Heuristics for sphere recognition

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joint w/ Frank H. Lutz & Mimi Tsuruga

The Algorithmic Problem

PL-HOMEOMORPHIC(d)

Given a finite simplicial complex of dimension d,
decide if it is PL-homeomorphic to $\mathbb{S}^d$.

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HOMEOMORPHIC $\leftrightsquigarrow$ PL-HOMEOMORPHIC

Morse Theory of Spheres and Manifolds

Theorem (Whitehead 1939; Forman 1998)

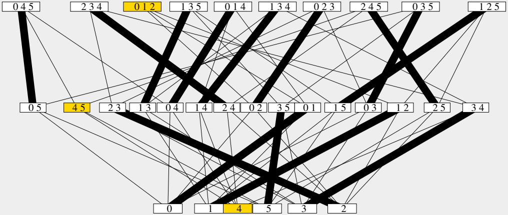
A pure d-dimensional simplicial complex K is a combinatorial d-manifold if and only if for any proper face F of K some subdivision L of the link of F in K admits a discrete Morse function with discrete Morse vector $(1, 0, \ldots, 0, 1)$, that is, with exactly two critical cells.

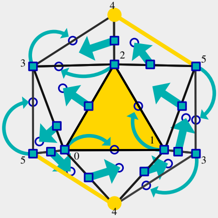
Discrete Morse Theory, I

K = finite simplicial complex

Definition (Forman 1998; Chari 2000)

Morse matching = matching in the Hasse diagram of K (seen as a directed graph) which does not induce a directed cycle





- unmatched nodes = critical faces

Discrete Morse Theory, II

Theorem (J. & Pfetsch 2004)

Given a finite simplicial complex K and an integer m, it is NP-complete *to decide whether there exists a Morse matching with at most m critical simplices.*

- Eğecioğlu & Gonzalez 1996
- Lewiner, Lopes, Tavares 2003
- Burton, Lewiner, Paixão & Spreer 2014+: parameterized complexity analysis

Sphere Recognition Heuristics

Input: K: triangulation of connected closed PL d-manifold, where $d \geq 3$
Output: Decision: Is K PL homeomorphic to $\mathbb{S}^d$?

compute Hasse diagram
for N rounds **do**
  if random discrete Morse vector spherical **then** **return** YES

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if homology not spherical **then** **return** NO

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        for N' rounds **do**
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```
compute Hasse diagram
for N rounds do
    if random discrete Morse vector spherical then return YES
compute homology
if homology not spherical then return NO
compute and simplify presentation of fundamental group π_1
if π_1 found to be trivial then
    if d ≠ 4 then
        return YES
    else
        for N′ rounds do
            perform random bistellar flip
            if boundary of simplex reached then return YES
return UNDECIDED
```

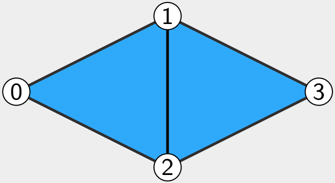
polymake Overview

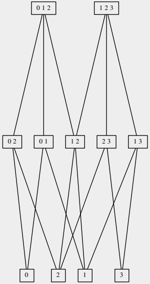
- software for research in mathematics
  - geometric combinatorics: convex polytopes, matroids, ...
  - linear/combinatorial optimization
  - toric/tropical geometry
  - combinatorial topology
- open source, GNU Public License
  - supported platforms: Linux, FreeBSD, MacOS X
  - about 100,000 uloc (C++, Perl, C, Java)
- co-authored (since 1996) w/ Ewgenij Gawrilow
  - contributions by Benjamin Assarf, Katrin Herr, Silke Horn, Lars Kastner, Georg Loho, Benjamin Lorenz, Andreas Paffenholz, Julian Pfeifle, Thomas Rehn, Thilo Rörig, Benjamin Schröter, Mimi Tsuruga, and others
- online version at `shell.polymake.org`

The Data Structure

Hasse diagram of face poset

- Ganter 1987: algorithm to enumerate the closed sets of a set system
  - linear in the size of the output
- Kaibel & Pfetsch 2002: face trees of convex polytopes
- implemented in `polymake`
  - with local updates $\rightsquigarrow$ possibly changing the combinatorics/topology





Homology Computation, I

Algorithms

- H.J.S. Smith 1861: elimination
- Iliopoulous 1989: modular method with polynomial running time
- Dumas, Heckenbach, Saunders & Welker 2003: survey of methods which may be superior for very large boundary matrices

Implementations

- Dumas et al.: `homology` / `GAP` / `LinBox`
- Mischaikow, Kalies, Mrozek, Pilarczyk et al.: `CHomP`
- Juda & Mrozek: `CAPD::RedHom`
- `polymake`

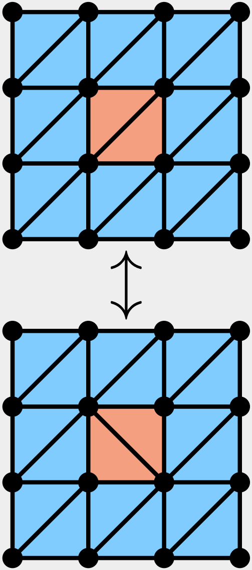
Homology Computation, II

∂_2	01	02	03	04	05	12	13	14	15	23	24	25	34	35	45
012	1	−1				1									
014	1			−1				1							
023		1	−1							1					
035			1		−1									1	
045				1	−1										1
125						1			−1			1			
134							1	−1					1		
135							1		−1					1	
234										1	−1		1		
245											1	−1			1

$(\partial_1)^{tr}$	01	02	03	04	05	12	13	14	15	23	24	25	34	35	45
0	−1	−1	−1	−1	−1										
1	1					−1	−1	−1	−1						
2		1				1				−1	−1	−1			
3			1				1			−1			−1	−1	
4				1				1			1		−1		1
5					1				1			1		1	1

Bistellar Flips in Combinatorial Manifolds

- Pachner 1987: PL-homeomorphic $\Longleftrightarrow$ bistellarly equivalent
- Björner & Lutz 2000: local search heuristics for sphere recognition
  - flip until boundary of $(d+1)$-simplex reached



Example: A Collapsible 5-Manifold and Its Boundary

Adiprasito, Benedetti & Lutz 2014+

N non-PL-5-manifold with

$$f = (5013, 72300, 290944, 495912, 383136, 110880)$$

- collapsible but not a ball
- Whitehead: PL + collapsible $\Rightarrow$ ball

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∂N = homology 4-sphere with

$$f = (5010, 65520, 212000, 252480, 100992)$$

- $\pi_1(\partial N)$ = binary icosahedral group
- recognizing all face links (to show manifold property) takes 7.5h

Strategies and Statistics for Discrete Morse Vectors

homology 4-sphere ∂N

discrete Morse vectors found in 100 runs (about 15sec each)

random	
$(1,3,5,2,1)$	29
$(1,4,6,2,1)$	24
(1,2,4,2,1)	16
$(1,5,7,2,1)$	10
$(1,6,8,2,1)$	10
$(1,8,10,2,1)$	2
$(1,3,6,3,1)$	1
$(1,3,7,4,1)$	1
$(1,4,7,3,1)$	1
$(1,7,9,2,1)$	1
$(1,11,13,2,1)$	1
$(2,4,5,2,1)$	1
$(2,8,9,2,1)$	1
$(2,9,10,2,1)$	1
$(2,14,15,2,1)$	1

random-lex	
(1,2,4,2,1)	41
$(1,3,5,2,1)$	33
$(1,4,6,2,1)$	12
$(1,5,7,2,1)$	4
$(1,6,8,2,1)$	4
$(1,3,6,3,1)$	2
$(1,2,5,3,1)$	1
$(1,2,6,4,1)$	1
$(1,4,7,3,1)$	1
$(1,7,9,2,1)$	1

random-revlex	
$(1,3,5,2,1)$	42
(1,2,4,2,1)	32
$(1,4,6,2,1)$	13
$(1,2,5,3,1)$	3
$(1,4,7,3,1)$	3
$(1,3,6,3,1)$	2
$(1,3,7,4,1)$	1
$(1,5,7,2,1)$	1
$(1,5,8,3,1)$	1
$(1,6,8,2,1)$	1
$(2,4,5,2,1)$	1

Conclusion

- sphere recognition often surprisingly easy in practice, even for large instances
- challenges remain:
  - e.g., Akbulut-Kirby spheres ($d = 4$)

- J.: *Computing invariants of simplicial manifolds*, `arXiv:math/0401176`
- J., Lutz & Tsuruga: *Heuristics for sphere recognition*, LNCS 8592: Proceedings ICMS 2014