Slater condition*.

Definition 5.4 (linearized Slater condition for (P)). *A control $u^*\in\mathcal{U}_{ad}$ satisfies the linearized Slater condition for* (P) *if there exist some $u_0\in\mathcal{U}_{ad}$ and some constant $c>0$ such that*

$$
y_a(x) + c \le \big(\mathcal{S}_2(u^*)\big)(x) + \big(\mathcal{S}_2'(u^*)(u_0-u^*)\big)(x) \le y_b(x) - c \qquad \forall\, x\in\overline{\Omega}.
$$

The formula for the derivative $\mathcal{S}_2'(u^*)(u_0-u^*)$ reads as in (4.24).

Theorem 5.5 (first order necessary optimality conditions for (P)). *Let* (A1)–(A4) *be satisfied. Moreover, let u^* be a local solution to* (P) *satisfying the linearized Slater condition and $(H^*,y^*)=\mathcal{S}(u^*)$. Then there exist Lagrange multipliers $\mu_a,\mu_b\in$*

$\mathcal{M}(\overline{\Omega})$ and an adjoint state $p^* \in W^{1,q'}(\Omega)$ (here q' is the conjugate of q and hence $1 \leq q' < \frac{3}{2}$) such that

$$\begin{cases} -\mathrm{div}(\kappa(x,y^*)\,\nabla p^*) + \dfrac{\partial \kappa}{\partial y}(x,y^*)\nabla y^* \cdot \nabla p^* = -\Delta y^* + \mathrm{div}\; z + (\mu_b - \mu_a)_{|\Omega} & \text{in } \Omega, \\ \left[\kappa(x,y^*)\dfrac{\partial p^*}{\partial \vec{n}}\right] + 4\sigma|y^*|^3\, G(p^*) = -\dfrac{\partial y^*}{\partial \vec{n}} + z\cdot\vec{n} + (\mu_b - \mu_a)_{|\Sigma} & \text{on } \Sigma, \\ \kappa(x,y^*)\dfrac{\partial p^*}{\partial \vec{n}} + 4\varepsilon\sigma|y^*|^3 p^* = (\mu_b - \mu_a)_{|\Gamma} & \text{on } \Gamma, \end{cases} \tag{5.8}$$

$$\mu_a \geq 0, \quad \mu_b \geq 0, \tag{5.9}$$

$$\int_\Omega \big(y_a - \mathcal{S}_2(u^*)\big)\, d\mu_a = \int_\Omega \big(\mathcal{S}_2(u^*) - y_b\big)\, d\mu_b = 0, \tag{5.10}$$

$$u^* = \mathbb{P}_{[u_a,u_b]}\left(-\frac{1}{\beta}(v^* + \rho h^*)\right), \tag{5.11}$$

$$v_j^* = \int_\Omega p^* r(\mathrm{Re}\;\mathrm{curl}\; H^* \cdot \mathrm{Re}\;\mathrm{curl}\; H_j + \mathrm{Im}\;\mathrm{curl}\; H^* \cdot \mathrm{Im}\;\mathrm{curl}\; H_j) \quad \forall j \in \{1,\dots,n\}, \tag{5.12}$$

$$h_j^* := \int_O \mathrm{Re}\;(H^* - H_d)\cdot \mathrm{Re}\; H_j + \mathrm{Im}\;(H^* - H_d)\cdot \mathrm{Im}\; H_j \quad \forall j \in \{1,\dots,n\}, \tag{5.13}$$

where H_j is defined as in Definition 4.1 *and $\mathbb{P}_{[u_a,u_b]}$ denotes the standard projection from $\mathbb{R}^n$ onto $[u_a,u_b]^n$.*

Proof. Since u^* satisfies the linearized Slater assumption, there exist Lagrange multipliers $\mu_a, \mu_b \in \mathcal{M}(\overline{\Omega})$ satisfying (5.5)–(5.7) (cf. [ZK79]). Let us demonstrate now that (5.5) is equivalent to the existence of $p^* \in W^{1,q'}(\Omega)$ with $1 \leq q' < \frac{3}{2}$ and $h^*, v^* \in \mathbb{R}^n$ satisfying (5.8) and (5.11)–(5.13). In view of Remark 4.5, the derivative of $\mathcal{S}_2$ at u^* in the direction $u \in \mathbb{R}^n$ is given by

$$\mathcal{S}_2'(u^*)u = y = \sum_{j=1}^n u_j X_q'(y^*)^{-1} K_j(H^*). \tag{5.14}$$

Taking (5.4) and (5.14) into account, we find that

$$\begin{aligned} \partial_u \mathscr{L}(u^*,\mu_a,\mu_b)(u-u^*) &= f'(u^*)(u-u^*) - \int_\Omega \mathcal{S}_2'(u^*)(u-u^*)d\mu_a + \int_\Omega \mathcal{S}_2'(u^*)(u-u^*)d\mu_b \\ &= \sum_{j=1}^n (u_j - u_j^*)\langle Ly^* - \widetilde{\mu}_a + \widetilde{\mu}_b, X_q'(y^*)^{-1}K_j(H^*)\rangle_{W^{1,q}(\Omega)^*,W^{1,q}(\Omega)} + (\rho h^* + \beta u^*)\cdot(u-u^*), \end{aligned}$$

where $\tilde{\mu}_a$, and $\tilde{\mu}_b$ denote the elements of $W^{1,q}(\Omega)^*$ associated with $\mu_a, \mu_b \in \mathcal{C}(\Omega)^* \hookrightarrow W^{1,q}(\Omega)^*$ (for $q > 3$) in the following sense:

$$\langle \widetilde{\mu}_a, v\rangle_{W^{1,q}(\Omega)^*,W^{1,q}(\Omega)} = \int_\Omega v\, d\mu_a, \quad \langle \widetilde{\mu}_b, v\rangle_{W^{1,q}(\Omega)^*,W^{1,q}(\Omega)} = \int_\Omega v\, d\mu_b \quad \forall v \in W^{1,q}(\Omega). \tag{5.15}$$

Thus, by (4.28), we infer that

$$\begin{aligned}
&\partial_u \mathscr{L}(u^*, \mu_a, \mu_b)(u - u^*) \\
&\quad = \sum_{j=1}^{n} (u_j - u_j^*) \langle (X_q'(y^*)^{-1})^* (Ly^* - \widetilde{\mu}_a + \widetilde{\mu}_b), K_j(H^*) \rangle_{W^{1,q'}(\Omega), W^{1,q'}(\Omega)^*} \\
&\qquad + (\rho h^* + \beta u^*) \cdot (u - u^*) \\
&\quad = \sum_{j=1}^{n} (u_j - u_j^*) \int_\Omega (X_q'(y^*)^{-1})^* (Ly^* - \widetilde{\mu}_a + \widetilde{\mu}_b) r (\text{Re curl } H^* \cdot \text{Re curl } H_j \\
&\qquad + \text{Im curl } H^* \cdot \text{Im curl } H_j) + (\rho h^* + \beta u^*) \cdot (u - u^*).
\end{aligned} \tag{5.16}$$

On the other hand, the weak formulation of (5.8) is given by

$$\begin{aligned}
&\int_\Omega \kappa(\cdot, y^*) \nabla p^* \cdot \nabla v + \int_\Omega \frac{\partial \kappa}{\partial y}(\cdot, y^*) \nabla y^* \cdot \nabla p^* \, v + 4 \int_\Sigma \sigma |y^*|^3 \, G(p^*) \, v \; + 4 \int_\Gamma \varepsilon \sigma |y^*|^3 p^* v \\
&\qquad = \int_\Omega (\nabla y^* - z) \cdot \nabla v - \int_{\overline{\Omega}} v \, d\mu_a + \int_{\overline{\Omega}} v \, d\mu_b \quad \forall v \in W^{1,q}(\Omega).
\end{aligned} \tag{5.17}$$

Recall that $X_q'(y^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ is given by

$$\begin{aligned}
\langle X_q'(y^*) v, \xi \rangle_{W^{1,q'}(\Omega)^*, W^{1,q'}(\Omega)} &= \int_\Omega \kappa(\cdot, y^*) \nabla v \cdot \nabla \xi + \int_\Omega \frac{\partial \kappa}{\partial y}(\cdot, y^*) \, v \, \nabla y^* \cdot \nabla \xi \\
&\quad + 4 \int_\Sigma G(\sigma |y^*|^3 \, v) \, \xi + 4 \int_\Gamma \varepsilon \, \sigma \, |y^*|^3 \, v \, \xi \quad \forall \xi \in W^{1,q'}(\Omega), \forall v \in W^{1,q}(\Omega).
\end{aligned}$$

Since the operator G is self-adjoint, the adjoint operator $X_q'(y^*)^* : W^{1,q'}(\Omega) \to W^{1,q}(\Omega)^*$ associated with $X_q'(y^*)$ is given by

$$\begin{aligned}
\langle X_q'(y^*)^* \xi, v \rangle_{W^{1,q}(\Omega)^*, W^{1,q}(\Omega)} &= \int_\Omega \kappa(\cdot, y^*) \nabla \xi \cdot \nabla v + \int_\Omega \frac{\partial \kappa}{\partial y}(\cdot, y^*) \nabla y^* \cdot \nabla \xi \, v \\
&\quad + 4 \int_\Sigma \sigma |y^*|^3 \, G(\xi) \, v + 4 \int_\Gamma \varepsilon \, \sigma \, |y^*|^3 \, \xi \, v \quad \forall \xi \in W^{1,q'}(\Omega), \forall v \in W^{1,q}(\Omega).
\end{aligned}$$

Altogether, we can write the weak formulation (5.17) as the following operator equation:

$$X_q'(y^*)^* p^* = Ly^* - \tilde{\mu}_a + \tilde{\mu}_b \quad \text{in } W^{1,q}(\Omega)^*. \tag{5.18}$$

In view of Theorem 4.3, the operator $X_q'(y^*) : W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$ is an isomorphism such that the adjoint operator $X_q'(y^*)^* : W^{1,q'}(\Omega) \to W^{1,q}(\Omega)^*$ is in turn an isomorphism. Thus, (5.18) admits a unique solution $p^* \in W^{1,q'}(\Omega)$ with $1 \le q' < \frac{3}{2}$ given by

$$p^* = \big(X_q'(y^*)^*\big)^{-1} \big(Ly^* - \tilde{\mu}_a + \tilde{\mu}_b\big) = \big(X_q'(y^*)^{-1}\big)^* \big(Ly^* - \tilde{\mu}_a + \tilde{\mu}_b\big).$$

Applying p^* to (5.16), we have

$$\begin{aligned}\partial_u \mathscr{L}(u^*, \mu_s, \mu_a, \mu_b)(u-u^*) &= \sum_{j=1}^n (u_j - u_j^*) \int_\Omega p^* r(\text{Re curl } H^* \cdot \text{Re curl } H_j \\ &\qquad + \text{Im curl } H^* \cdot \text{Im curl } H_j) + (\rho h^* + \beta u^*)\cdot(u-u^*) \\ &= (v^* + \rho h^* + \beta u^*)\cdot (u-u^*),\end{aligned}$$

where $v^* \in \mathbb{R}^n$ is specified by

$$v_j^* = \int_\Omega p^* r(\text{Re curl } H^* \cdot \text{Re curl } H_j + \text{Im curl } H^* \cdot \text{Im curl } H_j) \qquad \forall j = 1, \dots, n.$$

Consequently, the variational inequality (5.5) implies that

$$0 \le \partial_u \mathscr{L}(u^*, \mu_s, \mu_a, \mu_b)(u-u^*) = (v^* + \rho h^* + \beta u^*)\cdot(u-u^*) \quad \forall u \in \mathcal{U}_{ad}.$$

By classical arguments, a pointwise evaluation of the above variational inequality yields the desired projection formula,

$$u^* = \mathbb{P}_{[u_a, u_b]}\left(-\frac{1}{\beta}(v^* + \rho h^*)\right).$$

This completes the proof. □

Appendix A. Tools for the Maxwell equations. Throughout this section, we consider a simply connected, bounded Lipschitz domain $O \subset \mathbb{R}^3$ such that

$$\overline{O} := \bigcup_{i=0}^m \overline{O_i},\ (m \ge 1), \quad O_0, \dots, O_m \in \mathcal{C}^{0,1}, \quad O_i \cap O_j = \emptyset,\ i \ne j\,. \tag{A.1}$$

Let $I_c \subset \{0, \dots, m\}$ be the index set of the conducting materials, and denote $O_c := \bigcup_{i \in I_c} O_i$, $O_{nc} := O \setminus O_c$.

We assume that

$$\operatorname{dist}(O_i, O_j) > 0 \quad \text{for } i, j \in I_c\,,\ i \ne j\,. \tag{A.2}$$

A.1. Embedding results. In order to deal with the weak formulation of the Maxwell equations, embedding results for vector fields that satisfy a curl, a div, and a γ_n or γ_t constraint are very important.

For $1 \le p,\ \alpha \le \infty$, we introduce

$$\begin{aligned}\mathcal{W}_n^{p,\alpha}(O) &:= \{\psi \in L^p_{\text{curl}}(O) \cap L^p_{\text{div}}(O) \,:\, \gamma_n(\psi) \in L^\alpha(\partial O)\}\,, \\ \mathcal{W}_t^{p,\alpha}(O) &:= \{\psi \in L^p_{\text{curl}}(O) \cap L^p_{\text{div}}(O) \,:\, \gamma_t(\psi) \in L^\alpha(\partial O)\}\,. \end{aligned}\tag{A.3}$$

In simply connected domains O, these are Banach spaces with respect to the graph norms

$$\|\psi\|_{\mathcal{W}_n^{p,\alpha}(O)} := \|\text{curl } \psi\|_{[L^p(O)]^3} + \|\operatorname{div}\psi\|_{L^p(O)} + \|\gamma_n(\psi)\|_{L^\alpha(\partial O)}\,,$$

$$\|\psi\|_{\mathcal{W}_n^{p,\alpha}(O)} := \|\text{curl } \psi\|_{[L^p(O)]^3} + \|\operatorname{div}\psi\|_{L^p(O)} + \|\gamma_t(\psi)\|_{L^\alpha(\partial O)}\,.$$

The following result has been proved in [Dru07].

LEMMA A.1. *Let $O \subset \mathbb{R}^3$ be a simply connected, bounded Lipschitz domain. Then there exists some $q_1 > 3$ such that for all $p \in [q_1', q_1]$, we have $\mathcal{W}_n^{p,\alpha}(O) \hookrightarrow [L^s(O)]^3$ with continuous embedding, $s := \min\{q_1, p^*, 3\,\alpha/2\}$ (p^* = Sobolev embedding exponent). If $\partial O \in \mathcal{C}^1$, then one can choose $q_1 = +\infty$. The same is valid for the space $\mathcal{W}_t^{p,\alpha}(O)$.*

We also need embedding results for the case that one of the constraints is perturbed by a measurable coefficient. For a function μ satisfying (2.11), we introduce

(A.4)
$$V_\mu(O) := \left\{ \psi \in [L^2(O)]^3 \,\middle|\, \text{curl}\;\psi \in [L^2(O)]^3,\, \text{div}(\mu\,\psi) \in L^2(O),\, \gamma_n(\mu\,\psi) = 0 \text{ on } \partial O \right\}.$$

We endow $V_\mu(O)$ with the graph norm

$$\|\psi\|_{V_\mu(O)} := \|\psi\|_{[L^2(O)]^3} + \|\text{curl}\;\psi\|_{[L^2(O)]^3} + \|\,\text{div}(\mu\,\psi)\|_{L^2(O)}\,.$$

Obviously, $V_\mu(O)$ is a Hilbert space in this topology.

LEMMA A.2. *Let O be a simply connected Lipschitz domain. Assume that μ satisfies (2.11) and that the domain O satisfies (2.6). Then there exists a number $s > 3$ such that $V_\mu(O) \hookrightarrow [L^s(O)]^3$ with continuous embedding. If $\partial O \in \mathcal{C}^1$, then one can choose $s = 6$. The embedding $V_\mu(O) \hookrightarrow [L^p(O)]^3$ is compact for all $1 \le p < s$.*

The following property of the spaces $\mathcal{H}^q(O)$ (cf. (2.3)) is easy to derive (see, for example, [Dru07] for a proof).

LEMMA A.3. *Let $O \subset \mathbb{R}^3$ have the structure (A.1) considered throughout the paper and satisfy (A.2). Then if $H \in \mathcal{H}^q(O)$, we have $\gamma_n(\text{curl}\;H) = 0$ on ∂O_i, $i \in I_c$.*

A.2. A decomposition lemma.

LEMMA A.4. *Let $O \subset \mathbb{R}^3$ be a simply connected, bounded Lipschitz domain with the property (A.2). Let r satisfy, in addition to (3.1), the condition $r \in \mathcal{C}(\overline{O_i})$ for $i \in I_c$.*

Then there exists a $q_1 > 3$ such that for all $q \in [q_1', q_1]$ and for all $f \in [L^q(O)]^3$ such that $f = 0$ a.e. in O_{nc}, there exist unique $A \in \{\mathcal{H}^q(O) : \text{div}\,A = 0,\, \gamma_t(A) = 0\}$ and $p_i \in W^{1,q}_M(O_i)$ (subscript M = mean-value zero), $i \in I_c$, such that

$$f = \text{curl}\;A + \frac{1}{r} \sum_{i \in I_c} \nabla p_i\,\chi_{O_i}\,. \tag{A.5}$$

In addition, we can find a positive constant $c = c(O,\, q,\, r)$ such that

$$\|A\|_{L^q_{\text{curl}}(O)} + \sum_{i \in I_c} \|p_i\|_{W^{1,q}(O_i)} \le c\,\|f\|_{[L^q(O)]^3}\,. \tag{A.6}$$

Proof. For each $i \in I_c$, we have $O_i \in \mathcal{C}^{0,1}$, and $1/r \in C(\overline{O_i})$ is bounded away from zero and from above.

According to the main result of the paper [ERS07] (cf. also Remark 3.18 of [ERS07]) there exists $q_1 > 3$ such that for $i \in I_c$ and for all $q \in [q_1', q_1]$, there is a unique $p_i \in W^{1,q}_M(O_i)$ satisfying

$$\int_{O_i} \frac{1}{r}\,\nabla p_i \cdot \nabla \xi = \int_{O_i} f \cdot \nabla \xi \tag{A.7}$$

for all $\xi \in W^{1,q'}_M(O_i)$, and the estimate

$$\|p\|_{W^{1,q}(O_i)} \leq c(q, O_i, r)\, \|f\|_{[L^q(O_i)]^3}\,.$$

Define

$$\text{(A.8)} \qquad \begin{cases} w = f - 1/r\, \nabla p_i & \text{in } O_i,\ i \in I_c, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $w \in [L^{q_1}(O)]^3$, and in view of (A.7), $\operatorname{div} w = 0$ in O, and $\gamma_n(w) = 0$ on ∂O in the weak sense.

We now prove that we can find $A \in \{L^q_{\text{curl}}\,(O) \,:\, \operatorname{div} A = 0,\ \gamma_t(A) = 0\}$ such that curl $A = w$.

We at first prove that

(A.9)
$$\{\psi \in [L^2(O)]^3 \,:\, \operatorname{div}\psi = 0,\ \gamma_n(\psi) = 0\} = \{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\}\,.$$

To verify the last identity, consider first $\psi :=$ curl A, where $A \in L^2_{\text{curl}}\,(O)$ satisfies $\gamma_t(A) = 0$. It is readily verified that $\operatorname{div}\psi = 0$, $\gamma_n(\psi) = 0$ in the weak sense. Thus

(A.10)
$$\{\psi \in [L^2(O)]^3 \,:\, \operatorname{div}\psi = 0,\ \gamma_n(\psi) = 0\} \supseteq \{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\}\,.$$

Consider now $\psi \in [L^2(O)]^3$ with $\operatorname{div}\psi = 0$, $\gamma_n(\psi) = 0$, and assume that

$$\int_O \psi \cdot \text{curl } A = 0 \quad \forall\ A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\,.$$

Then, by definition, curl $\psi = 0$ in the weak sense. Since also $\operatorname{div}\psi = 0$, $\gamma_n(\psi) = 0$, and since O is simply connected, it follows that $\psi = 0$. We deduce that

$$\{\psi \in [L^2(O)]^3 \,:\, \operatorname{div}\psi = 0,\ \gamma_n(\psi) = 0\} \cap \{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\}^{\perp} = \emptyset\,.$$

This, combined with (A.10), proves (A.9).

We can further show that

(A.11)
$$\{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\} = \{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \operatorname{div}(A) = 0,\ \gamma_t(A) = 0\}\,.$$

As a matter of fact, given $A \in L^2_{\text{curl}}\,(O)$, $\gamma_t(A) = 0$, one finds a unique $a \in W^{1,2}_0(O)$ such that

$$\int_O \nabla a \cdot \nabla \xi = \int_O A \cdot \nabla \xi$$

for all $\xi \in W^{1,2}_0(O)$. Define the vector field $\bar{A} := A - \nabla a$. Then curl $A =$ curl $\bar{A}$, showing that

$$\{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \gamma_t(A) = 0\} \subseteq \{\text{curl } A \,:\, A \in L^2_{\text{curl}}\,(O),\ \operatorname{div}(A) = 0,\ \gamma_t(A) = 0\}\,.$$

In view of (A.9) and (A.11), there exists $A \in L^2_{\text{curl}}\,(O)$ such that $\operatorname{div} A = 0$, $\gamma_t(A) = 0$, and $w =$ curl A.

From the definition (A.8) of w, we deduce that curl A belongs to $[L^q(O)]^3$.

Using the notation (A.3), we can write that $A \in \mathcal{W}_t^{q,\infty}(O)$. Thus, by the embedding result of Lemma A.1, $A \in [L^q(O)]^3$, and

$$\|A\|_{L^q_{\mathrm{curl}}(O)} \le c(q,\, O,\, r)\, \|f\|_{[L^q(O)]^3}\,,$$

proving the estimate.

Finally, we easily verify from (A.8) that curl $A = 0$ in $O \setminus O_c$, which leads to $A \in \mathcal{H}^q(O)$. ∎

Appendix B. Essential properties of the radiation operators. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\overline{\Omega} = \bigcup_{i=0}^m \overline{\Omega_i}$, $m \ge 1$, where $\{\Omega_i\}_{i=0,\ldots,m}$ is a family of bounded Lipschitz domains such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Assume that Ω_0 is enclosed in Ω in the sense that every $x \in \partial\Omega_0$ is an interior point of Ω. Set $\Sigma := \partial\Omega_0$ and $\Gamma = \partial\Omega$.

We introduce the linear integral operator K defined by

$$\big(K(R)\big)(z) := \int_\Sigma w(z,y)\, R(y)\, dS_y \quad \text{for } z \in \Sigma\,, \tag{B.1}$$

where the kernel $w : \Sigma \times \Sigma \to \mathbb{R}$, called the *view factor* in the context of radiation theory, is given by

$$w(z,y) := \begin{cases} \dfrac{\vec{n}(z)\cdot(y-z)\ \vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,\Theta(z,y) & \text{if } z \neq y\,,\\ 0 & \text{if } z = y\,, \end{cases} \tag{B.2}$$

where Θ is the *visibility function* that penalizes the presence of opaque obstacles

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\subset \Omega_0\,,\\ 0 & \text{else.} \end{cases}$$

With the symbol $]z,y[$, we denote the set $conv\{z,y\} \setminus \{z,y\}$, and $\vec{n}$ is a unit normal to Σ.

Under mild assumptions on the geometry and on the emissivity ε (cf. Lemma B.2 (3)), the solution operator of the radiosity equation $(I - (1-\varepsilon)\, K)^{-1}$ is well defined. We can then define another linear operator,

$$G := (I - K)\,(I - (1-\varepsilon)\, K)^{-1}\, \varepsilon. \tag{B.3}$$

We recall some basics about the nonlocal radiation operators K, G. For Banach spaces X, Y, we denote by $\mathcal{L}(X,Y)$ the set of all linear bounded operators from X into Y. The following lemma has been proved in [Han02] for polyhedral surfaces, and in [Tii97] for piecewise $\mathcal{C}^1$ boundaries.

LEMMA B.1. *Let $\Sigma \in \mathcal{C}^1$ piecewise. Let $w : \Sigma \times \Sigma \to \mathbb{R}$ denote the view factor* (B.2)*. Then, for a.a. $z \in \Sigma$,*

$$\int_\Sigma w(z,\, y)\, dS_y \le 1\,.$$

In addition, equality is valid if and only if the enclosure condition (2.5) *is satisfied.*

The following lemma states easily derived, but essential, consequences of Lemma B.1.

LEMMA B.2. *Let the hypotheses of Lemma* B.1 *be valid.*

(1) *For each* $1 \leq p \leq \infty$ *the operator* K *extends to a bounded linear operator from* $L^p(\Sigma)$ *into itself, and the norm estimate* $\|K\|_{\mathcal{L}(L^p(\Sigma),\, L^p(\Sigma))} \leq 1$ *is valid.*
(2) *The operator* K *is positive, in the sense that* $K(f) \geq 0$ *a.e. on* Σ*, whenever* $f \geq 0$ *a.e. on* Σ*. Moreover,* K *is positive semidefinite and self-adjoint in* $L^2(\Sigma)$*.*
(3) *If* $\varepsilon : \Sigma \to \mathbb{R}$ *is such that*

$$0 < \varepsilon_l \leq \varepsilon(z) \leq 1 \quad \text{on } \Sigma\,,$$

then the operator $[I - (1-\varepsilon)K]$ *has an inverse in* $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$*.*
(4) *The operator* G *is positive semidefinite and self-adjoint in* $L^2(\Sigma)$*. The operator* $\mathbb{H} := I - G$ *is positive, self-adjoint in* $L^2(\Sigma)$*, and satisfies for all* $1 \leq p \leq \infty$ *the norm estimate* $\|\mathbb{H}\|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} \leq 1$ *.*
(5) *Assume that* (2.5) *is valid. Then the kernel of the operator* G *consists of the functions constant a.e. on* Σ*. The range of* G *consists of functions with mean-value zero over* Σ*.*

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