Math. Program., Ser. B
DOI 10.1007/s10107-010-0373-7

FULL LENGTH PAPER

On the dominant of the s-t-cut polytope: Vertices, facets, and adjacency

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Received: 15 August 2008 / Accepted: 10 August 2009

Abstract The natural linear programming formulation of the maximum s-t-flow problem in path variables has a dual linear program whose underlying polyhedron is the dominant $P^{\uparrow}_{s\text{-}t\text{-cut}}$ of the s-t-cut polytope. We present a complete characterization of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ with respect to vertices, facets, and adjacency.

Keywords Flows · Cuts · Polyhedral combinatorics

Mathematics Subject Classification (2000) 90C05 · 90C27 · 90C35 · 90C57

1 Introduction

We study the dominant of the s-t-cut polytope denoted by $P^{\uparrow}_{s\text{-}t\text{-cut}}$. This polyhedron occurs as the set of feasible dual solutions when formulating the maximum s-t-flow problem as a linear program in path variables. The primal pricing and dual separation problem of this pair of linear programs is a shortest s-t-path problem. This is one way to reduce the maximum s-t-flow problem to a series of shortest path computations. This connection has already been pointed out by Ford and Fulkerson [6] in the more general context of the maximum multiflow problem. The dual linear program is the most natural linear programming formulation of the minimum s-t-cut problem. With linear programming duality, this primal-dual pair of linear programs also yields the famous max-flow min-cut theorem [4,5].

Supported by the DFG research center Matheon in Berlin.

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Published online: 09 May 2010

Within the past 50 years, polyhedral combinatorics has proved to be a tremendously successful tool for tackling structural as well as algorithmic problems arising in combinatorial optimization. Polyhedra corresponding to many basic combinatorial optimization problems have been extensively studied in the literature. Surprisingly, and despite its fundamental role in network flow theory and related areas, not much is known about the polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$. The only work we are aware of is by Garg and Vazirani [9,10] who study an extended linear programming formulation of the minimum s-t-cut problem in directed graphs. They characterize vertices and edges of the set of feasible solutions which is a lifted version of $P^{\uparrow}_{s\text{-}t\text{-cut}}$.

In this paper we provide a complete characterization of the vertices, facets, and adjacency structure of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ for undirected as well as directed graphs.

Notation Let $G = (V, E)$ be an undirected or directed graph and $s, t \in V$ two distinct source and target nodes. Throughout this paper we assume that G is connected and, if G is a directed graph, that there is a directed s-t-path in G. Moreover, $\mathscr{P}$ and $\mathscr{C}$ denote the set of all s-t-paths and s-t-cuts, respectively, in G. We use the convention that s-t-paths are simple and that s-t-cuts are defined by

$$\mathscr{C} := \{C \subseteq E \mid C = \delta(U) \text{ for some } U \subseteq V\backslash\{t\} \text{ with } s \in U\}\,.$$

Here $\delta(U)$ denotes the set of edges connecting U to $V\backslash U$—for the case of directed graphs we let $\delta(U) := \delta^+(U) := \{(u, v) \in E \mid u \in U \text{ and } v \in V\backslash U\}$. For arbitrary subsets of nodes $X_1, X_2 \subseteq V$ we let $E(X_1, X_2) = E_G(X_1, X_2)$ denote the set of edges connecting X_1 to X_2 in G. In particular, $\delta(U) = E(U, V\backslash U)$. An s-t-cut $C \in \mathscr{C}$ is called *inclusionwise minimal*, or simply *minimal*, if there is no $C' \in \mathscr{C}$ with $C' \subsetneq C$.

The incidence vector of an s-t-cut $C \in \mathscr{C}$ is denoted by $\chi^C \in \{0, 1\}^E$. Analogously, $\chi^P \in \{0, 1\}^E$ denotes the incidence vector of an s-t-path $P \in \mathscr{P}$. For a subset of nodes $X \subseteq V$ we denote by $G[X]$ the subgraph of G induced by X. We say that X is *connected* if the graph $G[X]$ is connected.

The polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$ With y_P denoting the amount of flow being sent along path $P \in \mathscr{P}$, the problem of finding a maximum s-t-flow obeying edge capacities $c \in \mathbb{R}^E_+$ can be formulated as the following linear program:

$$\begin{aligned} \max \sum_{P \in \mathscr{P}} & y_P \\ \text{s.t.} \sum_{P \in \mathscr{P}} & y_P \chi^P \le c \\ & y \ge 0 \end{aligned}$$

The corresponding dual linear program is:

$$\begin{array}{lll} \min & x^\top c & \\ \text{s.t.} & x^\top \chi^P \ge 1 & \text{for all } P \in \mathscr{P} \\ & x \ge 0 & \end{array} \tag{1}$$

We study the associated polyhedron that is defined by the constraints of the dual linear program. It is not difficult to see that this polyhedron is the dominant of the s-t-cut polytope

$$P_{s\text{-}t\text{-cut}} := \operatorname{conv}\{\chi^C \mid C \in \mathscr{C}\} \subseteq \mathbb{R}^E.$$

That is,

$$\begin{aligned} P^{\uparrow}_{s\text{-}t\text{-cut}} &:= P_{s\text{-}t\text{-cut}} + \mathbb{R}^E_+ \\ &= \{x \in \mathbb{R}^E \mid x \geq 0,\ x^\top \chi^P \geq 1 \text{ for all } P \in \mathscr{P}\}; \end{aligned}$$

see [12, Corollary 13.1b]. The vertices of this polyhedron are integral (0/1) as they are incidence vectors of s-t-cuts. We refer to the book of Schrijver [12, Chapter 13] for further details.[1]

Results from the literature There is a close connection between $P^{\uparrow}_{s\text{-}t\text{-cut}}$ and the dominant of the s-t-path polytope

$$P_{s\text{-}t\text{-path}} := \operatorname{conv}\left\{\chi^P \mid P \in \mathscr{P}\right\}$$

that is given by

$$\begin{aligned} P^{\uparrow}_{s\text{-}t\text{-path}} &:= P_{s\text{-}t\text{-path}} + \mathbb{R}^E_+ \\ &= \{y \in \mathbb{R}^E \mid y \geq 0,\ y^\top \chi^C \geq 1 \text{ for all } C \in \mathscr{C}\}. \end{aligned}$$

The two polyhedra $P^{\uparrow}_{s\text{-}t\text{-path}}$ and $P^{\uparrow}_{s\text{-}t\text{-cut}}$ form a blocking pair of polyhedra. This is one interesting way to prove the max-flow min-cut theorem; see, e.g., [11, Section 9.2] for details.

Chapter 13.1a of Schrijver's book [12][1] gives a complete characterization of vertices, adjacency, and facets of the polyhedron $P^{\uparrow}_{s\text{-}t\text{-path}}$. The vertices of $P^{\uparrow}_{s\text{-}t\text{-path}}$ are precisely the incidence vectors of s-t-paths. Moreover, two vertices are adjacent if and only if the symmetric difference of the corresponding s-t-paths is an undirected circuit consisting of two internally node-disjoint (directed) paths. For $C \in \mathscr{C}$, the inequality $y^\top \chi^C \geq 1$ determines a facet of $P^{\uparrow}_{s\text{-}t\text{-path}}$ if and only if C is an (inclusion-wise) minimal s-t-cut.

Surprisingly, and in contrast to the situation for the polyhedron $P^{\uparrow}_{s\text{-}t\text{-path}}$, much less is known about its blocking polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$. While some information on the vertices and facets of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ can be easily derived from the facets and vertices of its blocking polyhedron $P^{\uparrow}_{s\text{-}t\text{-path}}$, nothing is known about the adjacency of vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$.

[1] While [12, Chapter 13] only deals with the case of directed graphs, it is not difficult to see that the results mentioned here hold for undirected graphs as well.

Garg and Vazirani [9,10] study a variant of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ for the case of directed graphs. Their interest lies on the polyhedron which is represented by the dual of the LP formulation of the maximum s-t-flow problem in edge-variables. This polyhedron lives in $\mathbb{R}^{E\cup V}$ and is given by the following constraints:

$$\begin{aligned} x_e + \pi_u - \pi_v &\geq 0 \qquad \text{for all } e=(u,v)\in E,\\ \pi_t - \pi_s &\geq 1 \\ x,\pi &\geq 0 \end{aligned} \tag{2}$$

It is easy to see that the projection of this polyhedron onto the subspace corresponding to the x-variables is precisely $P^{\uparrow}_{s\text{-}t\text{-cut}}$. In other words, the linear programming formulation of the minimum s-t-cut problem considered by Garg and Vazirani is an extended formulation of the linear program (1) of polynomial size.

Garg and Vazirani show that the vertices of the polyhedron (2) correspond exactly to s-t-cuts in which the s-side is connected. Moreover, two distinct vertices are adjacent if and only if the corresponding s-t-cuts $\delta^+(X_1)$ and $\delta^+(X_2)$ have the property that, up to exchanging X_1 and X_2, the set X_1 is properly contained in X_2 and $X_2\backslash X_1$ is connected. It can be observed that the stated results hold for undirected graphs as well.

A related object that has received considerable attention in the literature is the dominant of the cut polytope which is given by

$$\mathrm{conv}\{\chi^{\delta(U)} \mid \emptyset \neq U \subsetneq V\} + \mathbb{R}^E_+.$$

See, for example, [1–3]. Compared to $P^{\uparrow}_{s\text{-}t\text{-cut}}$, much less is known about the facial structure of this polyhedron which is also considerably more complicated.

Our contribution We give a complete characterization of vertices, facets, and adjacency for the polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$. This closes a surprising gap in the literature on geometric representations of paths, flows, and cuts.

From what is known about the blocking polyhedron $P^{\uparrow}_{s\text{-}t\text{-path}}$, it follows that the inequalities in (1) are all facet-defining for $P^{\uparrow}_{s\text{-}t\text{-cut}}$. For the case of undirected graphs, the vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ correspond exactly to s-t-cuts $\delta(X)$ in which X and $V\backslash X$ are connected. For directed graphs, the vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ correspond exactly to s-t-cuts $\delta^+(X)$ with the following property: For each edge $(u,v)\in\delta^+(X)$ there is an s-u-path in $G[X]$ and a v-t-path in $G[V\backslash X]$. These preliminary observations are presented in Sect. 2.

In Sect. 3 we give a complete characterization of the adjacency of vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$. For the case of undirected graphs, two distinct vertices are adjacent if and only if the corresponding s-t-cuts $\delta(X_1)$ and $\delta(X_2)$ have the property that, up to exchanging X_1 and X_2, the set X_1 is properly contained in X_2 and $X_2\backslash X_1$ is connected; see Sect. 3.1. Notice that this adjacency structure is identical to the one observed by Garg and Vazirani for the lifted polyhedron (2). In Sect. 3.2 we consider directed graphs. Surprisingly, the adjacency structure turns out to be considerably more complicated in this case. The necessary and sufficient condition for the adjacency of two s-t-cuts in the undirected case is only necessary but no longer sufficient in the directed case. We obtain a more elaborate condition which is necessary and sufficient for the adjacency of two s-t-cuts in directed graphs.

2 Vertices and facets

In this section, we characterize the vertices and facets of the polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$. The following observation is an immediate consequence of well known results on the blocking polyhedron $P^{\uparrow}_{s\text{-}t\text{-path}}$.

Observation 1 *A vector x is a vertex of $P^{\uparrow}_{s\text{-}t\text{-}cut}$ if and only if $x = \chi^C$ for some minimal s-t-cut C. For each s-t-path $P \in \mathscr{P}$, the inequality $x^\top \chi^P \geq 1$ determines a facet of $P^{\uparrow}_{s\text{-}t\text{-}cut}$.*

Proof For a blocking pair of polyhedra Q_1, Q_2, Fulkerson [7,8] shows that the vertices of Q_1 correspond exactly to the facets of Q_2 and the facets of Q_1 correspond exactly to the vertices of Q_2; see also [11, Sect. 9.2]. Since $P^{\uparrow}_{s\text{-}t\text{-path}}$ and $P^{\uparrow}_{s\text{-}t\text{-cut}}$ form a blocking pair of polyhedra, the claimed results follow from the characterization of vertices and facets of $P^{\uparrow}_{s\text{-}t\text{-path}}$ discussed above in Sect. 1. □

Since the nonnegativity constraints also determine facets of $P^{\uparrow}_{s\text{-}t\text{-cut}}$, the following constraints from linear program (1) form a minimal description of $P^{\uparrow}_{s\text{-}t\text{-cut}}$:

$$\begin{aligned} x^\top \chi^P &\geq 1 \quad \text{for all } P \in \mathscr{P}, \\ x &\geq 0 \end{aligned}$$

Not surprisingly, the vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ are, in general, highly degenerate. Consider a minimal s-t-cut C and the corresponding vertex χ^C. The number of inequalities $x^\top \chi^P \geq 1$, $P \in \mathscr{P}$, which are tight at vertex χ^C is equal to the number of s-t-paths $P \in \mathscr{P}$ which cross the s-t-cut C exactly once. In the worst case, this number is exponential in the dimension $|E|$ of the polyhedron $P^{\uparrow}_{s\text{-}t\text{-cut}}$. There is, however, a somewhat canonical way of choosing $|E|$ linearly independent inequalities from (1) that define χ^C. This will be discussed in more detail after Corollary 1 below and will turn out to be useful for proving adjacency of certain vertices later on in Sect. 3.

In the remainder of this section we give a more detailed characterization of the vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$ by deriving necessary and sufficient conditions for an s-t-cut to be minimal. In the following, a minimal s-t-cut is also called a *basic s-t-cut*. We start with the case of undirected graphs.

Corollary 1 *For an undirected graph and a point $x \in \mathbb{R}^E$, the following statements are equivalent:*

(i) *x is a vertex of $P^{\uparrow}_{s\text{-}t\text{-}cut}$,*
(ii) *$x = \chi^C$ for some basic s-t-cut C,*
(iii) *$x = \chi^C$ for some s-t-cut $C = \delta(X)$ with X and $V \setminus X$ being connected.*

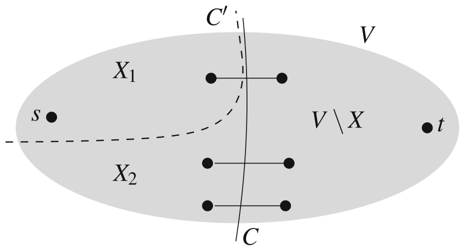

Fig. 1 If X is not connected, then the s-t-cut $C = \delta(X)$ is not basic. For the connected component X_1 of X containing s, the s-t-cut $C' := \delta(X_1)$ is a proper subset of C

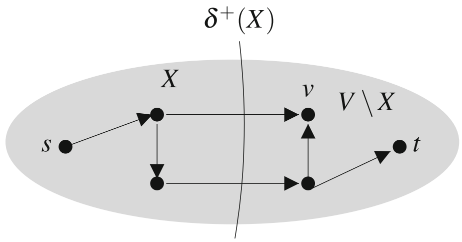

Fig. 2 The sets X and $V\backslash X$ are both connected but the cut $C = \delta^+(X)$ is not basic; notice that $\delta^+(X \cup \{v\}) \subsetneq \delta^+(X)$

As a consequence of property (iii), it is easy to determine a subset of $|E|$ linearly independent inequalities from (1) that define χ^C for a minimal s-t-cut C: Take the nonnegativity constraints corresponding to edges in $E\backslash C$ and, for each $e \in C$, the inequality $x^\top \chi^{P_e} \geq 1$ for some s-t-path P_e with $P_e \cap C = \{e\}$.

Proof It remains to prove the equivalence of (ii) and (iii). That is, an s-t-cut $\delta(X)$ is basic if and only if X and $V\backslash X$ are connected.
(iii)⇒(ii): Let $C = \delta(X)$ with X and $V\backslash X$ connected. We assume by contradiction that the s-t-cut $C = \delta(X)$ is not basic. That is, there exists an edge $e = uv \in \delta(X)$ and an s-t-cut $C' \subseteq C\backslash\{e\}$. Because X is connected and $s, u \in X$, there is an s-u-path that does not intersect $\delta(X) \supset C'$. Thus, u is also on the s-side of cut C'. Similarly, v is on the t-side of C'. This yields the contradiction $e = uv \in C'$.
(ii)⇒(iii): We assume that X is not connected and prove that C is not basic in this case; the other case that $V\backslash X$ is not connected is symmetric. Let X_1 and X_2 be nonempty such that $X = X_1 \,\dot\cup\, X_2$, $s \in X_1$, and $E(X_1, X_2) = \emptyset$; see Fig. 1. Since G is connected, $E(X_2, V\backslash X) \neq \emptyset$. Thus, the s-t-cut

$$C' := \delta(X_1) = C\backslash E(X_2, V\backslash X) \subsetneq C$$

shows that C is not basic. This concludes the proof. □

For the case of directed graphs, the equivalence of the minimality of an s-t-cut $\delta^+(X)$ and the connectivity of the sets X and $V\backslash X$ no longer holds; a small counterexample is given in Fig. 2. In the following we present a stronger condition.

Corollary 2 *For a directed graph and a point $x \in \mathbb{R}^E$, the following statements are equivalent:*

(i) *x is a vertex of $P^{\uparrow}_{s\text{-}t\text{-}cut}$,*

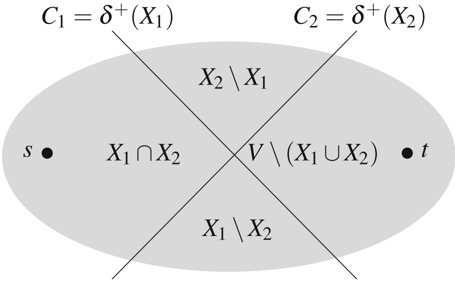

Fig. 3 $C_1 = \delta(X_1)$ and $C_2 = \delta(X_2)$ are crossing s-t-cuts if $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$ (i.e., $X_1 \backslash X_2 \neq \emptyset$ and $X_2 \backslash X_1 \neq \emptyset$). The s-t-cuts $\delta(X_1 \cap X_2)$ and $\delta(X_1 \cup X_2)$ are noncrossing

(ii) $x = \chi^C$ *for some basic s-t-cut C,*

(iii) $x = \chi^C$ *for some s-t-cut $C = \delta^+(X)$ with the following property: for each edge $e = (u, v) \in C$ there exists a directed s-u-path in $G[X]$ and a directed v-t-path in $G[V \backslash X]$.*

Proof It remains to prove the equivalence of (ii) and (iii). That is, an s-t-cut $C = \delta^+(X)$ is basic if and only if for each edge $e = (u, v) \in C$ there exists a directed s-u-path in $G[X]$ and a directed v-t-path in $G[V \backslash X]$. The proof of the direction (iii)⇒(ii) is identical to the corresponding part in the proof of Corollary 1.
(ii)⇒(iii): Suppose that there exists an edge $e = (u, v) \in C$ such that there exists no directed s-u-path in $G[X]$—the case where there is no directed v-t-path in $G[V \backslash X]$ is symmetric. Let

$$Y := \{w \in X \mid \text{there is a directed } s\text{-}w\text{-path in } G[X]\} \subseteq X \backslash \{u\}.$$

By definition, $E(Y, X \backslash Y) = \emptyset$. We conclude that the s-t-cut $\delta^+(Y) \subseteq C \backslash \{e\}$ is a proper subset of C. In particular, C is not basic. This concludes the proof. □

3 Adjacency

We characterize when two basic s-t-cuts C_1 and C_2 correspond to adjacent vertices χ^{C_1} and χ^{C_2} of $P^{\uparrow}_{s\text{-}t\text{-cut}}$. In this case we also say that the two basic s-t-cuts C_1 and C_2 are *adjacent*. The case of undirected graphs is treated in Sect. 3.1. Results for the more complicated case of directed graphs are presented in Sect. 3.2.

3.1 Undirected graphs

Throughout this section let $G = (V, E)$ be an undirected graph.

Definition 1 Let $C_1 = \delta(X_1)$ and $C_2 = \delta(X_2)$ be two s-t-cuts in G. We say that C_1 and C_2 are *crossing* if $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$, i.e., $X_1 \backslash X_2 \neq \emptyset$ and $X_2 \backslash X_1 \neq \emptyset$. Otherwise, C_1 and C_2 are called *noncrossing*.

Figure 3 illustrates the idea of crossing cuts. It is not difficult to show that crossing basic s-t-cuts are not adjacent.

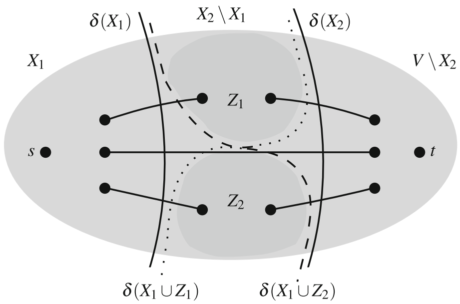

Fig. 4 If $X_2 \backslash X_1$ can be decomposed into subsets Z_1 and Z_2 with $E(Z_1, Z_2) = \emptyset$, then the s-t-cuts $\delta(X_1)$ and $\delta(X_2)$ are not adjacent since $\chi^{\delta(X_1)} + \chi^{\delta(X_2)} = \chi^{\delta(X_1 \cup Z_1)} + \chi^{\delta(X_1 \cup Z_2)}$

Lemma 1 *Let $\delta(X_1)$ and $\delta(X_2)$ be two basic s-t-cuts. If $\delta(X_1)$ and $\delta(X_2)$ are crossing, then they are not adjacent.*

Proof Let $\delta(X_1)$ and $\delta(X_2)$ be crossing basic s-t-cuts. Assume by contradiction that $\chi^{\delta(X_1)}$ and $\chi^{\delta(X_2)}$ are adjacent vertices of $P^{\uparrow}_{s\text{-}t\text{-cut}}$. Then there exists a vector $c \in \mathbb{R}^E_+$ such that $\delta(X_1)$ and $\delta(X_2)$ are the only two minimum cuts with respect to c. By submodularity of the cut function we know that

$$c(\delta(X_1 \cap X_2)) + c(\delta(X_1 \cup X_2)) \leq c(\delta(X_1)) + c(\delta(X_2)).$$

In particular, $\delta(X_1 \cap X_2)$ and $\delta(X_1 \cup X_2)$ are minimum s-t-cuts as well. But since X_1 and X_2 are both connected, $\delta(X_1 \cap X_2)$ is different from $\delta(X_1)$ and $\delta(X_2)$. This is a contradiction and concludes the proof. □

We have shown that adjacent basic s-t-cuts are noncrossing. Consequently, the cut-defining node set of a basic cut is contained in or contains the cut-defining node set of an adjacent basic cut. Now we will have a closer look at these cut-defining node sets.

Lemma 2 *Let $\delta(X_1)$ and $\delta(X_2)$ be two adjacent basic s-t-cuts with $X_1 \subsetneq X_2$. Then, $X_2 \backslash X_1$ is connected.*

Proof Suppose by contradiction that there exist two nonempty disjoint subsets $Z_1, Z_2 \subseteq X_2 \backslash X_1$ with $Z_1 \dot\cup Z_2 = X_2 \backslash X_1$ and $E(Z_1, Z_2) = \emptyset$; see Fig. 4 for an illustration. Then

$$\chi^{\delta(X_1)} + \chi^{\delta(X_2)} = \chi^{\delta(X_1 \cup Z_1)} + \chi^{\delta(X_1 \cup Z_2)}.$$

This leads to the same contradiction as in the proof of Lemma 1. □

We have shown that the adjacency of two basic s-t-cuts implies that they are noncrossing and that the set difference of the cut-defining node sets is connected. Now we show the reverse direction.

Lemma 3 *Let $C_1 = \delta(X_1)$ and $C_2 = \delta(X_2)$ be two basic s-t-cuts with $X_1 \subsetneq X_2$. If $X_2 \backslash X_1$ is connected, then C_1 and C_2 are adjacent.*

Proof It suffices to find $|E|-1$ linearly independent inequalities from the system

$$\begin{aligned} x^\top \chi^P &\geq 1 \quad \text{for all } P \in \mathscr{P}, \\ x &\geq 0 \end{aligned}$$

that are simultaneously tight for $x=\chi^{C_1}$ and for $x=\chi^{C_2}$. Obviously,

$$\left(\chi^{C_1}\right)_e = \left(\chi^{C_2}\right)_e = 0 \qquad \text{for each } e \in E\backslash(C_1 \cup C_2).$$

The nonnegativity constraints corresponding to edges in $E\backslash(C_1 \cup C_2)$ build the first part of the solution.

It remains to find $|C_1 \cup C_2|-1$ inequalities corresponding to s-t-paths that are tight for $x=\chi^{C_1}$ and for $x=\chi^{C_2}$. For each $e \in C_1 \cap C_2$ let P_e be an s-t-path with the property that

$$P_e \cap C_1 = P_e \cap C_2 = \{e\}. \tag{3}$$

Notice that such an s-t-path exists since X_1 and $V\backslash X_2$ are connected; see Corollary 1 (iii). Due to (3), it holds that

$$\left(\chi^{C_1}\right)^\top \chi^{P_e} = \left(\chi^{C_2}\right)^\top \chi^{P_e} = 1 \quad \text{for each } e \in C_1 \cap C_2.$$

The corresponding tight constraints constitute the second part of the solution.

It remains to find another $|C_1\backslash C_2| + |C_2\backslash C_1| - 1$ tight inequalities. Notice that $C_1\backslash C_2$ and $C_2\backslash C_1$ cannot be empty since C_1 and C_2 are basic and distinct. Consider the complete bipartite graph H on the set of nodes $(C_1\backslash C_2) \,\dot\cup\, (C_2\backslash C_1)$ and a spanning tree T of H. Notice that T contains $|C_1\backslash C_2|+|C_2\backslash C_1|-1$ edges; the edge set of T is denoted by $E(T)$. For each $e_1e_2 \in E(T)$ with $e_1 \in C_1\backslash C_2$ and $e_2 \in C_2\backslash C_1$, let $P_{e_1e_2}$ be an s-t-path with the property that

$$P_{e_1e_2} \cap C_1 = \{e_1\} \quad \text{and} \quad P_{e_1e_2} \cap C_2 = \{e_2\}. \tag{4}$$

Such an s-t-path exists since X_1, $X_2\backslash X_1$, and $V\backslash X_2$ are connected. Moreover, due to (4), it holds that

$$\left(\chi^{C_1}\right)^\top \chi^{P_{e_1e_2}} = \left(\chi^{C_2}\right)^\top \chi^{P_{e_1e_2}} = 1 \quad \text{for each } e_1e_2 \in E(T).$$

The corresponding tight constraints constitute the third and last part of the solution.

It remains to show that the chosen $|E|-1$ constraints are linearly independent. This can easily be seen as follows. Choose an arbitrary edge $e_0 \in C_1\backslash C_2$ and assume that the tree T is rooted at e_0. We describe a sorting of the remaining edges $e \in E\backslash\{e_0\}$ and a sorting of the chosen tight constraints with the following property: the resulting $(|E|-1) \times (|E|-1)$-matrix whose columns correspond to edges $e \in E\backslash\{e_0\}$ and whose rows correspond to tight constraints is lower triangular with diagonal entries all one.

– First take the edges $e \in E \setminus (C_1 \cup C_2)$ in any order. The same order is used for the corresponding tight nonnegativity constraints. Thus, the upper left corner of the final matrix is an identity matrix.
– Then, add the edges $e \in C_1 \cap C_2$ in any order. Use the same order for the tight constraints corresponding to s-t-paths P_e, $e \in C_1 \cap C_2$. The diagonal block corresponding to this second part is again an identity matrix. Notice that there can be additional non-zero entries in previous columns corresponding to edges $e \in E \setminus (C_1 \cup C_2)$.
– Finally, sort the edges $e \in (C_1 \setminus (C_2 \cup \{e_0\})) \dot\cup (C_2 \setminus C_1)$ in order of nondecreasing distance from the root e_0 in T. Sort the tight constraints corresponding to s-t-paths P_{e_1,e_2}, $e_1 e_2 \in E(T)$, accordingly. More precisely, if we assume that the edges $e_1 e_2 \in E(T)$ are directed away from the root e_0, we sort them according to the given sorting of their head nodes e_2. In particular, we get a lower triangular matrix and also the diagonal entries of this last block are all one.

This concludes the proof of the theorem. □

The main result of this section is summarized in the following theorem.

Theorem 1 *For undirected graphs, two basic s-t-cuts $\delta(X_1)$ and $\delta(X_2)$ are adjacent if and only if $X_1 \subsetneq X_2$ and $X_2 \setminus X_1$ is connected, or $X_2 \subsetneq X_1$ and $X_1 \setminus X_2$ is connected.*

3.2 Directed graphs

Throughout this section let $G = (V, E)$ be a directed graph. As in the case of undirected graphs, we need the concept of crossing s-t-cuts. However, in the directed case, the definition is slightly more complicated.

Definition 2 Let C_1 and C_2 be two s-t-cuts in the directed graph G. We say that C_1 and C_2 are *crossing* if $X_1 \nsubseteq X_2$ and $X_2 \nsubseteq X_1$ for all $X_1, X_2 \subseteq V$ with $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$. Otherwise, C_1 and C_2 are *noncrossing*.

In other words, in order for two s-t-cuts to cross, for *any* pair of cut-defining node sets one set must not contain or be contained in the other one. The example in Fig. 5 illustrates why this more complicated definition is essential in the case of directed graphs.

It is easy to observe that an equivalent definition of crossing s-t-cuts is as follows. For an s-t-cut C, let $X_C \subseteq V$ be the inclusionwise minimal subset of nodes

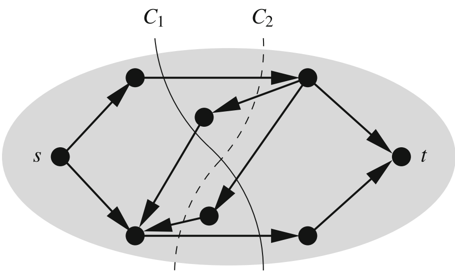

Fig. 5 The two depicted s-t-cuts C_1 and C_2 seem to cross but are indeed identical. Both just contain the two horizontal edges that go from left to right

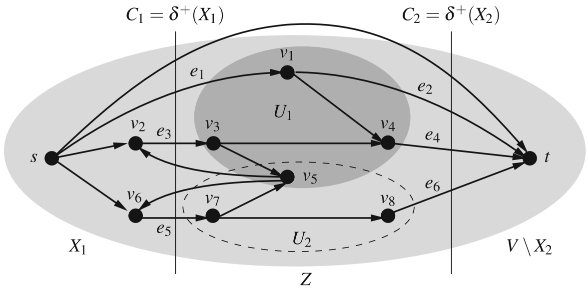

Fig. 6 An example of a directed graph with two s-t-cuts

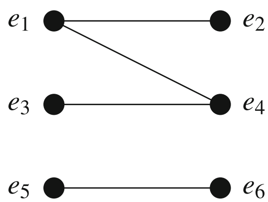

Fig. 7 The undirected bipartite graph H corresponding to the two s-t-cuts depicted in Fig. 6

with $C = \delta^+(X_C)$; notice that X_C is the set of nodes which can be reached from s via a directed path not containing edges from C. Then two s-t-cuts C_1 and C_2 are crossing if and only if $X_{C_1} \nsubseteq X_{C_2}$ and $X_{C_2} \nsubseteq X_{C_1}$.

Lemma 4 *Let $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$ be two basic s-t-cuts. If C_1 and C_2 are crossing, then they are not adjacent.*

Proof The proof is almost identical to the proof of Lemma 1. The only difference is the argument for $\delta^+(X_1 \cap X_2)$ being distinct from $\delta^+(X_1)$ and $\delta^+(X_2)$. For directed graphs this follows directly from the refined definition of crossing s-t-cuts; see Definition 2. □

As in the undirected case, we show now that the adjacency of two (noncrossing) basic s-t-cuts implies that the node set in-between is connected.

Lemma 5 *Let $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$ be two adjacent basic cuts with $X_1 \subsetneq X_2$. Then $X_2 \backslash X_1$ is connected.*

Proof The proof is identical to the proof of Lemma 2. □

For the case of directed graphs, however, we derive an even stronger result. Given two noncrossing s-t-cuts we define a bipartite graph as follows; an illustrating example is given in Figs. 6 and 7.

Definition 3 Let $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$ be two s-t-cuts with $X_1 \subsetneq X_2$ and let $Z := X_2 \backslash X_1$. Let H be the (undirected) bipartite graph with node set $V(H) := (C_1 \backslash C_2) \cup (C_2 \backslash C_1)$ and the following edge set: $e_1 \in C_1 \backslash C_2$ and $e_2 \in C_2 \backslash C_1$ are connected by an edge $e_1 e_2$ in H if and only if there is a directed head(e_1)-tail(e_2)-path in $G[Z]$.

Lemma 6 *If $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$ are adjacent basic s-t-cuts, then H is connected.*

Proof By contradiction assume that H is not connected, i.e., there exist disjoint non-empty subsets $U_1', U_2' \subseteq V(H)$ with $V(H) = U_1' \mathbin{\dot\cup} U_2'$ and $E_H(U_1', U_2') = \emptyset$. As in Definition 3, we set $Z := X_2 \setminus X_1$. For $i = 1, 2$, let $U_i \subseteq Z$ denote the set of nodes that can be reached in $G[Z]$ from a head-node of some edge $e \in C_1 \cap U_i'$ via a directed path. That is,

$$U_i := \{v \in Z \mid \exists\, e \in C_1 \cap U_i' \colon \exists \text{ directed head}(e)\text{-}v\text{-path in } G[Z]\}.$$

Before we proceed with the proof, we shortly discuss the definition of U_1 and U_2 for the example depicted in Figs. 6 and 7. The graph H in Fig. 7 is not connected and we can set $U_1' := \{e_1, e_2, e_3, e_4\}$ and $U_2' := \{e_5, e_6\}$. Thus, $U_1 = \{v_1, v_3, v_4, v_5\}$ and $U_2 = \{v_5, v_7, v_8\}$; see Fig. 6.

We show that the s-t-cuts $C_3 := \delta^+(X_1 \cup U_1)$ and $C_4 := \delta^+(X_1 \cup U_2)$ are different from C_1 and C_2 and satisfy

$$\chi^{C_1} + \chi^{C_2} = \chi^{C_3} + \chi^{C_4}. \tag{5}$$

This leads to the same contradiction as in the proof of Lemma 1. In order to prove (5), we show that

$$C_3 = (C_1 \cap C_2) \mathbin{\dot\cup} (C_1 \cap U_2') \mathbin{\dot\cup} (C_2 \cap U_1') \tag{6}$$

and

$$C_4 = (C_1 \cap C_2) \mathbin{\dot\cup} (C_1 \cap U_1') \mathbin{\dot\cup} (C_2 \cap U_2'). \tag{7}$$

Since $(C_1 \cup C_2) \setminus (C_1 \cap C_2) = U_1' \mathbin{\dot\cup} U_2'$, Eq. (5) follows immediately from (6) and (7). Notice that (6) and (7) hold for the example depicted in Figs. 6 and 7. Here we get $C_3 = \{e_2, e_4, e_5\}$ and $C_4 = \{e_1, e_3, e_6\}$.

It remains to prove (6)—the proof of (7) is symmetric. By definition of C_3 we get

$$\begin{aligned} C_3 &= \delta^+(X_1 \mathbin{\dot\cup} U_1) = E(X_1 \mathbin{\dot\cup} U_1, (V \setminus X_2) \mathbin{\dot\cup} (Z \setminus U_1)) \\ &= \underbrace{E(X_1, V \setminus X_2)}_{\overset{!}{=} C_1 \cap C_2} \mathbin{\dot\cup} \underbrace{E(X_1, Z \setminus U_1)}_{\overset{!}{=} C_1 \cap U_2'} \mathbin{\dot\cup} \underbrace{E(U_1, V \setminus X_2)}_{\overset{!}{=} C_2 \cap U_1'} \mathbin{\dot\cup} \underbrace{E(U_1, Z \setminus U_1)}_{\overset{!}{=} \emptyset}. \end{aligned}$$

We thus have to prove the four equations "$\overset{!}{=}$". Again, for the example depicted in Figs. 6 and 7, those equations hold.

An illustration of the general situation is given in Fig. 8. It is clear that $E(X_1, V \setminus X_2) = C_1 \cap C_2$. Moreover, $E(U_1, Z \setminus U_1) = \emptyset$ by definition of U_1.

We now show that $E(X_1, Z \setminus U_1) = C_1 \cap U_2'$. Since head$(e) \in U_1$ for each $e \in C_1 \cap U_1'$ by definition of U_1, it remains to prove the following claim.

Claim head$(e) \notin U_1$ for each $e \in C_1 \cap U_2'$.

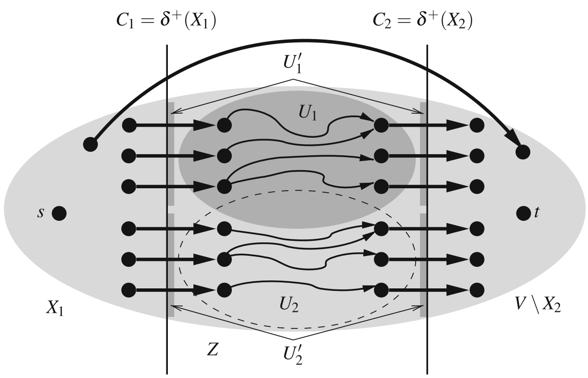

Fig. 8 An illustration of the proof of Eq. (6)

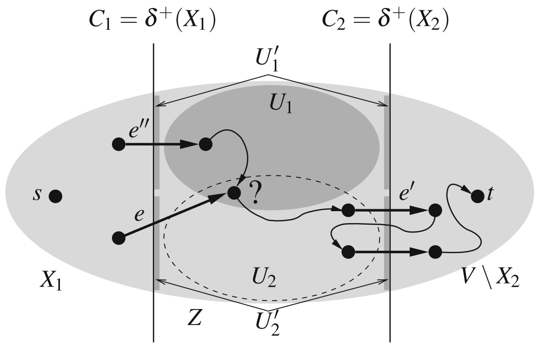

Fig. 9 An illustration of the proof that head$(e) \notin U_1$ if $e \in C_1 \cap U_2'$

Since C_1 is a basic cut, it follows from Corollary 2 (iii) that there is a directed path in $G[V \setminus X_1]$ from head(e) to the target node t. This path crosses the cut C_2. Let e' be the first edge on this path that is contained in C_2; see Fig. 9 for an illustration. Then, by Definition 3, e and e' are connected by an edge in H and thus $e' \in U_2'$ as well. If, by contradiction, head(e) was contained in U_1, there must exist an edge $e'' \in C_1 \cap U_1'$ and a directed head(e'')-head(e)-path in $G[Z]$. Concatenating this path with the directed path from head(e) to tail(e') yields a directed head(e'')-tail(e')-path in $G[Z]$. Thus, by Definition 3, $e''e'$ is an edge in H which is a contradiction since $e'' \in U_1'$ and $e' \in U_2'$ are in different connected components of H. This concludes the proof of the claim.

Finally, $E(U_1, V \setminus X_2) = C_2 \cap U_1'$ since U_1 contains tail(e) for each $e \in C_2 \cap U_1'$ (again due to Corollary 2 (iii)) but U_1 does not contain tail(e) for any $e \in C_2 \cap U_2'$. This concludes the proof of the lemma. $\square$

Next we will show the reverse direction of Lemma 6.

Lemma 7 *Let $C_1 = \delta^+(X_1)$ and $C_2 = \delta^+(X_2)$ be two basic cuts with $X_1 \subsetneq X_2$. If the bipartite graph H is connected, then C_1 and C_2 are adjacent.*

Proof The proof is almost identical to the proof of Lemma 3. We therefore only give a rough sketch; all remaining details are analogous to the proof of Lemma 3.

Again, we have to find $|E|-1$ linearly independent inequalities from the system

$$x^\top \chi^P \geq 1 \quad \text{for all } P \in \mathscr{P},$$
$$x \geq 0$$

that are simultaneously tight for $x=\chi^{C_1}$ and for $x=\chi^{C_2}$.

As above, the nonnegativity constraints corresponding to edges in $E\backslash(C_1 \cup C_2)$ build the first part of the solution. The second part consists again of tight path-constraints, one for each $e \in C_1 \cap C_2$. More precisely, for each $e \in C_1 \cap C_2$, let P_e be a directed s-t-path with the property that $P_e \cap C_1 = P_e \cap C_2 = \{e\}$. In the directed case, such an s-t-path exists due to Corollary 2 (iii).

For the third part of the solution we consider the connected bipartite graph H from Definition 3 and a spanning tree T of H. For each $e_1e_2 \in E(T)$ with $e_1 \in C_1\backslash C_2$ and $e_2 \in C_2\backslash C_1$, let $P_{e_1e_2}$ be an s-t-path with the property that $P_{e_1e_2} \cap C_1 = \{e_1\}$ and $P_{e_1e_2} \cap C_2 = \{e_2\}$. Such an s-t-path exists by definition of H and Corollary 2 (iii).

As in the proof of Lemma 3 it can be shown that the described $|E|-1$ constraints are linearly independent. This concludes the proof. □

The main result of this section is summarized in the following theorem.

Theorem 2 *For directed graphs, two basic s-t-cuts $\delta(X_1)$ and $\delta(X_2)$ are adjacent if and only if $X_1 \subsetneq X_2$ (or $X_2 \subsetneq X_1$) and the bipartite graph H from Definition 3 is connected.*

Acknowledgments The authors are much indebted to two anonymous referees whose helpful comments led to an improved presentation of the paper.

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