Weierstrass representation of discrete surfaces

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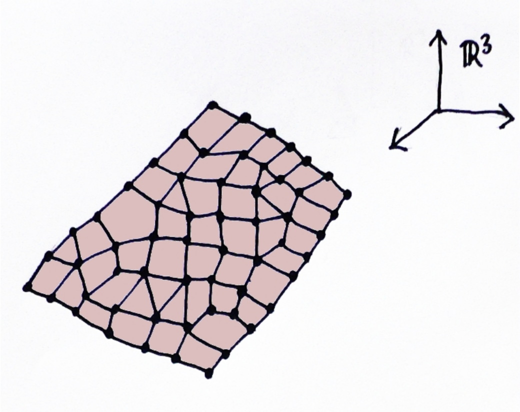
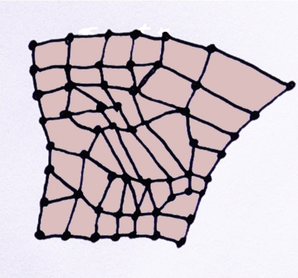
Joint work with C. Bohle

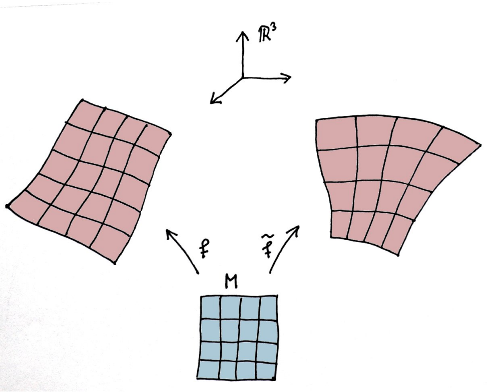
Conformality of discrete surfaces





When should one consider two discrete surfaces with the same combinatorics as conformal to each other?

Conformal smooth surfaces



$\mathbb{R}^3 = span\{i, j, k\} \subset \mathbb{H}$

f and $\tilde{f}$ are conformal to each other

$$\Leftrightarrow \exists \quad \lambda : M \to \mathbb{H}$$

such that

$$d\tilde{f} = \bar{\lambda} df \lambda.$$

Spin transformations

- Find $\tilde{f}$ conformal to f from λ if

$$0 = d(\bar{\lambda} df \lambda) = d\bar{\lambda} \wedge df \lambda - \bar{\lambda} df \wedge d\lambda$$

- Special solutions if

$$df \wedge d\lambda = 0$$

- Special property of these solutions:

$$\tilde{H}|d\tilde{f}| = H|df|$$

Special case: Minimal surfaces

- $M = \mathbb{C}, \qquad f(z) = jz$
- $\lambda = \phi + \psi j, \qquad \phi, \psi : \mathbb{C} \to \mathbb{C}.$

$\rightsquigarrow$

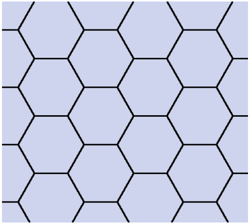
- $df \wedge d\lambda = 0 \quad \Leftrightarrow \quad \phi, \psi$ holomorphic
- $d\tilde{f} = 2i\,\mathrm{Im}(\phi\psi dz) + j(\psi^2 dz - \bar{\psi}^2 d\bar{z})$

Weierstrass representation!

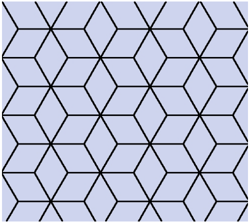
Discrete Spin structures: Combinatorics

- Start with an arbitrary cell decomposition $(\hat{V}, \hat{E}, \hat{F})$ of an oriented surface $\hat{M}$.
- Construct the quad-surface M with vertices and faces

$$V = \hat{V} \cup \hat{F} \qquad F = \hat{E}.$$

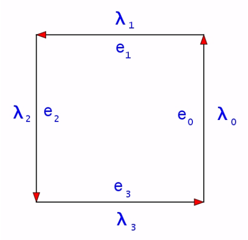


$\rightsquigarrow$



Discrete Spinors: Analysis

- Assume M is realized as a discrete surface in $\mathbb{R}^3$, i.e we have a function $f : V \longrightarrow \mathbb{R}^3$.
- Then on each quad $q \in F$ the differential df is represented by four edge vectors e_0, e_1, e_2, e_3.



- Spinors are functions

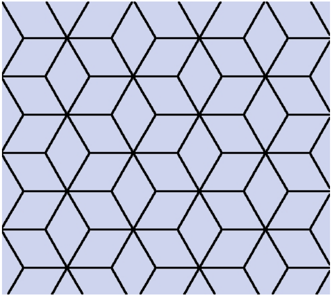
$$\lambda : E \longrightarrow \mathbb{H}$$

(see talks by C. Mercat and D. Cimasoni).

Discrete spin structures

For each quad $q \in F$ choose a two dimensional subspace $U_q \subset \mathbb{H}^4$ containing $(1,1,1,1)$ such that if the four λ-values on the edges of q lie in U_q then

$$\bar{\lambda}_0 e_0 \lambda_0 + \bar{\lambda}_1 e_1 \lambda_1 + \bar{\lambda}_2 e_2 \lambda_2 + \bar{\lambda}_3 e_3 \lambda_3 = 0.$$



- λ is called *holomorphic* if this holds for all quads $q \in F$.
- The choice of these U_q defines a holomorphic spin structure on M.
- There is a canonical choice.

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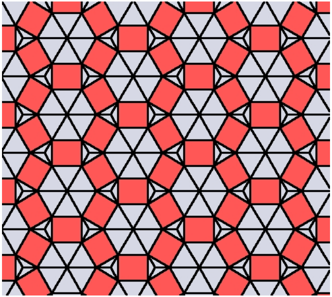

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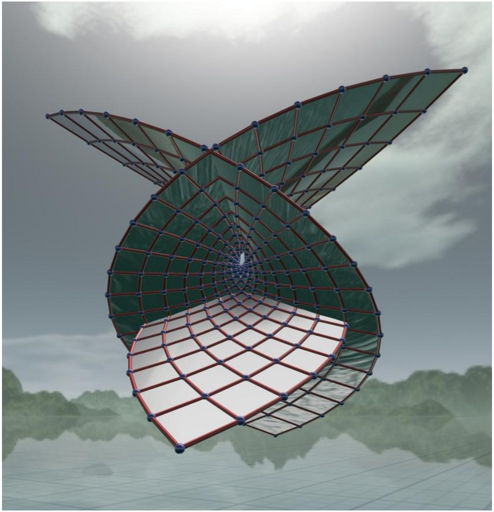
Applications



- Minimal surfaces with arbitrary quad-combinatorics
- Spin-transformations of the cylinder $\rightsquigarrow$ discrete tori for which $H^2|df|^2$ is a flat metric
- Discrete Dirac spheres: $H^2|df|^2$ has constant curvature.
- Discrete isothermic surfaces without curvature line grid

Enneper