Polygonal smoke

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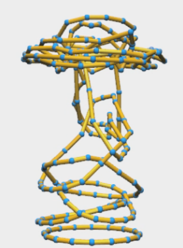
jointly with Steffen Weißmann

DFG Research Center MATHEON
Mathematics for key technologies

Smoke

Smoke rings

- Claim: The whole smoke can be modelled as a collection of entangled smoke rings.
- Smoke rings move on their own, but they also interact.
- Interaction can even imply a topology change (reconnection)



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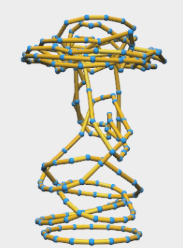

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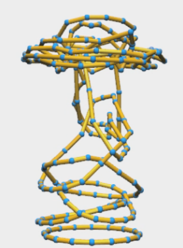
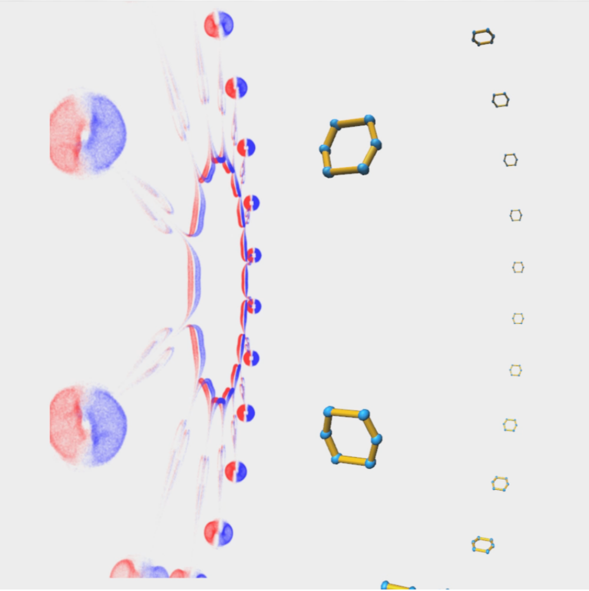


Vortex filaments

Colliding vortex rings



- A velocity vector field v is uniquely determined by its vorticity

$$\omega = \operatorname{curl} v$$

- v is given by the Biot-Savart formula:

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \omega(y) \times \frac{x - y}{|x - y|^3} \, dy$$

- In an ideal fluid ω flows with the velocity v it generates:

$$\dot{\omega} = \operatorname{curl}(v \times \omega) = [\omega, v]$$

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- Just swept along with the flow
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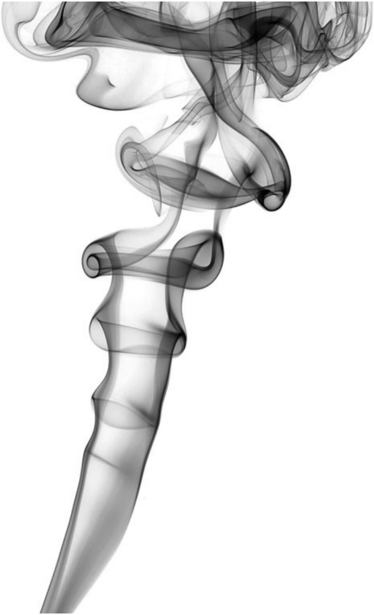

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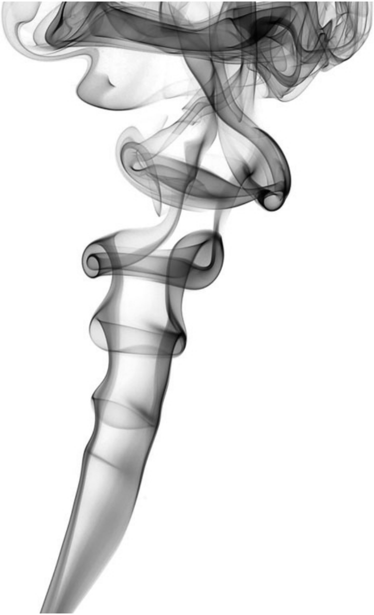

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- Extends back to where it took off
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Vortex sheet roll up

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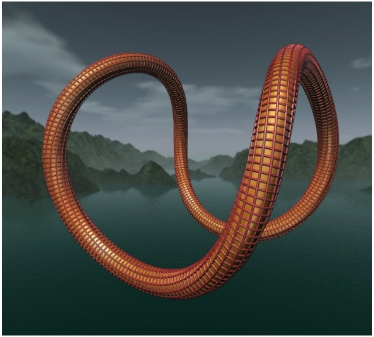
- Suppose all vorticity is concentrated in a small tube of radius R around a space curve γ

  (like water flowing through the tube).

- Then away from γ the velocity field is given by

$$v(x) = K \oint \frac{\gamma' \times (x - \gamma)}{|x - \gamma|^3}$$



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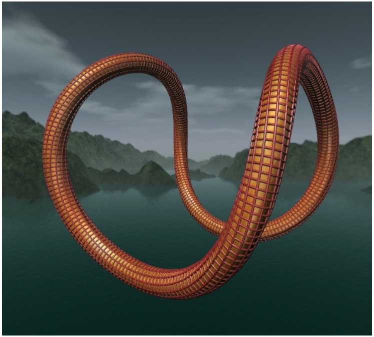

- Evolution of γ: Evaluate velocity v on $\gamma \rightsquigarrow$

$$\dot{\gamma} \approx C_f \; K \; \log(R) \; \gamma' \times \gamma''$$

- Scale down K as $R \to 0 \rightsquigarrow$ smoke ring flow
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- Curve moves orthogonal to its osculating plane.

- Speed is proportional to the curvature.

- Length is constant.

  Area vector is constant.

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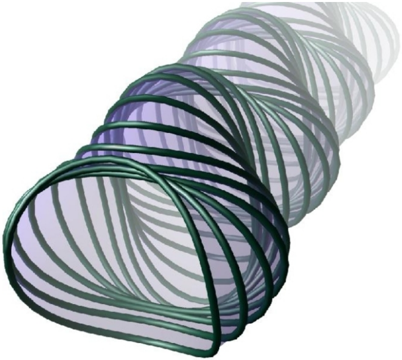

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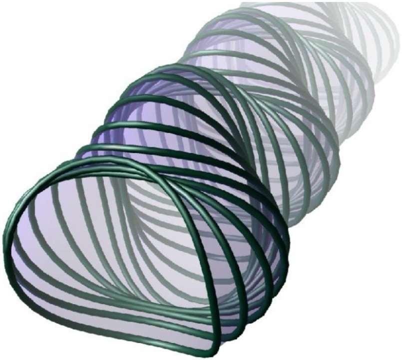

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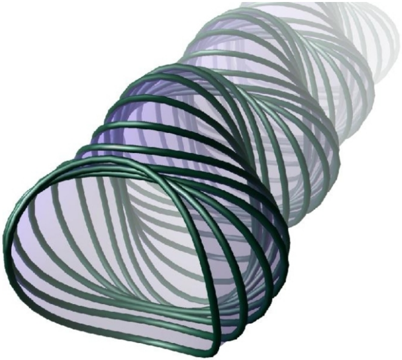

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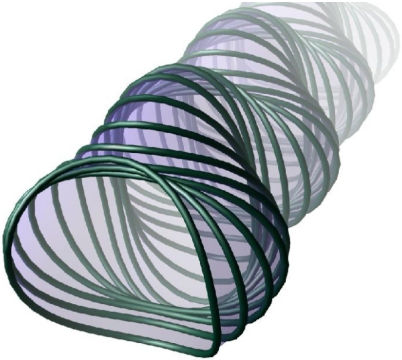

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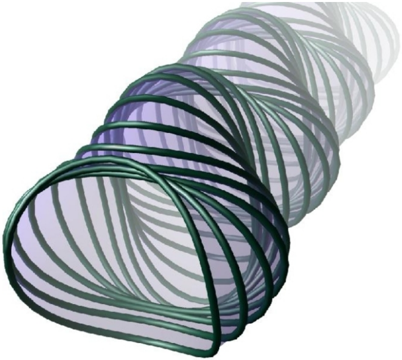

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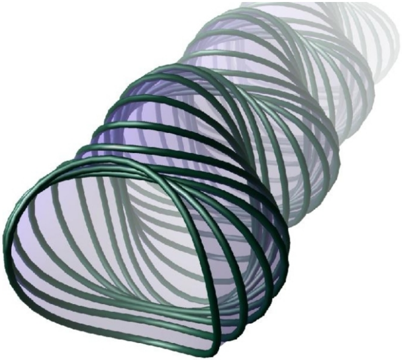


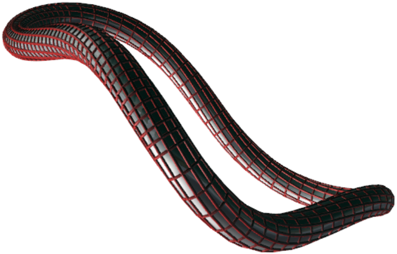
- For a space curve:

$$A = \frac{1}{2} \oint \gamma \times \gamma'$$

- For a closed polygon:

$$A = \frac{1}{2} \sum_{i=1}^{n} \gamma_i \times \gamma_{i+1}$$

- b a unit vector $\rightsquigarrow \langle A, b \rangle$ is the algebraic area of the orthogonal projection of γ onto a plane with normal vector b



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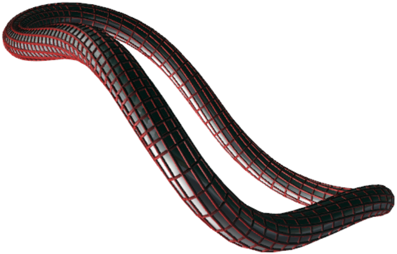

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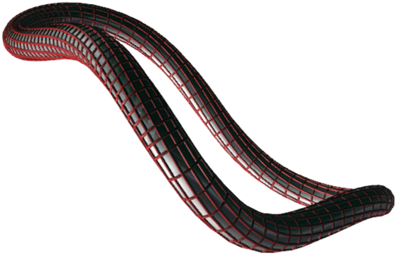

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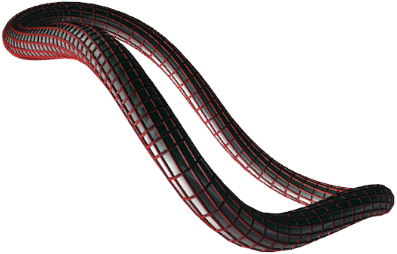
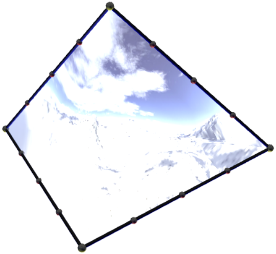


- A quadrilateral $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $\mathbb{R}^3$ is called a *skew parallelogram* of twist τ if the difference vector

$$V = \frac{\gamma_3 + \gamma_1}{2} - \frac{\gamma_2 + \gamma_0}{2}$$

between the centers of its diagonals is a multiple of its area vector:

$$V = \tau A$$



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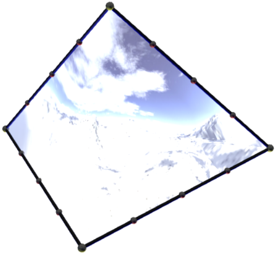

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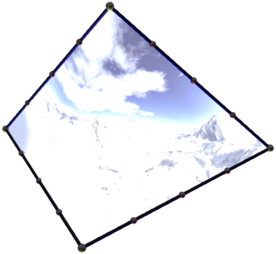


- Opposite sides of a skew parallelogram have the same length.

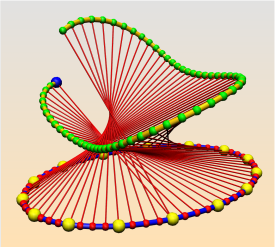
- A polygon η is called a Darboux transform with rod-length ρ and twist τ if all quadrilaterals

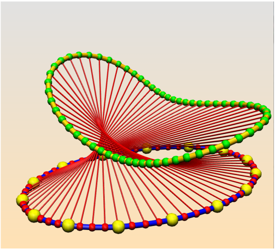
$$\begin{matrix} \eta_i & \eta_{i+1} \\ \gamma_i & \gamma_{i+1} \end{matrix}$$

are skew parallelograms with

$$|\eta_i - \gamma_i| = \rho \text{ and twist } \tau.$$

- For generic ρ, τ every closed polygon has exactly two closed Darboux transforms.
- Hoffmann 2000





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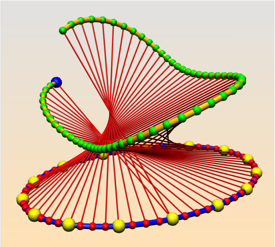

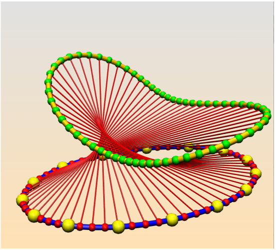

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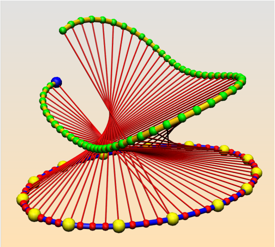

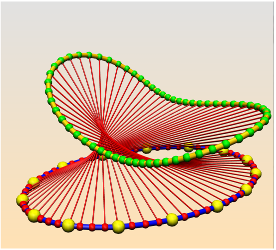

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- Closed Darboux transforms of a closed polygon have the same:

  length

  Area vector

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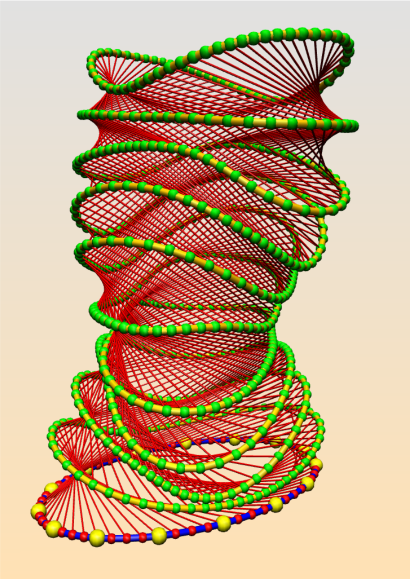
- Iterating Darboux transforms with the same ρ and τ gives "discrete Lund-Regge surfaces" (Schief 2007)



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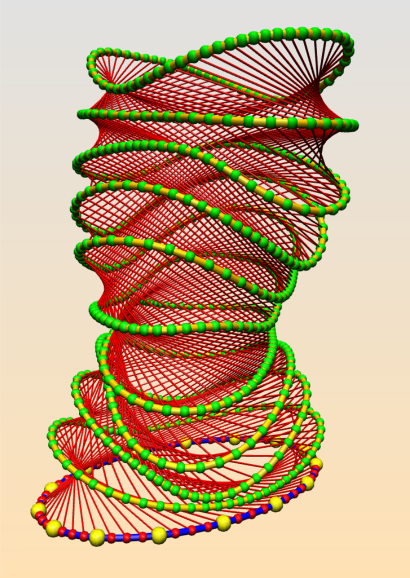

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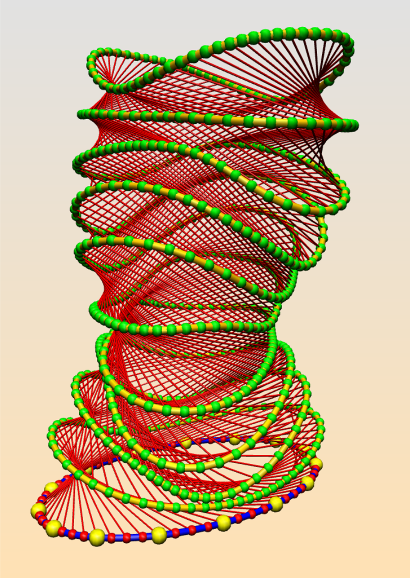

Iterated Darboux transforms

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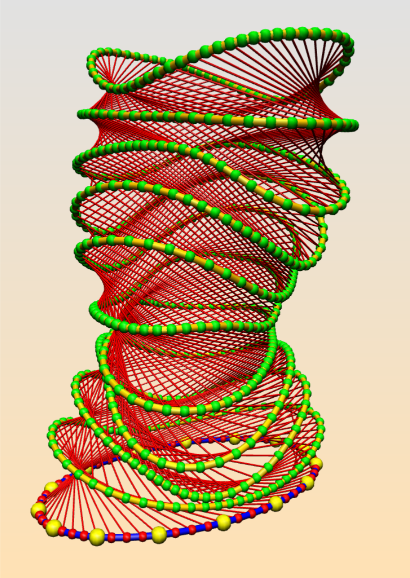

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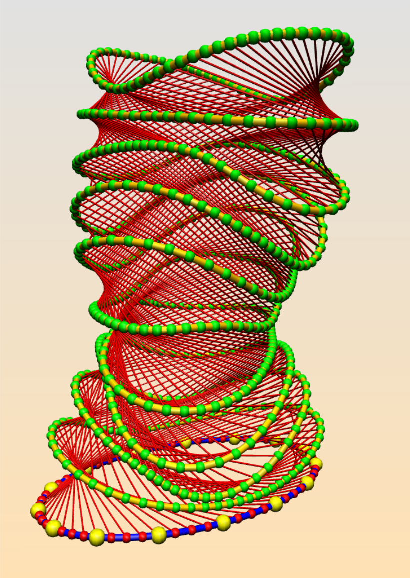
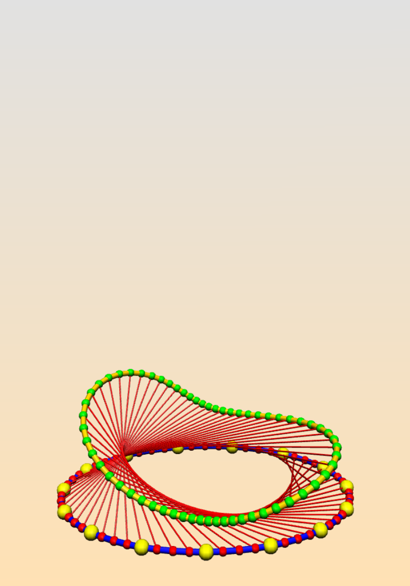


- Using twists τ and $-\tau$ in an alternating fashion preserves reflectional symmetries.
- Forget the odd iterations.
- Excellent discrete version of the smoke-ring flow.



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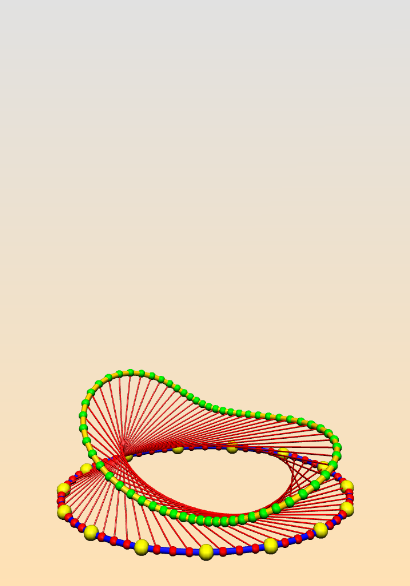

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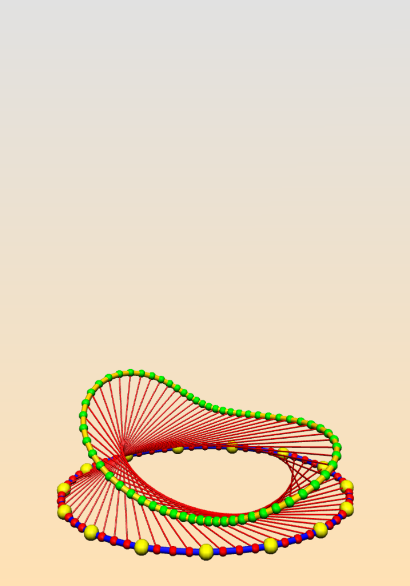

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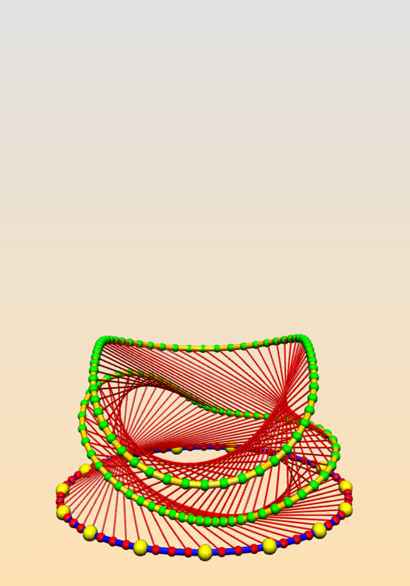

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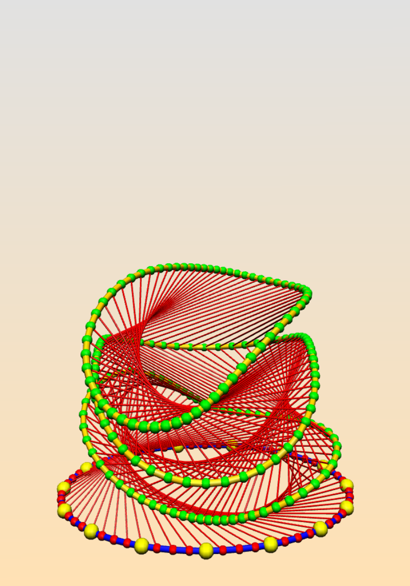

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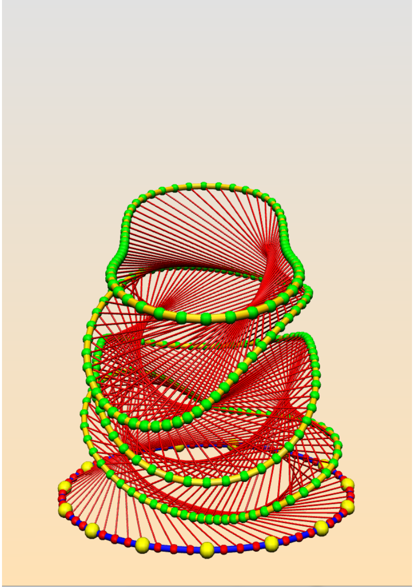

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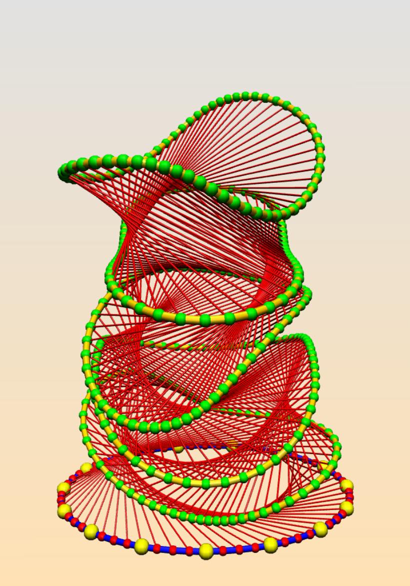

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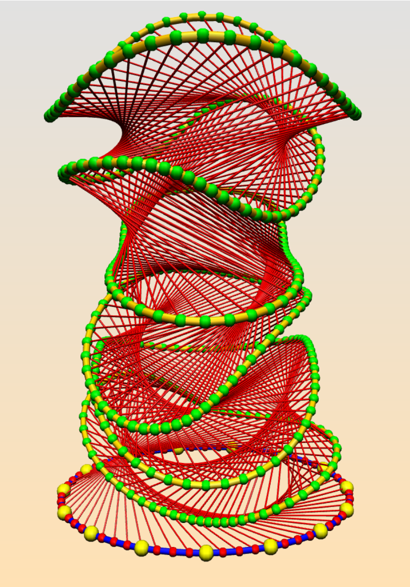
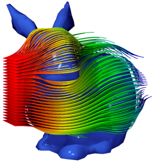
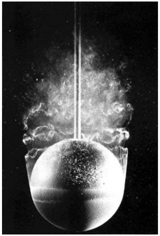
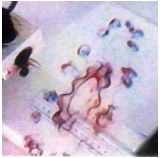


To make this practical, several features have to be included:



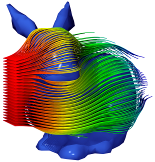


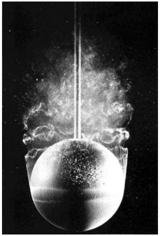


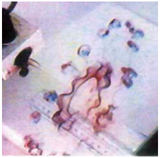
- Interaction between thick vortex rings
- Obstacles
- Vorticity generation at obstacle boundaries ("vortex shedding")
- Topology changes ("vortex reconnection")

To make this practical, several features have to be included:

- ▷ Interaction between thick vortex rings
- ▷ Obstacles
- ▷ Vorticity generation at obstacle boundaries (“vortex shedding”)
- ▷ Topology changes (“vortex reconnection”)



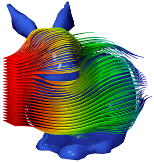


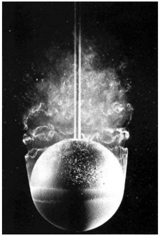


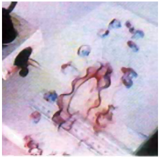
What is missing?

To make this practical, several features have to be included:

- ▷ Interaction between thick vortex rings
- ▷ Obstacles
- ▷ Vorticity generation at obstacle boundaries ("vortex shedding")
- ▷ Topology changes ("vortex reconnection")

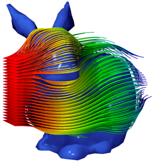


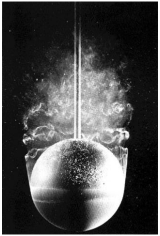


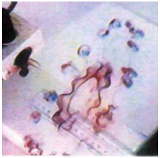


To make this practical, several features have to be included:

- ▷ Interaction between thick vortex rings
- ▷ Obstacles
- ▷ Vorticity generation at obstacle boundaries (“vortex shedding”)
- ▷ Topology changes (“vortex reconnection”)







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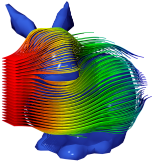
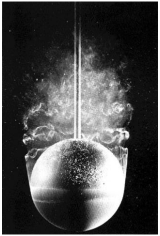
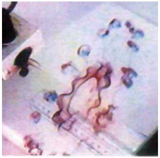


- ▷ Interaction between thick vortex rings



- ▷ Obstacles
- ▷ Vorticity generation at obstacle boundaries (“vortex shedding”)



- ▷ Topology changes (“vortex reconnection”)