Lattice Polygons and Real Roots

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Overview

1. Triangulations of Lattice Polytopes
   - Foldability
   - Lattice Polygons
2. Why Should We Care?
   - Real Roots of (Very Special) Polynomial Systems
3. Odds and Ends
   - Computational Experiments
   - Pick's Theorem
   - Products

Regularity, Denseness, Foldability

A triangulation of a lattice d-polytope is . . .

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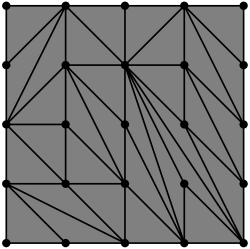

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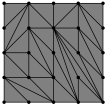
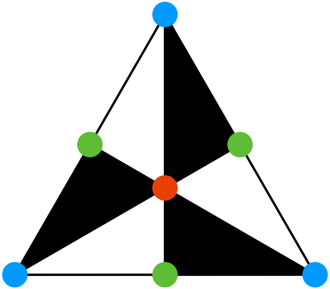
- foldable iff the graph of the triangulation is $(d+1)$-colorable
  - $\Leftrightarrow$ dual graph bipartite





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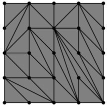
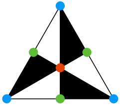
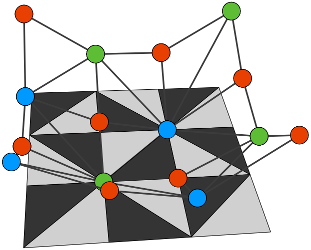
- regular iff the triangulation can be lifted to the lower hull of a $(d+1)$-polytope







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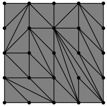
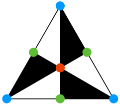
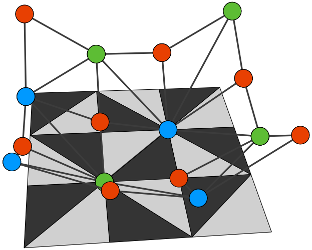






Definition

rdf = regular & dense & foldable

Lattice Edges in the Plane

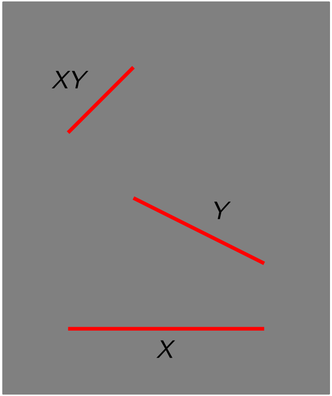
Let p and q be lattice points in $\mathbb{Z}^2$.

Definition

line segment $[p, q] = \mathrm{conv}(p, q)$ is *of type*

$$\begin{cases} X & \text{if first coordinate of } p - q \text{ odd} \\ & \text{and second even} \\ Y & \text{if first coordinate even} \\ & \text{and second odd} \\ XY & \text{if both coordinates odd} \end{cases}$$

No type defined if both coordinates even.

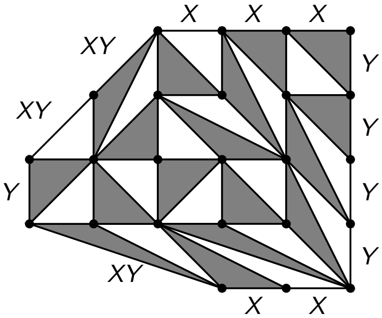


Foldable Triangulations of Lattice Polygons

Let Δ be a dense and foldable triangulation of a lattice polygon.

Theorem (J. & Ziegler, 2012)

The signature $\sigma(\Delta)$ equals the absolute value of the difference between the numbers of black and of white boundary edges of type τ, for any fixed $\tau \in \{X, Y, XY\}$.



signature

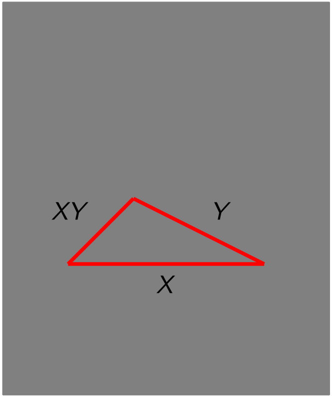
$\sigma(\Delta) = |\#\text{black facets} - \#\text{white facets}|$

A Simple Observation

Let T be a lattice triangle of *odd normalized area* in the plane.

Lemma

Then T has precisely one edge of type X, one of type Y and one of type XY.



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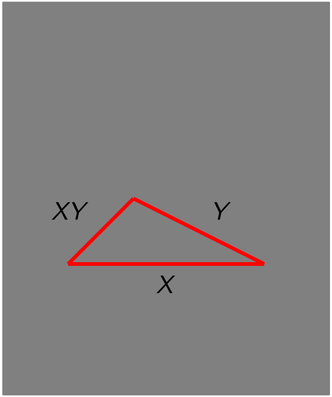


Proof.

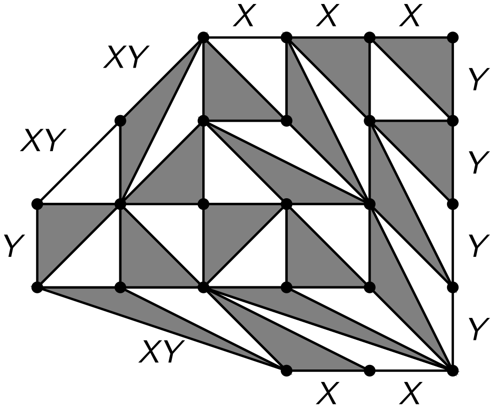
For integer a, b, c, d consider

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

odd and check cases. $\square$

The Proof

- dense $\Rightarrow$ all triangles unimodular



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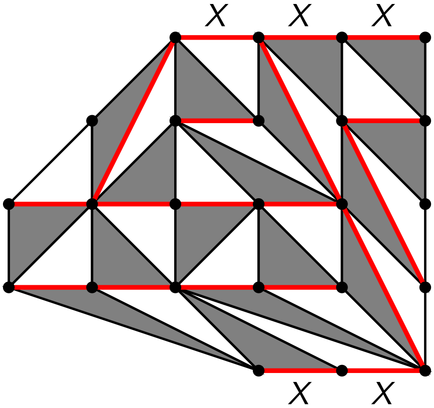

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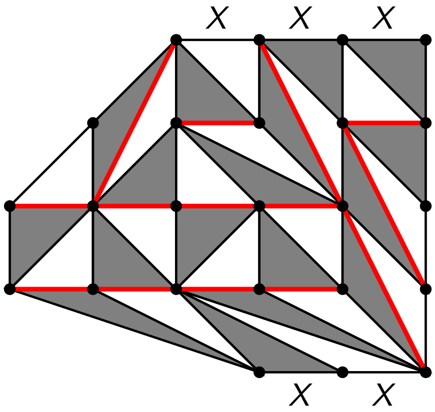
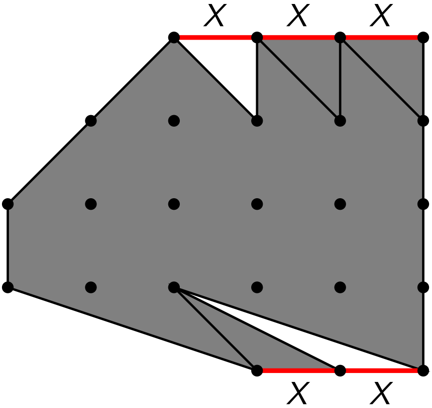


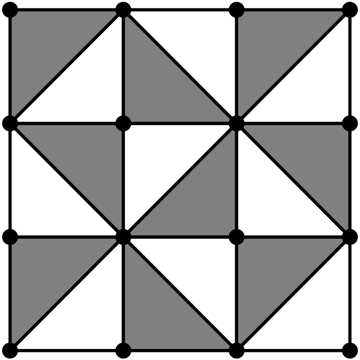
The Proof

- dense $\Rightarrow$ all triangles unimodular
- pick $\tau \in \{X, Y, XY\}$
- interior τ-edges form partial matching in dual graph of Δ
- remove matched pairs of triangles



A Special Case

Let P be an axis-parallel lattice rectangle in the plane.



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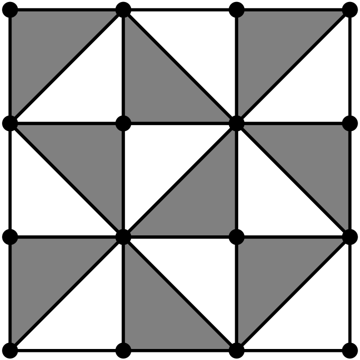

Corollary

The signature of any dense and foldable triangulation of P vanishes.

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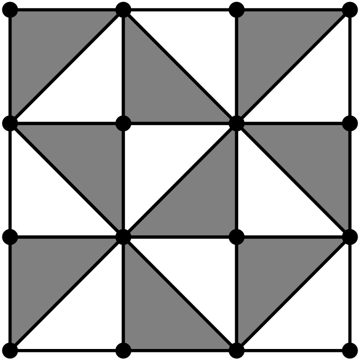

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The signature of any dense and foldable triangulation of P vanishes.

Proof.

There are no XY edges in the boundary. $\square$

Wronski Polynomial Systems

Let P be a lattice d-polytope with an rdf lattice triangulation $\mathcal{T}$ with coloring $c : P \cap \mathbb{Z}^d \to \{0, \ldots, d\}$ (and lifting $\lambda : P \cap \mathbb{Z}^d \to \mathbb{N}$).

Wronski polynomial for $\alpha_i \in \mathbb{R}$ and parameter $s \in (0, 1]$:

$$\sum_{m \in P \cap \mathbb{Z}^d} s^{\lambda(m)} \alpha_{c(m)} x^m \qquad \in \mathbb{R}[x_1, \ldots, x_d]$$

Wronski system : system of d Wronski polynomials w.r.t. $\mathcal{T} = P^\lambda$ and *generic* coefficients $\alpha_0^{(k)}, \ldots, \alpha_d^{(k)}$

Theorem (Bernstein, 1975; Kushnirenko, 1976; Khovanskii, 1977)

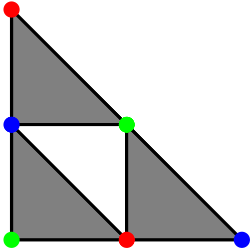
$\#$ *complex roots* $= d! \operatorname{vol}(P) =: \nu(P)$

generic : no multiple complex roots

Lower Bounds for the Number of Real Roots

Theorem (Soprunova & Sottile, 2006)

Each Wronski system w.r.t. P^λ has at least $\sigma(P^\lambda)$ real roots,

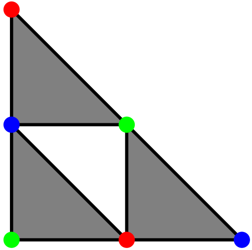


$a(1+xy)+b(x+y^2)+c(x^2+y)$

Lower Bounds for the Number of Real Roots

Theorem (Soprunova & Sottile, 2006)

Each Wronski system w.r.t. P^λ has at least $\sigma(P^\lambda)$ real roots, provided that certain additional geometric conditions are satisfied.



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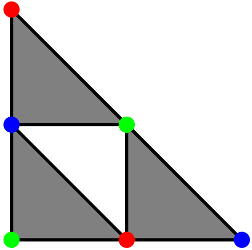

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$\sigma(P^\lambda) = 3 - 1 = 2$

signature $\sigma(P^\lambda) = |\#\text{black facets} - \#\text{white facets}|^*$

$*$: only odd normalized volume $\nu(F)$

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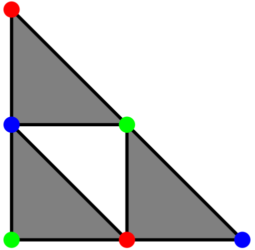

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$$\lambda(0,1) = \lambda(1,0) = \lambda(1,1) = 0$$
$$\lambda(0,0) = \lambda(2,0) = \lambda(0,2) = 1$$

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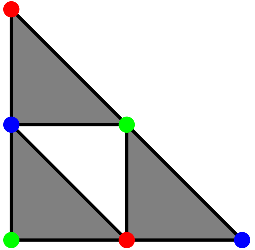
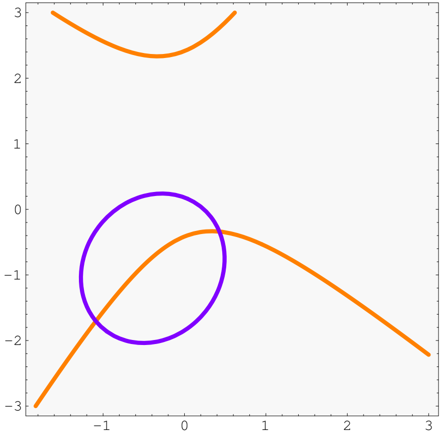


$$a(1+xy)+b(x+y^2)+c(x^2+y)$$
$$\sigma(P^\lambda)=3-1=2$$

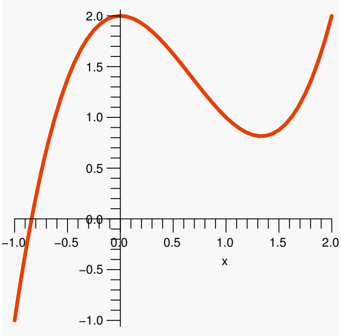
$\boxed{a_1=1,\ b_1=-1,\ c_1=2}$

$\boxed{a_2=-1,\ b_2=3,\ c_2=5}$



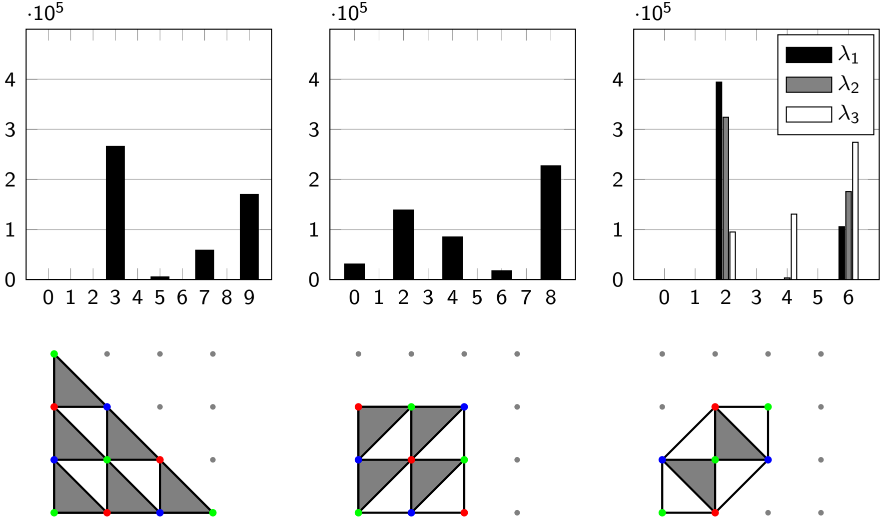
The Case $d = 1$

- one polynomial p in one indeterminate
- generic $\Rightarrow$ all coefficients non-vanishing
- Newton polytope $=$ interval $[0, r]$, where $r = \deg p$
- unique rdf triangulation into unit intervals
- signature $=$ parity of r



Computational Experiments

with `polymake` and `Singular` [Assarf, J. & Paffenholz]



Counting Lattice Points in Polygons

Let P be a lattice polygon.

A : Euclidean area
B : number of boundary lattice points
I : number of interior lattice points

Theorem (Pick, 1899)

$B = 2 \cdot (A - I + 1)$

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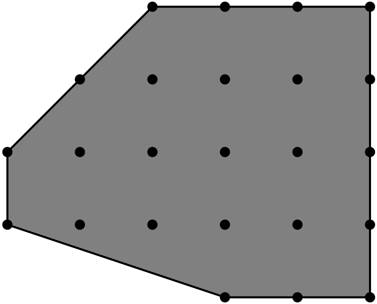

Theorem (Pick, 1899)

$B = 2 \cdot (A - I + 1)$

$A = 33/2$
$B = 13$
$I = 11$

Bounding the Signature

Let Δ be a dense and foldable triangulation of a lattice polygon.

Corollary

$\sigma(\Delta) \leq \frac{2}{3}(A - I + 1)$

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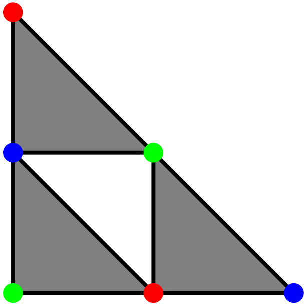


$A = 4/2$
$B = 6$
$I = 0$

Bounding the Signature

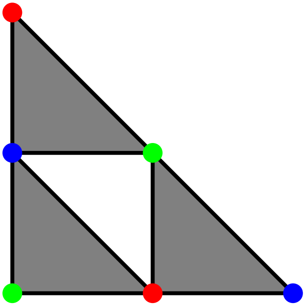
Let Δ be a dense and foldable triangulation of a lattice polygon.

Corollary

$\sigma(\Delta) \leq \frac{2}{3}(A - I + 1)$

Proof.

There is at least one type $\tau \in \{X, Y, XY\}$ with at most $B/3$ boundary edges. $\square$



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The Simplicial Product

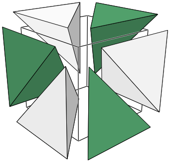
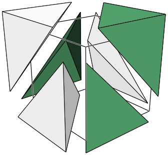
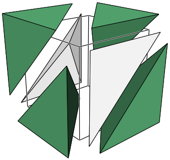
K, L : simplicial complexes
V_K, V_L : respective vertex sets with orderings O_K, O_L

$O := O_K \times O_L$: product partial ordering

Definition

$$K \times_{\text{stc}} L := \left\{ F \subseteq V_K \times V_L \;\middle|\; \begin{array}{l} \pi_K(F) \in K \text{ and } \pi_L(F) \in L\,, \\ \text{and } O\,|_F \text{ is a total ordering} \end{array} \right\}$$

- Eilenberg & Steenrod, 1952: *Cartesian product*
- Santos, 2000: *staircase refinement*

Products of Polytopes

P^λ : rdf-triangulation of m-dimensional lattice polytope $P \subset \mathbb{R}^m$
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Theorem (J. & Witte, 2007)

For color consecutive vertex orderings of the factors the simplicial product $P^\lambda \times_{\text{stc}} Q^\mu$ is an rdf-triangulation of the polytope $P \times Q$ with signature

$$\sigma(P^\lambda \times_{\text{stc}} Q^\mu) = \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu)\ .$$

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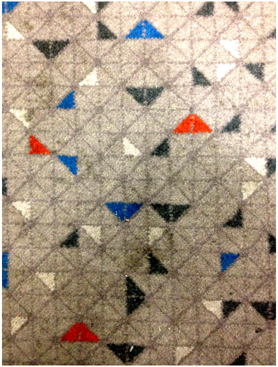
$$\sigma_{m,n} = \sigma(\text{stc}(\Delta_m \times \Delta_n)) = \begin{cases} \binom{(m+n)/2}{m/2} & \text{if } m \text{ and } n \text{ even} \\ \binom{(m+n-1)/2}{m/2} & \text{if } m \text{ even and } n \text{ odd} \\ 0 & \text{if } m \text{ and } n \text{ odd} \end{cases}$$

Conclusion

- very special triangulations of Newton polytopes allow to read off lower bound for number of real roots (for very special systems of polynomials)
- bivariate case easy to analyze
- behaves well with respect to forming products



Soprunova & Sottile, *Adv. Math.* 204 (2006)
J. & Witte, *Adv. Math.* 210 (2007)
J. & Ziegler, *Amer. Math. Monthly* 121 (2014)

The "Additional Geometric Conditions"

- $P \subset \mathbb{R}^d_{\geq 0}$: lattice d-polytope with N lattice points, λ as above ...

$$\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [t^v \mid v \in P \cap \mathbb{Z}^d] ,$$

  - *toric variety* $X_P =$ (Zariski) closure of image
  - *real part* $Y_P = X_P \cap \mathbb{RP}^{N-1}$, lift Y_P^+ to $\mathbb{S}^{N-1}$

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- *Wronski projection*

$$\begin{aligned} \pi : \quad \mathbb{CP}^{N-1} \setminus E \;&\to\; \mathbb{CP}^d \\ [x_v \mid v \in P \cap \mathbb{Z}^d] \;&\mapsto\; \Big[\sum_{v \in c^{-1}(i)} x_v \mid i = 0, 1, \ldots, d \Big] \end{aligned}$$

with *center*
$E = \left\{ x \in \mathbb{CP}^{N-1} \mid \sum_{v \in c^{-1}(i)} x_v = 0 \quad \text{for } i = 0, 1, \ldots, d \right\}$

S&S 2006

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$$\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [t^v \mid v \in P \cap \mathbb{Z}^d] \,,$$

  - *toric variety* $X_P =$ (Zariski) closure of image
  - *real part* $Y_P = X_P \cap \mathbb{RP}^{N-1}$, lift Y_P^+ to $\mathbb{S}^{N-1}$ must be oriented
- *s-deformation* $s.Y_P$ (for $s \in (0,1]$) = closure of the image of

$$s.\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [s^{\lambda(v)}\, t^v \mid v \in P \cap \mathbb{Z}^d]$$

- *Wronski projection*

$$\pi : \begin{array}{c} \mathbb{CP}^{N-1} \setminus E \;\to\; \mathbb{CP}^d \\ [x_v \mid v \in P \cap \mathbb{Z}^d] \mapsto [\displaystyle\sum_{v \in c^{-1}(i)} x_v \mid i = 0, 1, \ldots, d\,] \end{array}$$

must avoid
$E = \left\{ x \in \mathbb{CP}^{N-1} \,\middle|\, \sum_{v \in c^{-1}(i)} x_v = 0 \quad \text{for } i = 0, 1, \ldots, d \right\}$

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