Designing fluid motion with obstacles

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Mathematics for key technologies

Vortex rings in fluids

Euler equations

- A velocity vector field v is uniquely determined by its vorticity

$$\omega = \operatorname{curl} v$$

- v is given by the Biot-Savart formula:

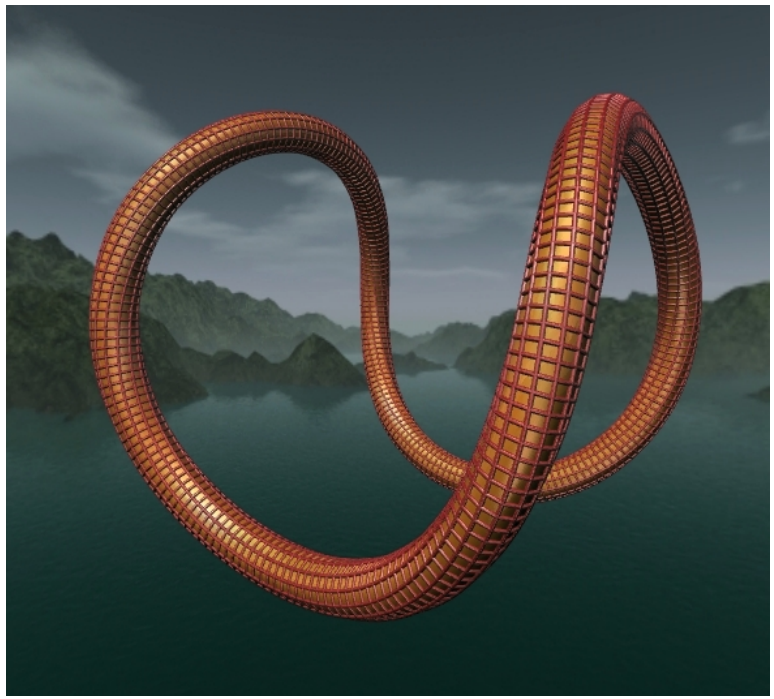
$$v(x) = \int_{\mathbb{R}^3} \frac{\omega(y) \times (x - y)}{|x - y|^3} dy$$

- In an ideal fluid ω flows with the velocity v it generates:

$$\dot{\omega} = [\omega, v]$$

- Suppose all vorticity is concentrated in a small tube of radius r around a space curve γ

  (like water flowing through the tube).



- Then the velocity field is given by

$$v(x) = K \oint \frac{\gamma' \times (x - \gamma)}{|x - \gamma|^3}$$

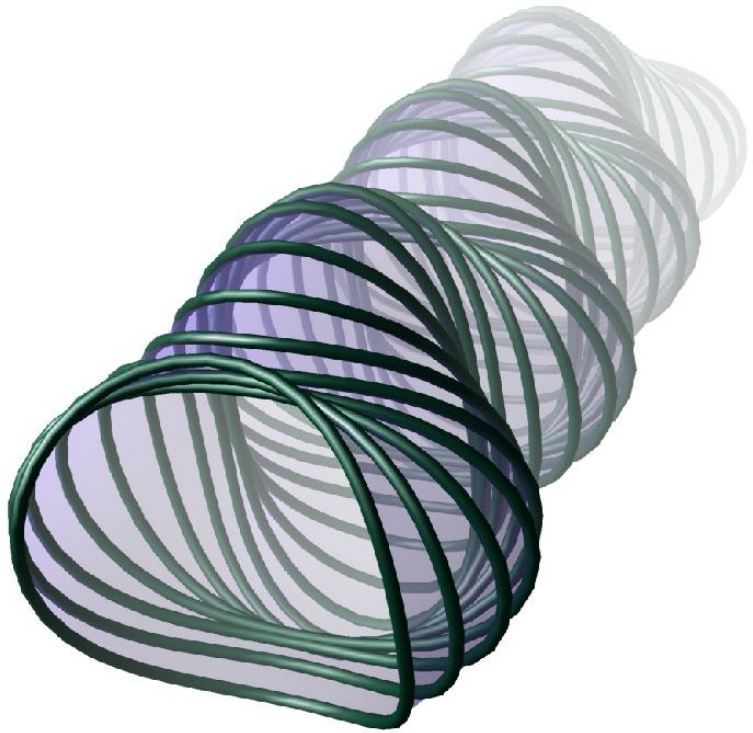
Smoke ring flow

- Scaling down the strength K as $R \to 0$ gives an evolution equation for the space curve γ:

$$\dot{\gamma} = \gamma' \times \gamma''$$

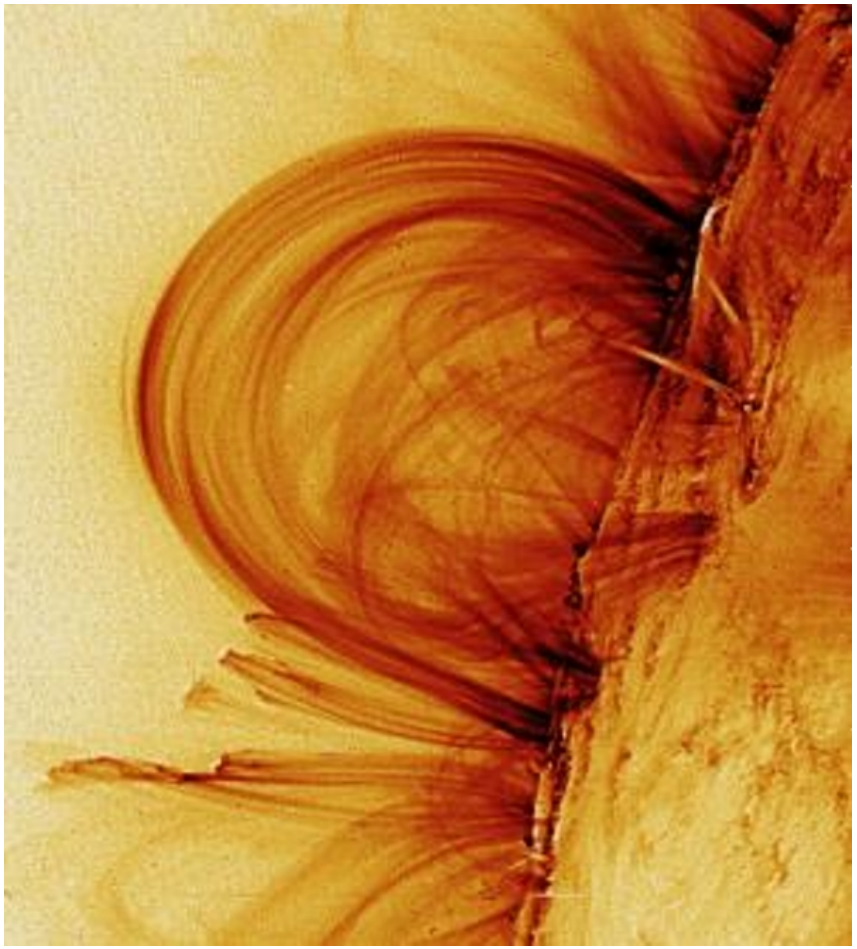
- γ moves orthogonal to its osculating plane with speed proportional to the curvature κ

- This *smoke ring flow* has infinitely many constants of the motion (like length, $\int \kappa^2 \ldots$)

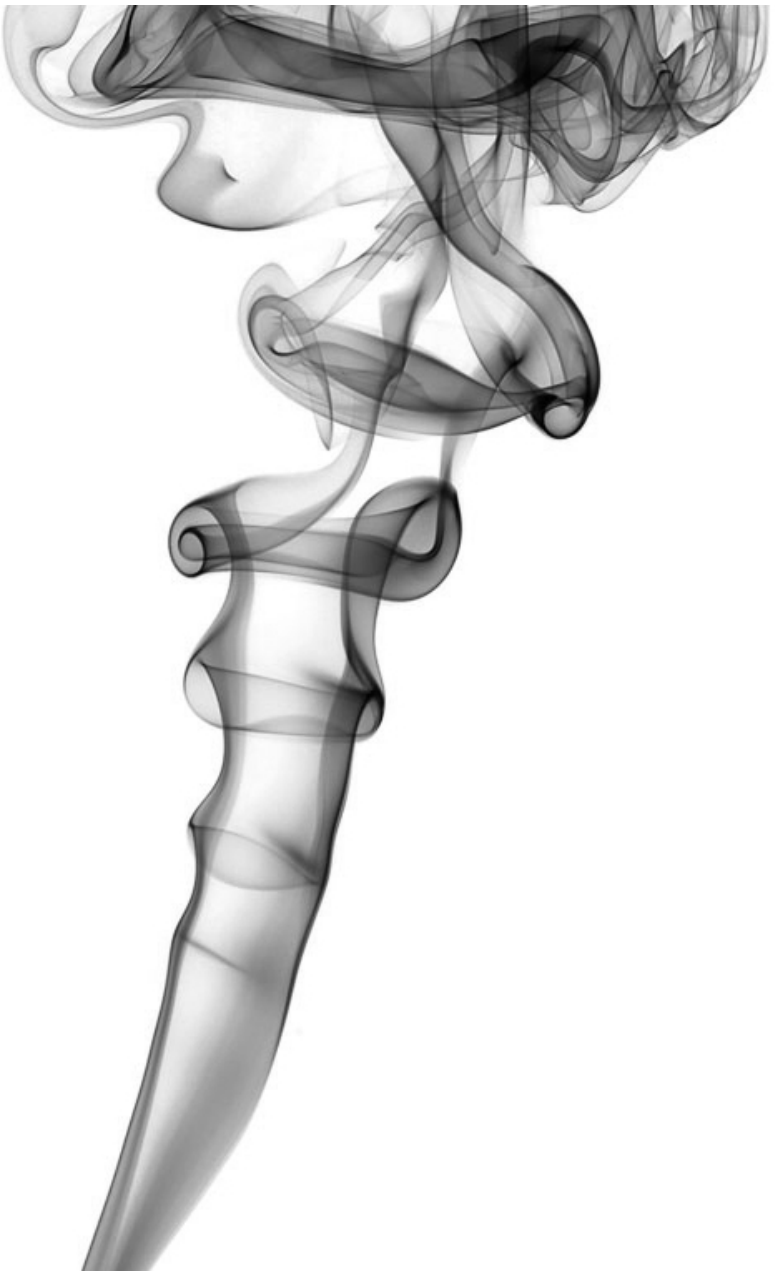


Every flow consists of smoke rings

- Every vorticity field can be approximated by a finite collection of vortex filaments, provided the field lines are closed curves
- Vorticity is usually generated in the boundary layer of bodies moving in the fluid $\rightsquigarrow$ field lines very often are indeed closed curves



Vortex sheet roll up

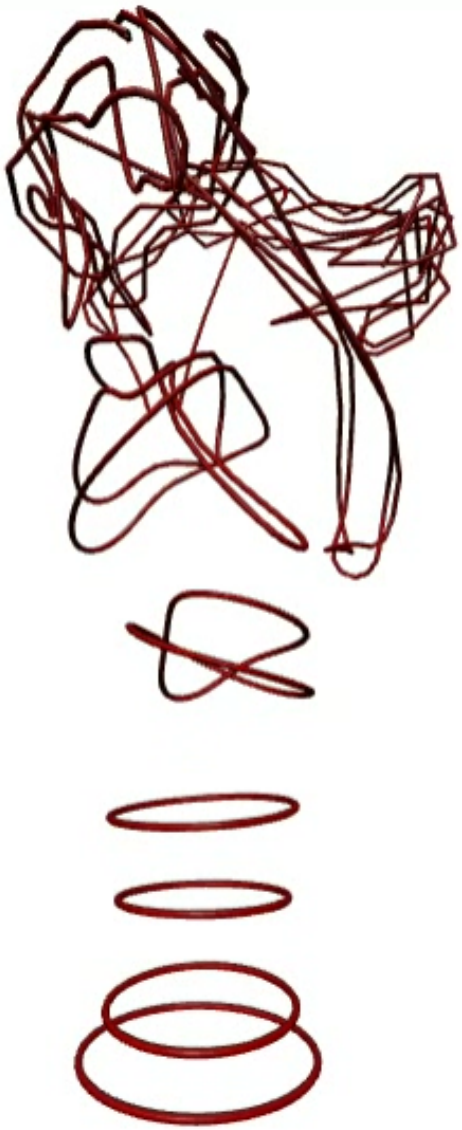


- Vorticity originates as 2-dimensional *vortex sheets*
- Vortex sheets roll up into 1-dimensional structures (“smoke rings”)

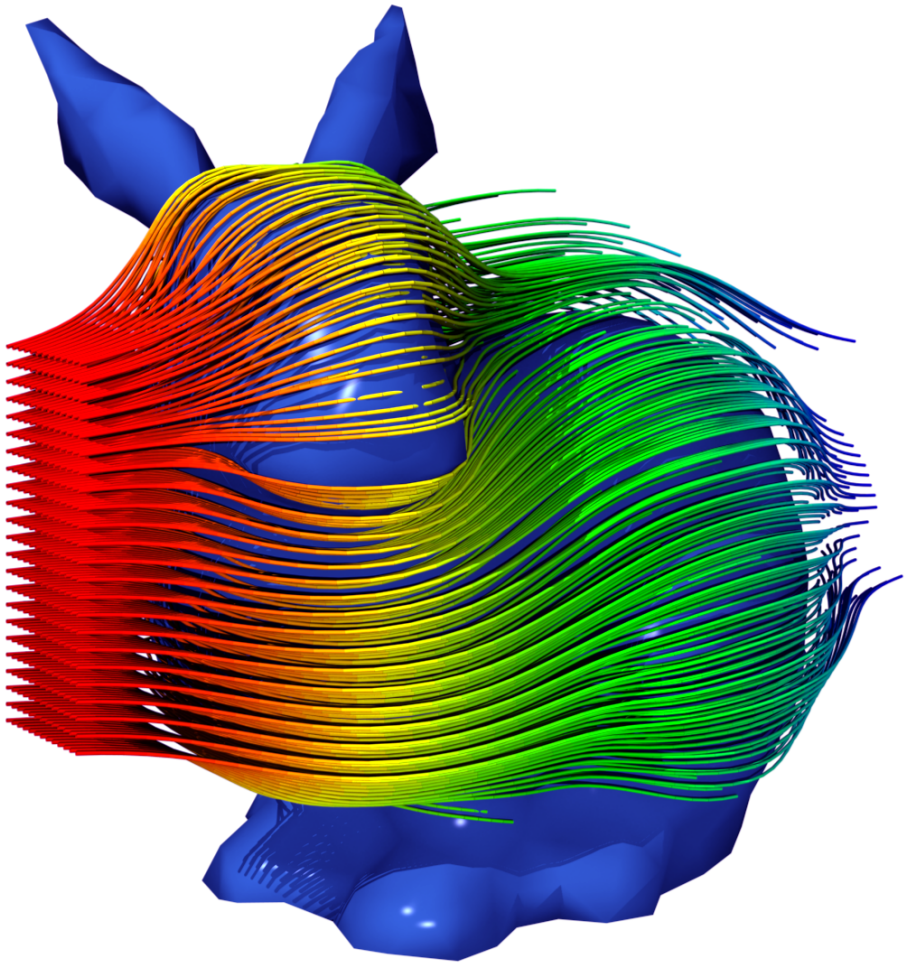
Vortex sheet roll up

Modelling a jet using smoke rings

- Fluid being ejected from a hole is called a *jet*.
- A jet can be modelled by a cylindrical vortex sheet.
- Can be realized by periodically generating smoke rings at the rim of the hole

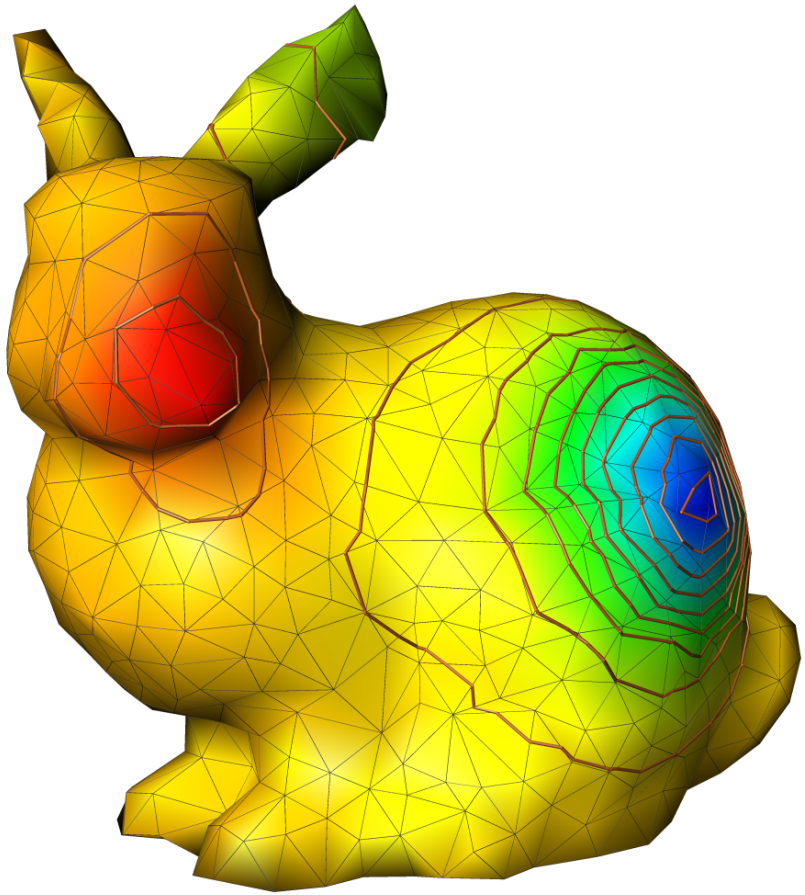


Making fluid flow around obstacles

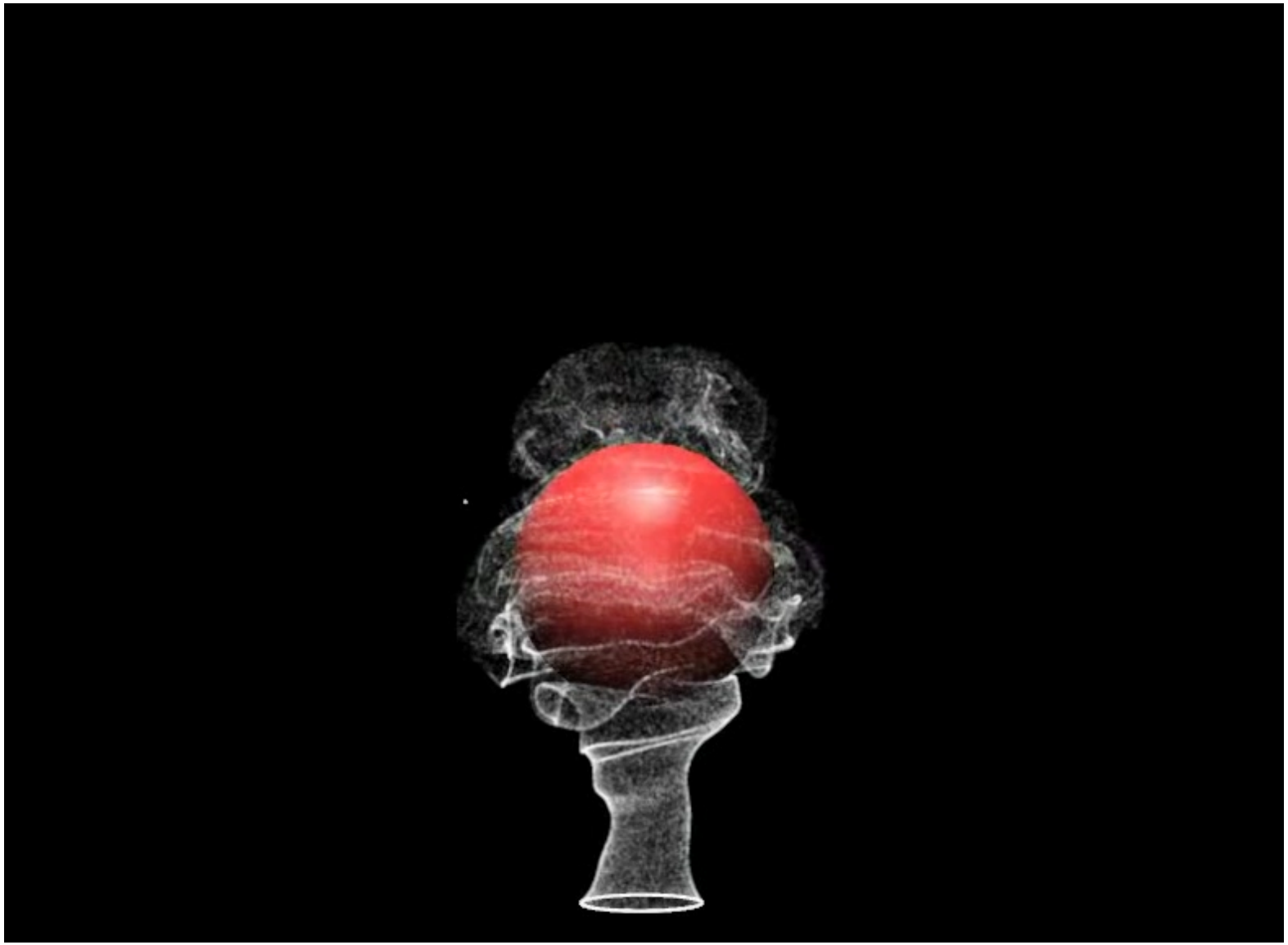


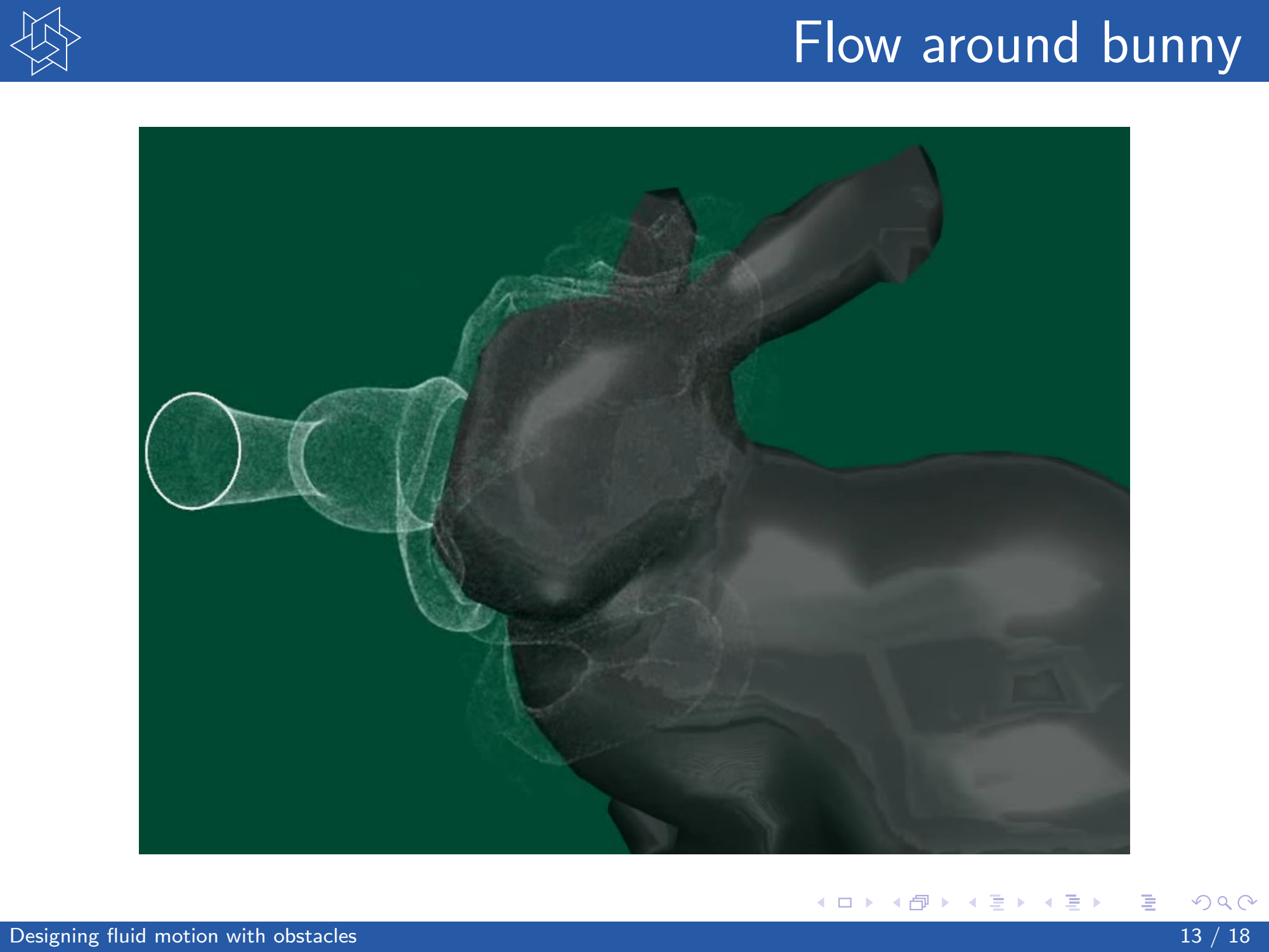
Obstacles as collections of smoke rings

- Near the surface of an obstacle the velocity drops to zero (boundary layer)
- Boundary layer is a vortex sheet with closed vortex lines (level curves of the stream function)
- Can be approximated by a finite number of smoke rings



Flow around sphere

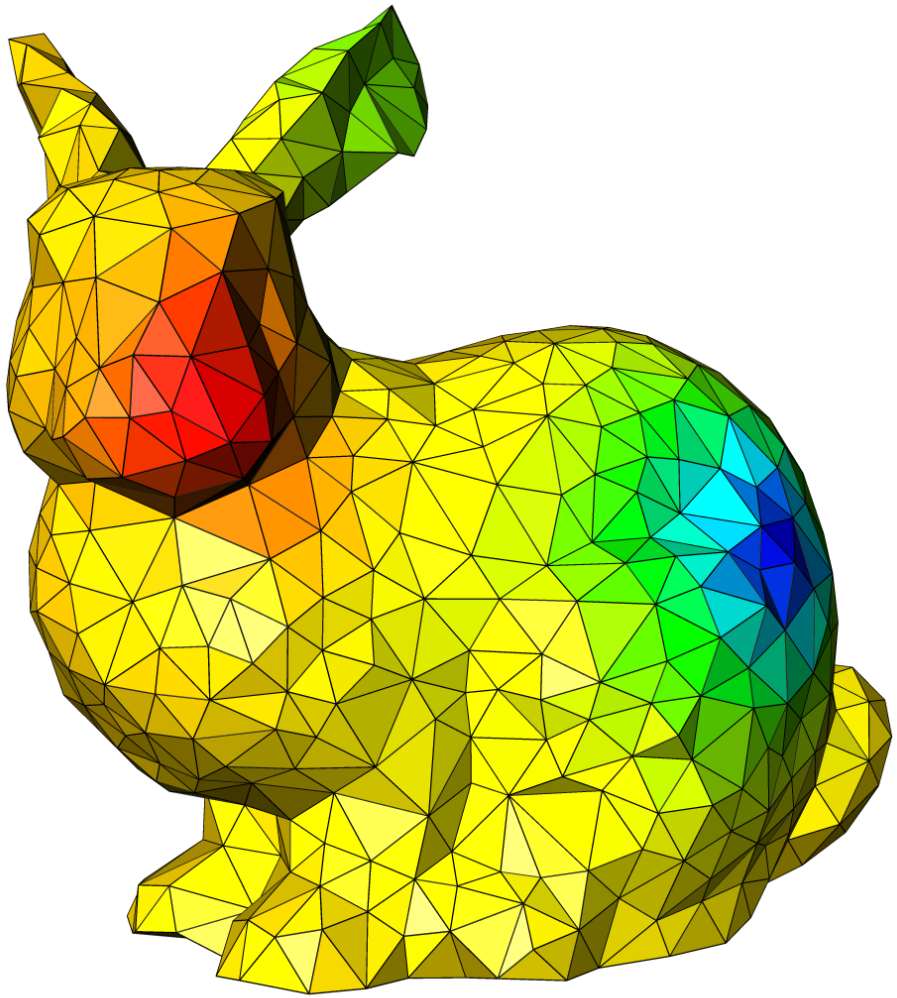




Flow around bunny

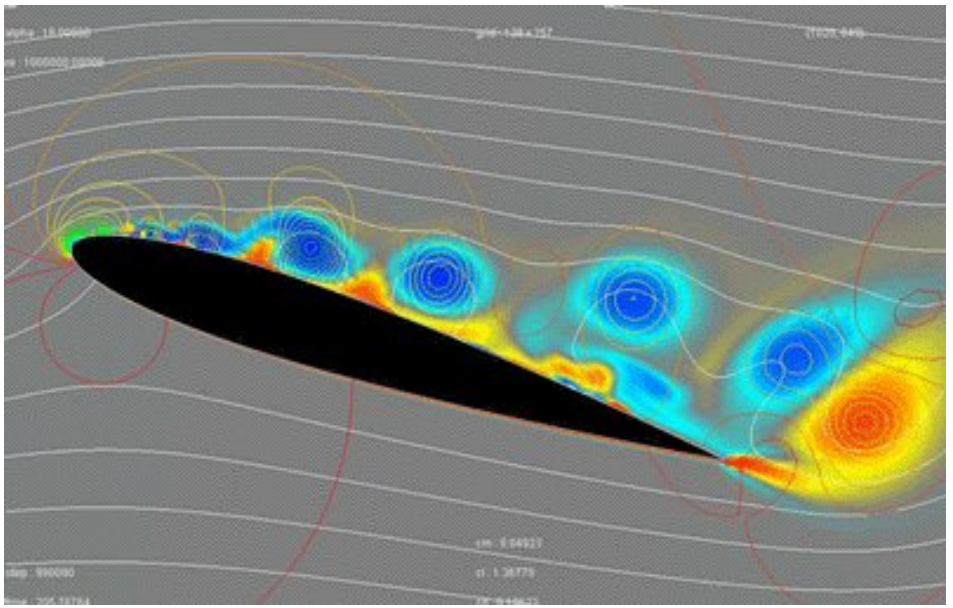
Moving obstacles

- Given a velocity field that does not see the obstacle and a moving body:
- Turn on all the faces of the body as smokerings
- adjust the strength of each face so that there is no flux through any face



Vorticity creation

- Vorticity is born in boundary layers that separate from the body
- Separation happens in chunks $\rightsquigarrow$ smoke rings
- Can be realized by lifting boundary layer smoke rings off the surface



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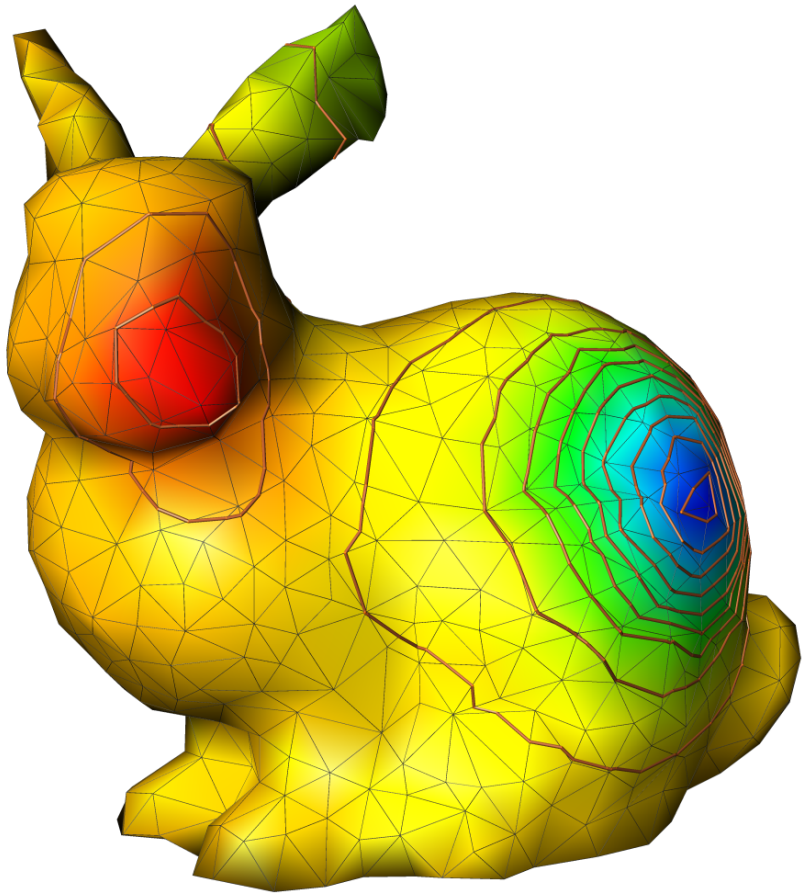

Modelling tools

- Infinite ground plane
- Constant wind background velocity
- Sources, sinks
- Manually placed Vortex rings (both moving and fixed) $\rightsquigarrow$ control of overall features of the flow
- Jets
- Explosions (brief shockwave generating vorticity on the bodies in the scene)