The Order Dimension of Planar Maps Revisited*

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Abstract. Schnyder characterized planar graphs in terms of order dimension. This seminal result found several extensions. A particularly far reaching extension is the Brightwell-Trotter Theorem about planar maps. It states that the order dimension of the incidence poset $\mathbf{P}_{\mathrm{M}}$ of vertices, edges and faces of a planar map $\mathbf{M}$ has dimension at most 4. The original proof generalizes the machinery of Schnyder-paths and Schnyder-regions. In this note we use a simple result about the order dimension of grid intersection graphs to show a slightly stronger result: $\dim(\mathsf{split}(\mathbf{P}_{\mathrm{M}})) \leq 4$. This may be the first result in the area that is obtained without using the tools introduced by Schnyder.

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1 Introduction

In the context of this work we think of a *planar map* as a planar multigraph, loops and multiple edges allowed, with a fixed plane embedding. With a planar map $\mathbf{M}$ we associate the *incidence poset* $\mathbf{P}_{\mathrm{M}}$ of vertices, edges and faces of $\mathbf{M}$, i.e., the inclusion order of vertices, edges and faces considered as closed sets. Figure 1 shows a map and its incidence poset.

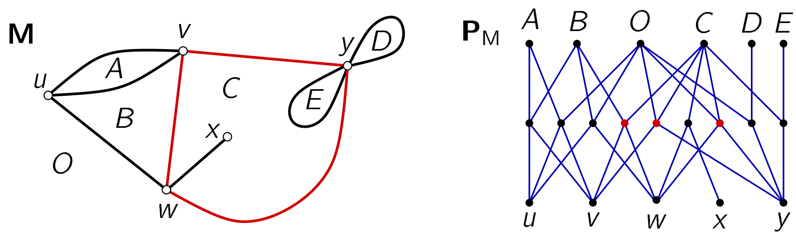

Figure 1: A planar map $\mathbf{M}$ and its incidence poset $\mathbf{P}_{\mathrm{M}}$. Three edges have been highlighted to improve readability.

The *order dimension* $\dim(\mathbf{P})$ of an order is the least t such that $\mathbf{P}$ is the intersection of t linear extensions of $\mathbf{P}$. A collection of linear extensions of $\mathbf{P}$ whose intersection is $\mathbf{P}$ is called a *realizer* for $\mathbf{P}$. Alternatively, $\dim(\mathbf{P})$ is the smallest dimension t such that $\mathbf{P}$ is a induced suborder of $(\mathbb{R}^t, \leq_{\mathrm{dom}})$, where the dominance relation is defined by $(x_1, \ldots, x_t) \leq_{\mathrm{dom}} (y_1, \ldots, y_t)$ iff $x_i \leq y_i$

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for all $i = 1, .., t$. Trotter's book [34] is the standard reference for the dimension theory of partial orders.

Theorem 1 (Brightwell and Trotter [10]) $\dim(\mathbf{P}_\mathrm{M}) \le 4$ *for every planar map* $\mathbf{M}$.

The *split* of an order $\mathbf{P} = (X, <)$ is defined as $\mathsf{split}(\mathbf{P}) = (X' \cup X'', <_s)$ where X' and X'' are two copies of X and $x' <_s y''$ iff $x \le y$. The split was defined by Kimble (c.f. [34]) who also proved that for all orders $\mathbf{P}$:

$$\dim(\mathbf{P}) \le \dim(\mathsf{split}(\mathbf{P})) \le \dim(\mathbf{P}) + 1.$$

Both inequalities are easily obtained by looking at critical pairs and alternating cycles, c.f. [34].

The main contribution of this note is a simple proof of the following:

Theorem 2 $\dim(\mathsf{split}(\mathbf{P}_\mathrm{M})) \le 4$ *for every planar map* $\mathbf{M}$.

The material is organized as follows: In the next section we discuss bipartite graphs in the context of order dimension. There we also proof that grid intersection graphs have order dimension at most four. In Section 3 we show that for a planar map M the bipartite comparability graph of $\mathsf{split}(\mathbf{P}_\mathrm{M})$ admits a representation as grid intersection graph. This implies Theorem 2. In the last section we collect some background information about dimension of orders and planarity and indicate further directions of research.

2 Bipartite graphs and orders

Let $G = (X, Y; E)$ be a bipartite graph. With G we can associate an order $\mathbf{Q}_\mathrm{G}$ of height two by converting an edge $\{x, y\} \in E$ with $x \in X$ into a comparability $x < y$. Since G is bipartite there are no transitive relations in $\mathbf{Q}_\mathrm{G}$. If we interchange the partition classes X and Y of G we obtain the dual order of $\mathbf{Q}_\mathrm{G}$. Since the dimension of an order and its dual equal each other we can take $\dim(\mathbf{Q}_\mathrm{G})$ as a well defined notion of dimension for a bipartite graph. Indeed for this research it was useful to think of bipartite graphs as orders of height 2 and conversely.

A *grid intersection graph* is a graph admitting an intersection representation with only vertical and horizontal segments. In our context we require that vertical segments and horizontal segments form two independent sets, so that grid intersection graphs are bipartite. This class of graphs is also known as *pure 2-DIR*.

Proposition 1 *If a bipartite graph* $G = (X, Y; E)$ *has a representation as grid intersection graph, then* $\dim(G) \le 4$*, i.e.,* $\dim(\mathbf{Q}_\mathrm{G}) \le 4$.

Proof. Assume a grid intersection representation of G with elements of X being represented as horizontal segments and the elements of Y as vertical segments such that no two endpoints of different segments share a coordinate.

In $\mathbf{Q}_\mathrm{G}$ the set X is the set of minimal elements and Y is the set of maximal elements. Define four linear extensions of $\mathbf{Q}_\mathrm{G}$ corresponding to the four directions up, left, down, right (we can also think of the geographic directions N, W, S, E). For each of the linear extensions we project all segments orthogonally onto a ray pointing in the given direction. Then we trace the ray with the projections and list the elements of $X \cup Y$ obeying the following rules: elements of X (minimals) are taken as soon as the first point of the projection of their segment is met while elements of Y (maximals) are taken when the last point of the projection of their segment is met, Figure 2 shows an example. Comparable elements correspond to crossing segments. The rules ensure that if $x < y$ is a comparable pair, then independent of the direction x is taken before y, hence the total orders corresponding to the directions are indeed linear extensions.

To show that L_N, L_W, L_S, and L_E form a realizer of $\mathbf{Q}_{\mathrm{G}}$, i.e, their intersection is $\mathbf{Q}_{\mathrm{G}}$ we have to show that a pair a, b of incomparable elements appear in either order in the linear extensions. Since a and b are incomparable their segments are disjoint. Hence, there is a line separating them. In the given setting where each of the segments is horizontal or vertical there is a horizontal or vertical separating line ℓ. If the separating line ℓ is vertical, then the order of a and b in L_W and L_E is reversed. If ℓ is horizontal, then L_N and L_S have a and b in different order. □

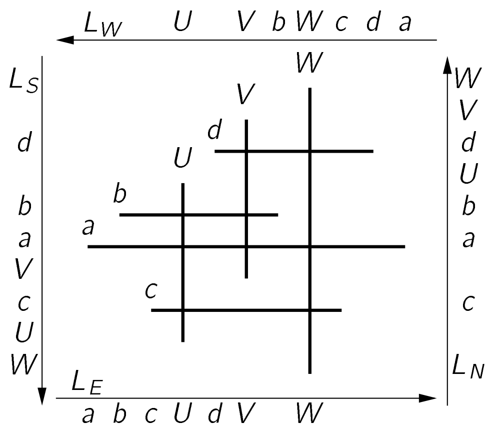

Figure 2: A grid intersection with bipartition classes $X = \{a, b, c, d\}$ (minimals) and $Y = \{U, V, W\}$ (maximals) with its 4-realizer L_S, L_E, L_N, L_W.

The applicability of the proposition is limited by the fact that recognizing grid intersection graphs is NP-hard [26]. Still in some interesting cases a grid intersection may be available or constructible. We will see such a case in the next section. The following corollary may also lead to applications of the proposition. Define a *cross matrix* to be a 3×3 matrix $\begin{pmatrix} * & 1 & * \\ 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}$, where the stars are place holder for unrestricted entries. In [5] grid intersection graphs have been characterized as bipartite graphs admitting an adjacency matrix that has no cross submatrix.

Corollary 1 *If a bipartite graph $G = (X, Y; E)$ has an ordering on X and Y such that the bipartite adjacency matrix A_G that has no induced cross submatrix, then* $\dim(G) \leq 4$.

Another result from [5] that should be mentioned in our context is that the class of grid intersection graphs and the class of bipartite intersection graphs of axis-aligned rectangles (a.k.a. bipartite graphs of boxicity 2) are the same. Indeed Proposition 1 was already known in the context of boxicity: in [1] it was shown that $\dim(P) \leq 2\mathrm{box}(G_P)$, where G_P is the comparability graph of P.

Proposition 1 has a natural generalization to intersection graphs of more general objects than segments. Let $\mathcal{S}$ be a class of geometric objects in $\mathbb{R}^d$ we say that $\mathcal{S}$ is *t-separable* if there exist a family $\mathcal{H} = \{H_1, \dots H_t\}$ of hyperplanes such that any two disjoint elements of $\mathcal{S}$ can be separated by a translate of one of the hyperplanes from $\mathcal{H}$.

Proposition 2 *If a bipartite graph $G = (X, Y; E)$ has a representation as intersection graphs of objects from a t-separable class, then* $\dim(G) \leq 2t$*, i.e.,* $\dim(\mathbf{Q}_{\mathrm{G}}) \leq 2t$.

The result is best possible as the standard example S_{2t} of a $2t$ dimensional order admits an intersection representation with axis aligned boxes in $\mathbb{R}^t$.

3 Planar maps and grid intersection graphs

In this section we show that starting from a planar map $\mathbf{M}$ we can construct a grid intersection graph B such that the associated order $\mathbf{Q}_\mathrm{B}$ equals the split of the incidence order $\mathbf{P}_\mathrm{M}$ of $\mathbf{M}$, i.e., $\mathbf{Q}_\mathrm{B} = \mathrm{split}(\mathbf{P}_\mathrm{M})$. Proposition 3 is a weaker intermediate result. The proof of this proposition, however, is very transparent and cleanly shows the key idea.

An angle of $\mathbf{M}$ is a triple (e_1, v, e_2) where e_1 and e_2 are consecutive edges in the clockwise order of edges around v, i.e., an adjacent pair in the rotation of v. Note that angles of $\mathbf{M}$ and of the dual map $\mathbf{M}^*$ are in bijection, indeed f is the face on the left side when leaving v along e_2, iff (e_2^*, f, e_1^*) is an angle of $\mathbf{M}^*$. The angular map $\mathbf{A}_\mathrm{M}$ associated with $\mathbf{M}$ has vertices $V \cup F$ where V and F are the vertices and faces of $\mathbf{M}$ respectively. The edges of $\mathbf{A}_\mathrm{M}$ correspond to incidences (v, f) at angles of $\mathbf{M}$, i.e., $\mathbf{A}_\mathrm{M}$ has as many edges connecting v and f as there are angles shared by the two. An example of a map $\mathbf{M}$ with its angular map is shown in Figure 3.

The angular map $\mathbf{A}_\mathrm{M}$ of a planar map is a again a planar map and it is bipartite, indeed every facial cycle has length four, i.e., $\mathbf{A}_\mathrm{M}$ is a plane quadrangulation. Although the drawing of $\mathbf{M}$ is fixed there remains some choices for the drawings of $\mathbf{A}_\mathrm{M}$ and of the dual map $\mathbf{M}^*$. However if a drawing of one of the two is fixed then there is an essentially unique consistent choice for the drawing of the other. We want to use Theorem 3 to get a grid intersection representation of the angle graph, however, the theorem only applies to simple graphs while $\mathbf{A}_\mathrm{M}$ may have multiple edges. Therefore, we remove all but one instance of every multiedge of $\mathbf{A}_\mathrm{M}$. This results in a *reduced angular map*. Since $\mathbf{A}_\mathrm{M}$ is a planar map it makes a difference which of the instances of a multiedge is kept so that the reduced angular map is not uniquely defined. Theorem 3 could directly be applied to the reduced angular map, most proofs of the theorem, however, assume that the planar bipartite in question is a quadrangulation. Therefore, we prefer to first augment the reduced angular map to a quadrangulation $\mathbf{A}^\diamond_\mathrm{M}$ by adding some new vertices and edges that are incident to at least one new vertex.

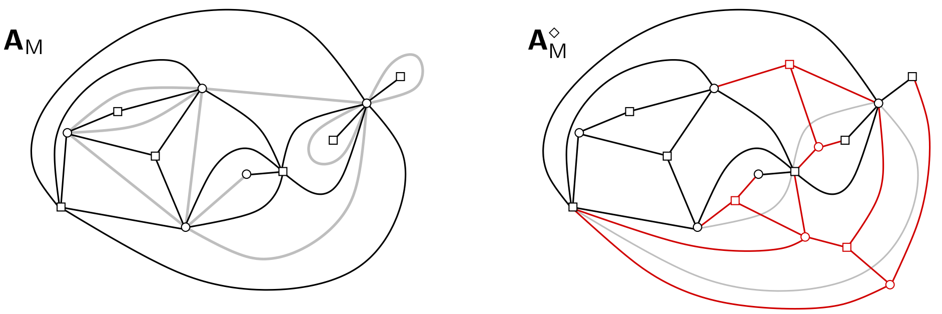

Figure 3: The planar map $\mathbf{M}$ from Fig. 1 with its angular map $\mathbf{A}_\mathrm{M}$ (left) and a quadrangular augmentation $\mathbf{A}^\diamond_\mathrm{M}$ of a reduced angular map (right).

Theorem 3 *Every planar bipartite map* $\mathbf{H}$ *admits a contact representation with interiorly disjoint horizontal and vertical segments.*

This was shown by Hartman et al. [25] and by de Fraysseix et al. [12]. A short inductive proof can be found in [11]. A survey of different proofs of the theorem is contained in [17].

Apply Theorem 3 to $\mathbf{A}^\diamond_\mathrm{M}$. In the resulting contact representation enlarge all segments slightly so that all contacts become intersections but without introducing additional intersections. From Proposition 1 it follows that $\dim(\mathbf{A}^\diamond_\mathrm{M}) \le 4$. From the definitions of $\mathbf{A}_\mathrm{M}$ and $\mathbf{A}^\diamond_\mathrm{M}$ it follows that if v is a vertex and f a face of $\mathbf{M}$ then there is an edge (v, f) in $\mathbf{A}^\diamond_\mathrm{M}$ if and only if v and f are incident in $\mathbf{M}$. This shows that the incidence order of vertices and faces of $\mathbf{M}$ is an induced suborder of $\mathbf{Q}_{A^{\diamond}_{M}}$. The monotonicity of order dimension with respect to induced suborders implies:

Proposition 3 *The order dimension of the incidence order of vertices and faces of a planar map is at most four.*

For the proof of Theorem 2 we have to work a little more. Recall that the order $\mathsf{split}(\mathbf{P}_M)$ contains points v', v'' for every vertex v, points e', e'' for every edge e, and points f', f'' for every face f of $\mathbf{M}$. The relations are $x' < y''$ if and only if $x \subseteq y$ when identifying x and y with their geometric extent in $\mathbf{M}$. The grid intersection graph constructed for the proof of Proposition 3 can be regarded a representation for the suborder of $\mathsf{split}(\mathbf{P}_M)$ induced on $V' \cup F''$. A representation of the suborder induced on $V' \cup V'' \cup F' \cup F''$ is easily obtained by adding to each segment representing an $x \in V' \cup F''$ a short segment that intersects x and nothing else. Hence the challenge is to find appropriate segments representing the elements of $E' \cup E''$. If e is an edge of $\mathbf{M}$ that is neither a loop nor a bridge, then e corresponds to a non-degenerate quadrangular face $\lozenge_e$ of $\mathbf{A}_M$, i.e., to a face with four different incident vertices. Assuming that the four edges of the face $\lozenge_e$ are not removed when reducing $\mathbf{A}_M$ to a simple graph, then we find a rectangle in the grid intersection representation that corresponds to $\lozenge_e$. Suppose that the horizontal segments bounding this rectangle correspond to the vertices of e and the vertical segments to the faces of the dual edge e^*, then we can introduce segments representing e' and e'' as a pair of crossing segments in the rectangle. The segment for e' is horizontal and intersects the vertical boundaries and the segment for e'' is vertical and intersects the horizontal boundaries of the rectangle. A special case is when the quadrangular face $\lozenge_e$ of $\mathbf{A}_M$ happens to be the outer face, however, in this case it is also quite obvious how to introduce segments representing e' and e'' as a pair of crossing segments in the unbounded face.

Suppose $\mathbf{M}$ is a planar map such that the angular map $\mathbf{A}_M$ has no multiple edges, then $\mathbf{A}_M$ has a segment contact representation such that the rectangular faces of this representation (including the outer face) are in bijection to the edges of $\mathbf{M}$. From this segment contact representation we obtain, with the above construction, a grid intersection representation of the comparability graph of $\mathsf{split}(\mathbf{P}_M)$. For an example see Figure 4. We have thus obtained:

Proposition 4 *If* $\mathbf{M}$ *is an at least 2-connected planar map, then* $\dim(\mathsf{split}(\mathbf{P}_M)) \leq 4$.

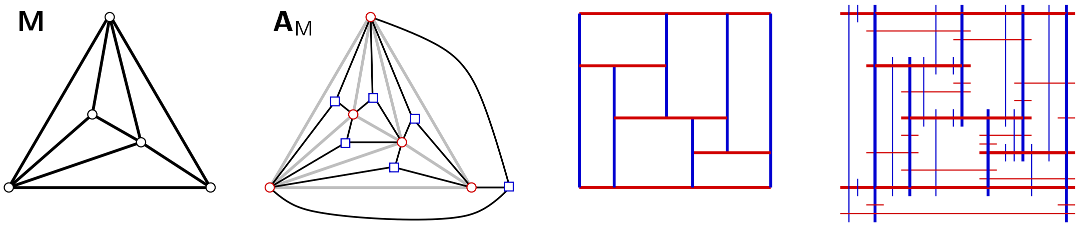

Figure 4: A 2-connected planar map $\mathbf{M}$, the angular map $\mathbf{A}_M$, a segment contact representation of $\mathbf{A}_M$, and the grid intersection representation of $\mathsf{split}(\mathbf{P}_M)$.

Finally we construct a grid intersection representation for $\mathsf{split}(\mathbf{P}_M)$ for general planar maps $\mathbf{M}$ that takes the previous proposition as a base case in an inductive construction.

Let $\mathbf{M}$ be a planar map such that $\mathbf{A}_M$ has multiple edges. The reason for the multiple edges are loops and cut vertices of $\mathbf{M}$, so we have to deal with them. First we subdivide each loop ℓ with a new vertex x_ℓ into a parallel pair ℓ_1, ℓ_2 of edges. Let $\mathbf{M}^+$ be the resulting map.

Fact S. $\mathsf{split}(\mathbf{P}_M)$ is an induced suborder of $\mathsf{split}(\mathbf{P}_{M^+})$.

For every loop ℓ of $\mathbf{M}$ delete the elements x'_ℓ and x''_ℓ as well as ℓ'_2 and ℓ''_2 from $\mathsf{split}(\mathbf{P}_{M^+})$. The remaining elements ℓ'_1 and ℓ''_1 in $\mathsf{split}(\mathbf{P}_{M^+})$ take the role of ℓ' and ℓ'' in $\mathsf{split}(\mathbf{P}_M)$. All the other elements and relations of split($\mathbf{P}_{\mathsf{M}}$) and split($\mathbf{P}_{\mathsf{M}^+}$) are in obvious bijection. △

Now consider a cut vertex v and break $\mathbf{M}^+$ at v into two maps $\mathbf{M}_1$ and $\mathbf{M}_2$ such that both retain a copy of v. We assume that the outer face of $\mathbf{M}_1$ equals the outer face of $\mathbf{M}$. By induction on the number of vertices we can assume grid intersection representations of split($\mathbf{P}_{\mathsf{M}_1}$) and split($\mathbf{P}_{\mathsf{M}_2}$). Moreover, if we follow the construction, the grid intersection representation of $\mathbf{M}_2$ has the horizontal segment s_v corresponding to v and the vertical segment s_f corresponding to the unbounded face. The segments s_v and s_f have both endpoints in the outer region of the representation. Since v and f_2 also belong to $\mathbf{M}_1$ and are incident there, the grid intersection representation of $\mathbf{M}_1$ has a corner where the segments of v and f_2 meet. An appropriately scaled copy of the representation of split($\mathbf{P}_{\mathsf{M}_2}$) fits into this corner to make a representation of split($\mathbf{P}_{\mathsf{M}^+}$). An example for the inductive step is shown in Figure 5. This completes the proof of our main result, Theorem 2.

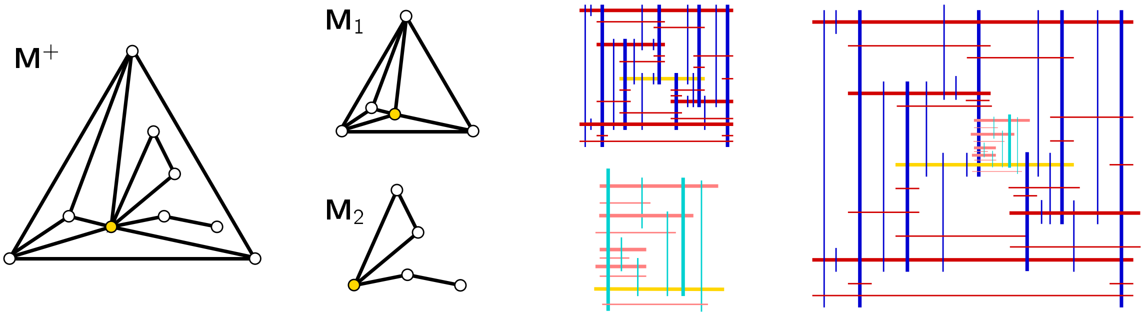

Figure 5: Map $\mathbf{M}^+$ is split at a cut vertex into $\mathbf{M}_1$ and $\mathbf{M}_2$. The grid intersection representation of $\mathbf{M}^+$ has a copy of the representation of $\mathbf{M}_2$ in a corner of the representation of $\mathbf{M}_1$.

4 Order dimension and planarity

In this final section we intend to give a brief survey on results connecting graph planarity and order dimension. Probably the first contribution can be found in a paper of Babai and Duffus [3], they prove that the order dimension of the incidence order of a non-planar graph is at least 4. In his seminal paper [30] Schnyder added the following results:

1. The dimension of the incidence order of vertices and faces of a graph G is at most 3 $\iff$ G is planar.
2. The dimension of the incidence order $\mathbf{P}_G$ of vertices, edges, and faces of a planar triangulation G is 4 and the dimension of the suborder obtained by deleting a single vertex or face from $\mathbf{P}_G$ is 3. (Note that this can be interpreted as a statement about the dimension of face lattices of simplicial 3-polytopes.)

Due to the count by Google-Scholar Schnyder's paper [30] and the followup [31] where he emphasizes on compact drawings of planar graphs, together have 700 citations (October 2013). The reason behind this huge number is that Schnyder introduced important auxiliary structures on planar triangulations. These structures, nowadays known as Schnyder woods (respectively Schnyder realizers) and Schnyder angle labelings have found many applications. For illustration we cite some applications to graph drawing models [27], [13], [6], [2] and to the enumeration and encoding of planar maps [8], and [24].

Regarding the limits of possible extensions of Schnyder's results there are two observations:

- Every graph of chromatic number at most 4 has an incidence order of dimension at most 4. This class includes graphs of arbitrarily high genus (Schnyder [30]).
- Skeleton graphs of simplicial 4-polytopes can contain arbitrarily large complete graphs (neighborly polytopes). Since the dimension of complete graphs has the growth rate of $\log\log(n)$ (see e.g. [14]), there is no constant bound for the dimension of 4-polytopes.

Ossona de Mendez [29] shows a generalization of **2** to stacked polytopes of higher dimensions. Brightwell and Trotter [9] and [10] proved strong extensions of Schnyder's results.

3. The dimension of the incidence order $\mathbf{P}_G$ of vertices, edges, and faces of a 3-connected planar graph G is 4 and the dimension of the suborder obtained by deleting a single vertex or face from $\mathbf{P}_G$ is 3, see [9]. (This can be interpreted as a statement about the dimension of face lattices of 3-polytopes.)

4. The dimension of the incidence order $\mathbf{P}_G$ of vertices, edges, and faces of a planar map $\mathbf{M}$ with loops and multiple edges allowed is at most 4, [10].

The key to the proofs of Brightwell and Trotter are Schnyder like structures but the proofs are rather long and technical. In [33], Problem 2, Trotter asked for a simple argument (in the sense that it avoids the use of Schnyder's structures) showing that the dimension of the vertex-edge-face poset of a planar map is at most 10^{10}.

A simplified proof for **3** was given in [15], the proof introduces a very useful generalization of Schnyder's structures to 3-connected planar graphs. The usefulness may be exemplified by some of the subsequent applications in graph drawing, see e.g. [7], [4]. Proposition 4 is stronger than the first part of **3** (dimension ≤ 4), it would be interesting to understand the drop in dimension upon removal of a face on the basis of the grid intersection representation.

Miller [28] pioneered the connection between Schnyder structures and orthogonal surfaces that subsequently lead to new and even simpler proofs of **3** in [16] and [23]. Orthogonal structures also open a line for generalizations to higher dimensions. This line of research is the subject of [18]. So far **4** has shown to be more elusive. The present paper seems to contain the first new proof of the result.

In [21] with Trotter we provide a characterization of outerplanar graphs in terms of order dimension. Again the key tool are Schnyder structures. The research was continued in [20] with characterizations of outerplanar maps whose vertex-face respectively vertex-edge-face order has order dimension at most 3.

Adjacency posets of planar graphs have been studied in [19]. The adjacency poset of a graph $G = (V, E)$ has two copies V' and V'' of V as vertices and comparabilities $x' < y''$ iff $\{x, y\} \in E$. For planar graphs the dimension of the adjacency poset is at most 8 while there exist examples of dimension 5. For outerplanar graphs the upper bound is 5 and the existence of examples of dimension 4 is shown. In [19] it is also shown that if the underlying graph of an order P of height 2 is planar, then $\dim(P) \leq 4$. The proof there is based **3**, i.e., on the use of Schnyder structures. The result implies Proposition 3.

The dimension of orders with a planar diagram respectively a planar cover graph also has received quite some attention. Since this is not central to this note we only provide two recent references where additional pointers can be found: [22] and [32].

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References

[1] A. Adiga, D. Bhowmick, and L. S. Chandran, *Boxicity and poset dimension*, SIAM J. Discrete Math., 25 (2011), pp. 1687–1698.

[2] M. J. Alam, T. Biedl, S. Felsner, M. Kaufmann, S. G. Kobourov, and T. Ueckerdt, *Computing cartograms with optimal complexity*, in Proc. ACM Symp. on Computational Geometry, 2012, pp. 21–30.

[3] L. Babai and D. Duffus, *Dimension and automorphism groups of lattices*, Algebra Universalis, 12 (1981), pp. 279–289.

[4] I. Bárány and G. Rote, *Strictly convex drawings of planar graphs*, Documenta Mathematica, 11 (2006), pp. 369–391.

[5] S. Bellantoni, I. B.-A. Hartman, T. M. Przytycka, and S. Whitesides, *Grid intersection graphs and boxicity*, Discrete Mathematics, 114 (1993), pp. 41–49.

[6] O. Bernardi and É. Fusy, *Schnyder decompositions for regular plane graphs and application to drawing*, Algorithmica, 62 (2012), pp. 1159–1197.

[7] N. Bonichon, S. Felsner, and M. Mosbah, *Convex drawings of 3-connected planar graphs*, Algorithmica, 47 (2007), pp. 399–420.

[8] N. Bonichon, C. Gavoille, and N. Hanusse, *An information-theoretic upper bound of planar graphs using triangulation*, in Proceedings STACS '02, vol. 2607 of Lecture Notes Comput. Sci., Springer-Verlag, 2003, pp. 499 – 510.

[9] G. Brightwell and W. T. Trotter, *The order dimension of convex polytopes*, SIAM J. Discrete Math., 6 (1993), pp. 230–245.

[10] G. Brightwell and W. T. Trotter, *The order dimension of planar maps*, SIAM J. Discrete Math., 10 (1997), pp. 515–528.

[11] J. Czyzowicz, E. Kranakis, and J. Urrutia, *A simple proof of the representation of bipartite planar graphs as the contact graphs of orthogonal straight line segments*, Inf. Process. Lett., 66 (1998), pp. 125–126.

[12] H. de Fraysseix, P. Ossona de Mendez, and J. Pach, *A left-first search algorithm for planar graphs*, Discr. Comput. Geom., 13 (1995), pp. 459–468.

[13] R. Dhandapani, *Greedy drawings of triangulations*, Discrete & Computational Geometry, 43 (2010), pp. 375–392.

[14] B. Dushnik and E. W. Miller, *Partially ordered sets*, American Journal of Mathematics, 63 (1941), pp. 600–610.

[15] S. Felsner, *Convex drawings of planar graphs and the order dimension of 3-polytopes*, Order, 18 (2001), pp. 19–37.

[16] S. Felsner, *Geodesic embeddings and planar graphs*, Order, 20 (2003), pp. 135–150.

[17] S. Felsner, *Rectangle and square representations of planar graphs*, in Thirty Essays in Geometric Graph Theory, J. Pach, ed., Springer, 2013, pp. 213–248.

[18] S. Felsner and S. Kappes, *Orthogonal surfaces and their cp-orders*, Order, 25 (2008), pp. 19–47.

[19] S. Felsner, C. M. Li, and W. T. Trotter, *Adjacency posets of planar graphs*, Discrete Mathematics, 310 (2010), pp. 1097–1104.

[20] S. Felsner and J. Nilsson, *On the dimension of outerplanar maps*, Order, 28 (2011), pp. 415–435.

[21] S. Felsner and W. T. Trotter, *Posets and planar graphs*, J. of Graph Theory, 49 (2005), pp. 262–272.

[22] S. Felsner, W. T. Trotter, and V. Wiechert, *The dimension of posets with planar cover graphs*, 2013. Graphs and Combinatorics, accepted.

[23] S. Felsner and F. Zickfeld, *Schnyder woods and orthogonal surfaces*, Discrete & Computational Geometry, 40 (2008), pp. 103–126.

[24] E. Fusy, D. Poulalhon, and G.Schaeffer, *Dissection and trees, with applications to optimal mesh encoding and random sampling*, ACM Trans. Algorithms, 4 (2008), pp. 1–48.

[25] I. B.-H. Hartman, I. Newman, and R. Ziv, *On grid intersection graphs*, Discrete Mathematics, 97 (1991), pp. 41–52.

[26] J. Kratochvíl, *A special planar satisfiability problem and a consequence of its NP-completeness*, Discrete Applied Mathematics, 52 (1994), pp. 233–252.

[27] C. Lin, H. Lu, and I.-F. Sun, *Improved compact visibility representation of planar graphs via Schnyder's realizer*, SIAM J. Discrete Math., 18 (2004), pp. 19–29.

[28] E. Miller, *Planar graphs as minimal resolutions of trivariate monomial ideals*, Documenta Math., 7 (2002), pp. 43–90.

[29] P. Ossona de Mendez, *Geometric realization of simplicial complexes*, in Graph Drawing, 1999, pp. 323–332.

[30] W. Schnyder, *Planar graphs and poset dimension*, Order, 5 (1989), pp. 323–343.

[31] W. Schnyder, *Embedding planar graphs on the grid*, in Proc. 1st ACM-SIAM Sympos. Discrete Algorithms, 1990, pp. 138–148.

[32] N. Streib and W. T. Trotter, *Dimension and height for posets with planar cover graphs*, Electronic Notes in Discrete Mathematics, 38 (2011), pp. 807–812.

[33] W. Trotter, *Progress and new directions in dimension theory for finite partially ordered sets*, in Extremal problems for finite sets, Bolyai Society, 1991, pp. 457–477.

[34] W. T. Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory*, Johns Hopkins Series in the Mathematical Sciences, The Johns Hopkins University Press, 1992.