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Geometry I
WS 09/10

Exercise Sheet 3

Exercise 1: Hyperbolic circles. (4 pts)

Define the circle of radius r around point $c \in \mathbb{H}^2$ as

$$C_r(c) := \{x \subset \mathbb{H}^2 \mid d(c,x) = r\},$$

where $d(\cdot,\cdot)$ denotes the hyperbolic distance. Find the length of $C_r(c)$. (Hint: It is enough to consider circles with center $(0,0,1)$. Why?)

Exercise 2: Orthogonal lines. (4 pts)

Let l_1 and l_2 be hyperbolic lines with unit normals n_1 and n_2. Show that there exists a unique hyperbolic line l_3 such that $l_1 \perp l_3$ and $l_2 \perp l_3$ if and only if $|\langle n_1, n_2\rangle| > 1$.

Exercise 3: Hyperbolic triangles. (4 pts)

Consider an hyperbolic triangle with sidelengths a, b, c and interior angles α, β, γ.

1. Prove the hyperbolic angle cosine theorem:

$$\cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

2. Assume that $\gamma = \pi/2$. Prove the following formulas:

$$\cosh c = \cosh a \cosh b, \qquad \text{(Pythagorean theorem)}$$
$$\sin\alpha = \frac{\sinh a}{\sinh c}.$$

Exercise 4: Scalar products. (6 pts)

1. Let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ be the space of $m \times n$ real matrices. By using the isomorphism $\mathrm{Mat}_{m\times n}(\mathbb{R}) \simeq \mathbb{R}^{mn}$ prove that the Euclidean scalar product on $\mathbb{R}^{mn}$ induces a scalar product on $\mathrm{Mat}_{m\times n}(\mathbb{R})$ given by $\langle A, B\rangle = \mathrm{trace}\left(A^T B\right)$, $A, B \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Prove that $\|AB\|_{m,m} \leq \|A\|_{m,n}\|B\|_{n,m}$, where $\|\cdot\|_{m,n}$ is the norm induced by this scalar product on $\mathrm{Mat}_{m\times n}(\mathbb{R})$.

2. Let $\rho_P : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ be the transformation defined by $\rho_P(A) = PAP^{-1}$, with $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and $P \in O(n)$ (ρ_P is called *conjugation* by P). Prove that ρ_P is an isometry of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ with respect to the scalar product $\langle A, B\rangle = \mathrm{trace}\left(A^T B\right)$ from part 1.

Due: Tutorial on 13.11.09