Conformal Equivalence of Triangulated Surfaces

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Mathematics for key technologies

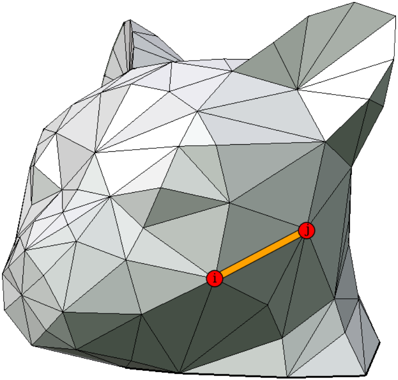
Let M be a combinatorial triangulated surface.

A *metric* on M assigns to every edge between adjacent vertices i, j a positive number

$$l_{ij} = e^{\lambda_{ij}/2}$$

such that for each triangle i, j, k the triangle inequalities hold:

$$l_{ij} \le l_{jk} + l_{ki}$$
$$l_{jk} \le l_{ki} + l_{ij}$$
$$l_{ki} \le l_{ij} + l_{jk}$$



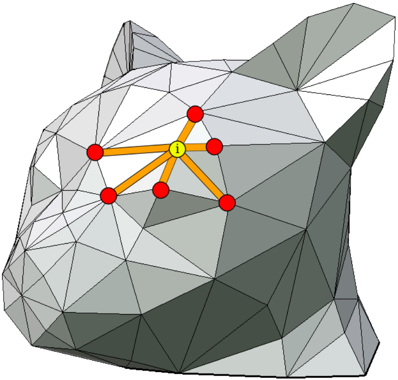
- Elementary conformal change: multiply the length of all edges adjacent to vertex i with the same positive factor $e^{u_i/2}$

$$\tilde{l}_{ij} = e^{u_i/2} l_{ij}$$

- General conformal change: For a function u on the vertex set define

$$\tilde{l}_{ij} = e^{(u_i+u_j)/2} l_{ij}$$

$$\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j$$



Two metrics on the same combinatorial surface are conformally equivalent

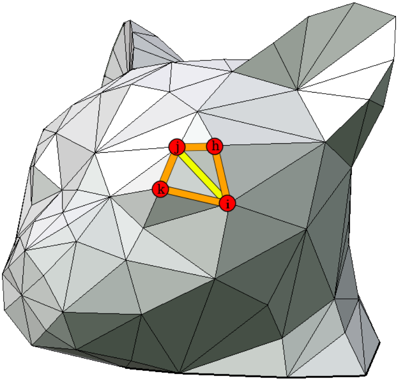
$$\Updownarrow$$

For each edge ij the cross ratios

$$cr_{ij} = \frac{l_{ih} l_{jk}}{l_{ik} l_{jh}}$$

coincide:

$$\tilde{cr}_{ij} = cr_{ij}$$



- Conformal structure on combinatorial surface M (equivalence class of metrics)

  $\updownarrow$

  Assignment of $cr_{ij} > 0$ to each edge such that for each vertex i

$$\prod cr_{ij} = 1$$

- For a compact surface of genus g:

$$\begin{aligned} dim\{\text{conformal structures}\} &= 2|V| + 6g - 6 \\ &= dim(\mathcal{T}_{g,|V|}) \end{aligned}$$

$\mathcal{T}_{g,|V|}$ = Teichmüller space of Riemann surfaces of genus g with $|V|$ punctures

$g : \mathbb{R}^n \to \mathbb{R}^n$ a Möbius transformation $\rightsquigarrow$

The metrics on M induced from embeddings

$$f : V \to \mathbb{R}^n$$

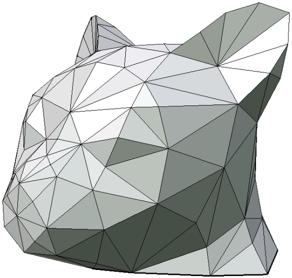
and

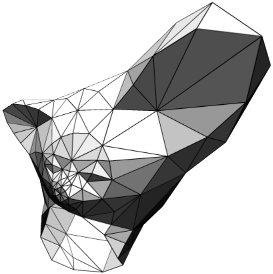
$$\tilde{f} = g \circ f$$

are conformally equivalent.

This follows from

$$\left|\frac{p}{|p|^2} - \frac{q}{|q|^2}\right| = \frac{1}{|p|} \cdot \frac{1}{|q|} \, |p - q|$$





Theorem (Trojanov):

On a compact Riemann surface M with boundary choose

$$\text{a metric } g \text{ on } \partial M$$

$$p_1, \ldots, p_n \in M$$

$$\alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

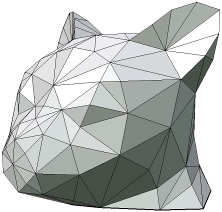
Then there is a unique flat metric on $M - \{p_1, \ldots, p_n\}$ (compatible with the conformal structure) with cone points of curvature α_j at p_j which induces the metric g on ∂M.

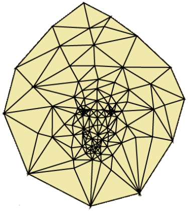
Why a discrete version would be useful

- Needed for computing texture coordinates in Computer Graphics

- Keeping the metric on the boundary yields minimal overall length distortion

- Allowing suitable cone points reduces distortion further





Back to Mathematics

Given:

- a metric with lengths $\tilde{l}_{ij}$
- Prescribed cone angles α_i for each vertex, reasonable in the sense that there exist $0 < \alpha^i_{jk} < \pi$ satisfying $\alpha^i_{jk} + \alpha^j_{ki} + \alpha^k_{ij} = \pi$ with

$$\sum_{j,k} \alpha^i_{jk} = \alpha_i$$

Look for:

- Conformal factors u_i such that the new lengths $l_{ij} = e^{(u_i+u_j)/2}\ \tilde{l}_{ij}$ result in the prescribed cone angles for

$$\alpha^i_{jk} = 2\tan^{-1}\sqrt{\frac{(l_{ij}+l_{jk}-l_{ki})(l_{jk}+l_{ki}-l_{ij})}{(l_{ki}+l_{ij}-l_{jk})(l_{jk}+l_{ki}+l_{ij})}}$$

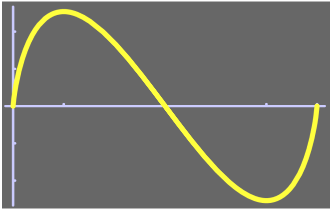
$u = (u_1, \ldots, u_n)$ solves these equations $\Leftrightarrow$
u is a critical point of the function

$$E(u) = \sum_{t_{ijk} \in T} \alpha^i_{jk}\lambda_{jk} + \alpha^j_{ki}\lambda_{ki} + \alpha^k_{ij}\lambda_{ij} - \pi(u_i + u_j + u_k)$$
$$+ 2\big(Л(\alpha^i_{jk}) + Л(\alpha^j_{ki}) + Л(\alpha^k_{ij}) \big)$$
$$+ \sum_{v_i \in V} \alpha_i u_i$$

Here $\lambda_{ij} = 2 \log \tilde{l}_{ij}$ and

$$Л(x) = -\int_0^x \log |2 \sin t| \, dt$$

denotes Milnor's Lobachevsky function.



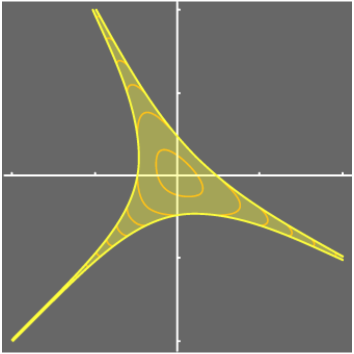
Normalize u by $\sum u_i = 0$.

Good news:

E is a strictly convex function of u.

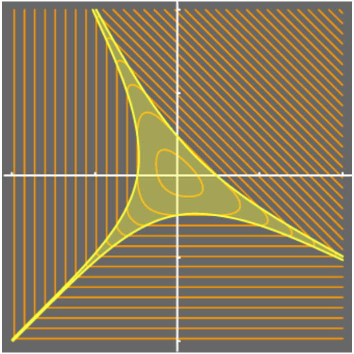
Bad news:

Due to the triangle inequalities the domain of definition of E is not convex.



Good news:

E can be extended
to a proper convex function
on the whole of $\mathbb{R}^n$



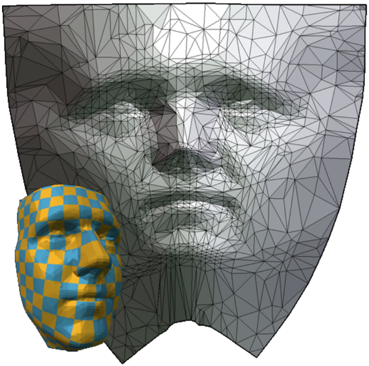
Corollary:

- ▷ There exists a solution u
- ▷ If the triangle inequalities are satisfied, then u is unique

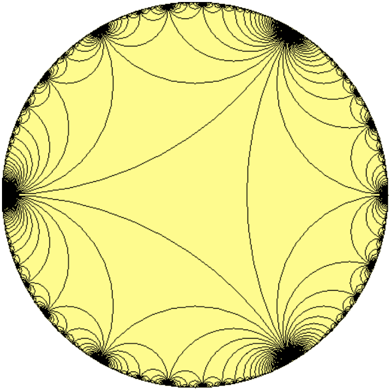
For practical purposes this is good enough:

- Convex optimization problem $\rightsquigarrow$ globally convergent Newton method

- Problems with triangle inequalities $\rightsquigarrow$ improve the combinatorics of the original triangulation in a few places.



- Start with a flat metric on a compact 2-manifold with finitely many cone singularities $\rightsquigarrow$ vertices $v_1, \ldots, v_n$
- Choose a Dirichlet triangulation of M (interiors of circumcirles contain no other vertices) $\rightsquigarrow$ lengths l_{ij} and cross ratios cr_{ij}
- Each triangle inherits from its circumcircle the metric of an ideal hyperbolic triangle (Klein model)
- Crossratios allow to glue all these triangles together to obtain a complete hyperbolic metric on $M - \{v_1, \ldots, v_n\}$ with cusps at v_i

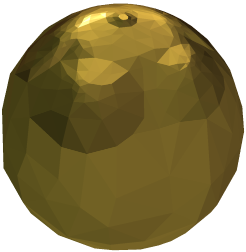


Definition: Two flat metrics with cone points $v_1, \ldots, v_n$ on a compact 2-manifold are *conformally equivalent* if the corresponding complete hyperbolic metrics on $M - \{v_1, \ldots, v_n\}$ with cusps at $v_1, \ldots, v_n$ are isometric.

Theorem (Rivin 1994): Every complete hyperbolic metric with on $S^2 - \{v_1, \ldots, v_n\}$ with cusps at $v_1, \ldots, v_n$ is isometric to the boundary of a unique convex ideal polyhedron in hyperbolic 3-space.

Corollary:

Every flat metric on $S^2 - \{v_1, \dots, v_n\}$ with cone points at $v_1, \dots, v_n$ is conformally equivalent to the boundary of a convex polyhedron in $\mathbb{R}^3$ with vertices on S^2 (unique up to Moebius transformations)



Conclusion

- Powerful (final?) definition of a Discrete Riemann Surface
- Highly efficient algorithms for Computer Graphics
- Hyperbolic geometry explains the appearance of Lobachevski function $Л$ (volume of ideal tetrahedra)

Outlook

- Extend uniformization results to higher genus and to surfaces with boundary
- Provide constructive proof of Rivin's Theorem