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Berlin Mathematical School

Geometry I
WS 09/10

Exercise Sheet 4

Exercise 1: Hyperbolic distance. (4 pts)

Consider the Poincaré disk model $\mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| < 1\}$. Given two points z and w in $\mathbb{D}^2$ prove that

$$|z - w| = \frac{\sinh\left(\frac{d(z,w)}{2}\right)}{\cosh\left(\frac{d(0,z)}{2}\right)\cosh\left(\frac{d(0,w)}{2}\right)},$$

where $d(z, w)$ is the hyperbolic distance between $z, w \in \mathbb{D}^2$ given by

$$d(z, w) := \log\left(\frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}\right).$$

Exercise 2: Regular n-gons. (4 pts)

Let $n \in \mathbb{N}$, $n \geq 3$. Consider n-sided polygons in the hyperbolic plane, all of whose sides are equal and all of whose angles have the value $2\pi/n$. Are there such polygons for any n?

Exercise 3: Hyperbolic quadrilaterals. (4 pts)

Find the fourth angle of a quadrilateral in the hyperbolic plane such that three of its angles are right and the lengths of the sides which join two right angles are a and b. In which limit does this fourth angle approach $\pi/2$?

Exercise 4: Circles and lines in $\mathbb{C}$. (2 pts)

Let C be either a circle or a straight line in $\mathbb{C}$. Show that C has the equation

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \qquad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C}.$$

Exercise 5: Geodesics. (2 pts)

Consider the half-plane model $\mathbb{H}^2_+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The geodesics in $\mathbb{H}^2_+$ are the semicircles orthogonal to the real axis and the vertical straight lines. For each of the following pairs of points in $\mathbb{H}^2_+$ find the equation of the geodesic between them:

1. $(z, w) = (-3 + 4\mathrm{i}, -3 + 5\mathrm{i})$;
2. $(z, w) = (-3 + 4\mathrm{i}, 3 + 4\mathrm{i})$;
3. $(z, w) = (-3 + 4\mathrm{i}, 5 + 12\mathrm{i})$.

Due: Tutorial on 20.11.09